Weak strong uniqueness to compressible Euler and similar systems

Piotr Gwiazda

Institute of Mathematics
Polish Academy of Sciences

Phoenix, 8th December 2015

Savage-Hutter model

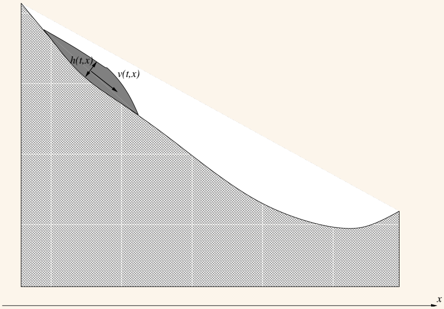
Original 1-D Savage-Hutter Model '89

Find

- the height $h : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$
- the (depth-averaged) velocity $u : \mathbb{R} \times \mathbb{R}^+ \longrightarrow \mathbb{R}$

satisfying the system of conservation laws

$$\begin{aligned} \partial_t h + \partial_x (hu) &= 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \beta h^2 \right) &= hg, \end{aligned} \qquad \text{(SH)}$$



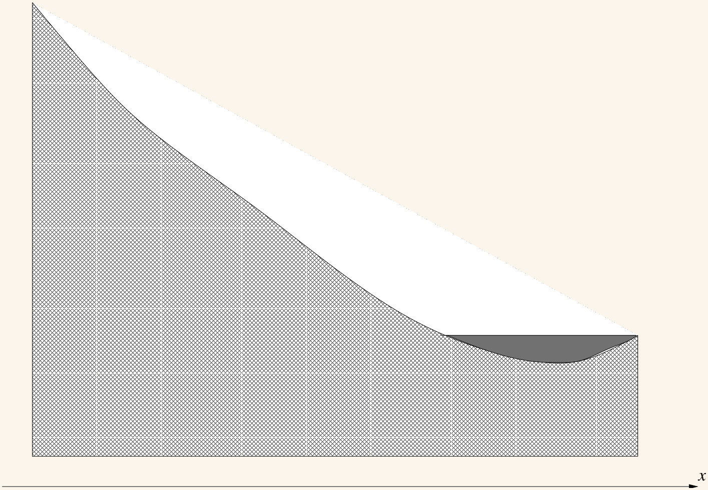
Here $\beta(x)$, $g(x,u)$ are defined by

$$\begin{aligned}\beta(x) &= k\cos\xi(x),\\ g(x,u) &= \sin\xi(x) - \mathrm{sign}(u)\cos\xi(x)\end{aligned}$$

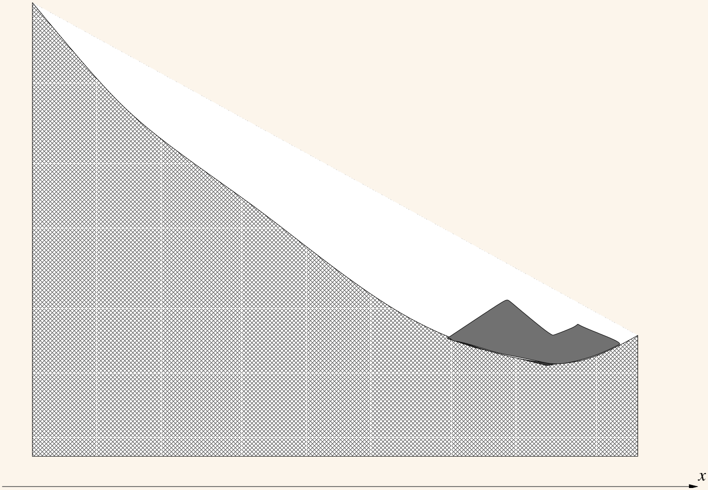
where $\xi(x)-$ inclination angle of bottom topography at point x.

$$\mathrm{sign}(u) = \begin{cases} -1 & \text{for} \quad u < 0 \\ [-1,1] & \text{for} \quad u = 0 \\ 1 & \text{for} \quad u > 0 \end{cases}$$

Possible (physical) formation of steady states for fluid



Possible (physical) formation of steady states for granular material



Steady state solution for Savage-Hutter

Consider static problem ($u = 0$) with $\xi = 0$

$\Rightarrow$ Governing equation:

$$kh_x \in [-1, 1]$$

$\Rightarrow$ h is Lipschitz with Lipschitz constant $\leq \frac{1}{k}$

Physically relevant solution!

2d System

$$\partial_t h + \operatorname{div}(h\mathbf{u}) = 0,$$

$$\partial_t(h\mathbf{u}) + \operatorname{div}(h\mathbf{u}\otimes\mathbf{u}) + \nabla(ah^2) = h\left(-\gamma\frac{\mathbf{u}}{|\mathbf{u}|} + \mathbf{f}\right),$$

where

$$\Omega = \left([0,1]|_{\{0,1\}}\right)^2.$$

initial conditions

$$h(0,\cdot) = h_0,\ \mathbf{u}(0,\cdot) = \mathbf{u}_0.$$

The term $\frac{\mathbf{u}}{|\mathbf{u}|}$ has to be understood as a multi-valued mapping, which for non-zero velocities takes the value $\frac{\mathbf{u}}{|\mathbf{u}|}$, whereas for $\mathbf{u} = 0$ takes the values in the whole closed unit ball.

Finite time dissipation of kinetic energy for weak admisible solution to Savage Huter model

Theorem (G., Świerczewska-Gwiazda, Wiedemann, Nonlinarity 2015)

Let $(h, \mathbf{u})$ be an admissible weak solution of the Savage-Hutter equations with initial energy E_0 and $\|f\|_\infty < d$. Then there exists a time $0 \leq T < \infty$ such that for almost every $t > T$, $(h, \mathbf{u})$ is stationary, i.e. $\mathbf{u}(t, x) = 0$ for almost every $t > T$ and $x \in \Omega$, $\partial_t h(t, x) = 0$ for almost every $t > T$, $x \in \Omega$, and

$$\left| \nabla h - \frac{f}{2a} \right| \leq \frac{\gamma}{2a}.$$

The finite-time runout of solutions is essentially used at the modelling stage as providing data for calibration of the system. This property was assumed in numerical simulations.

Existence of weak solutions

Theorem (Feireisl, G., Świerczewska-Gwiazda to appear Comm. PDE)

Let $T > 0$ *and the initial data* h_0, $\mathbf{u}_0$ *be given. Suppose that* $\mathbf{f} \in C^1([0,T] \times \Omega; R^2)$.
Then the problem (S-H) admits infinitely many weak solutions in $(0,T) \times \Omega$. *The weak solutions belong to the class*

$$h,\ \partial_t h, \nabla h \in C^1([0,T] \times \Omega),$$

$$\mathbf{u} \in C_{\rm weak}([0,T]; L^2(\Omega; R^2)) \cap L^\infty((0,T) \times \Omega; R^2).$$

$$\operatorname{div} \mathbf{u} \in C([0,T] \times \Omega), \mathbf{B} \in L^\infty((0,T) \times \Omega; R^2)$$

Admissible solutions

Definition

We say that $[h, \mathbf{u}, \mathbf{B_u}]$ is an *admissible* weak solution to the Savage-Hutter system if in addition the energy inequality holds for a.a. $\tau \in (0, T)$.

$$E_{\rm tot}(\tau) \equiv \int_\Omega \left[\frac{1}{2} h |\mathbf{u}|^2 + a h^2 \right] (\tau, \cdot) \ {\rm d}x + \int_0^\tau \int_\Omega h \gamma \mathbf{B_u} \cdot \mathbf{u} \ {\rm d}x \ {\rm d}t$$

$$\leq \int_\Omega \left[\frac{1}{2} h_0 |\mathbf{u}_0|^2 + a h_0^2 \right] \ {\rm d}x + \int_0^\tau \int_\Omega h \mathbf{f} \cdot \mathbf{u} \ {\rm d}x \ {\rm d}t,$$

Weak-strong uniqueness

Savage-Hutter. Weak-strong uniqueness

Theorem (Feireisl, G., Świerczewska-Gwiazda to appear in Comm. PDE)

Let $[h, \mathbf{u}, \mathbf{B_u}]$ *be an admissible weak solution of the Savage-Hutter system in* $(0, T) \times \Omega$. *Let* $[H, \mathbf{U}, \mathbf{B_U}]$, $H > 0$ *be a globally Lipschitz (strong) solution of the same problem, with*

$$h_0 = H(0, \cdot), \ \mathbf{u}_0 = \mathbf{U}(0, \cdot).$$

Then

$$h = H, \ \mathbf{u} = \mathbf{U} \ \textit{a.e. in } (0, T) \times \Omega.$$

Measure-valued solutions

Weak-strong uniqueness for mvs

- Y. Brenier, C. De Lellis, L. Székelyhidi, Jr., Weak-strong uniqueness for measure-valued solutions. Comm. Math. Phys. 2011,
  Incompressible Euler -oscillation and concentration measure, general hyperbolic systems - only oscillation measure, both in weak formulation and entropy inequality
- S. Demoulini, D. M. A. Stuart, A. E. Tzavaras, Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics. Arch. Ration. Mech. Anal. 2012
  In weak formulation only oscillation measure, in entropy inequality there appears non-negative concentration measure
- P. G, A. Świerczewska-Gwiazda, E. Wiedemann, Weak-Strong Uniqueness for Measure-Valued Solutions of Some Compressible Fluid Models, Nonlinearity, 2015
  Oscillatory and vector-valued concentration measure both in weak formulation and entropy inequality

Generalized Young measures

A (generalized) Young measure on $\mathbb{R}^d$ with parameters in $\mathbb{R}^d \times \mathbb{R}^+$ is a triple $(\nu_{x,t}, m, \nu_{x,t}^\infty)$, where

- $\nu_{x,t} \in \mathcal{P}(\mathbb{R}^d)$ for a.e. $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$ (oscillation measure)
- $m \in \mathcal{M}^+(\mathbb{R}^d \times \mathbb{R}^+)$ (concentration measure)
- $\nu_{x,t}^\infty \in \mathcal{P}(\mathcal{S}^{d-1})$ for m-a.e. $(x, t) \in \mathbb{R}^d \times \mathbb{R}^+$ (concentration-angle measure)

R. J. DiPerna and A. J. Majda, Oscillations and concentrations in weak solutions of the incompressible fluid equations, Comm. Math. Phys. 1987.

J. J. Alibert and G. Bouchitté, Non-uniform integrability and generalized Young measures, J. Convex Anal. 1997.

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Compressible Euler equations

Formulation (CE)

$$\partial_t h + \mathrm{div}(hu) = 0$$
$$\partial_t(hu) + \mathrm{div}(hu \otimes u) + \nabla(\kappa h^\gamma) = hG.$$

Here, $h : [0, T] \times \mathbb{T}^n \to \mathbb{R}$, $u : [0, T] \times \mathbb{T}^n \to \mathbb{R}^n$, and $G : [0, T] \times \mathbb{T}^n \to \mathbb{R}^n$, $\gamma > 1$.

Remark

The global existence of measure-valued solutions was proved by Neustupa in '93 (see also Kröner & Zajączkowski '96 for polytropic fluid). However he used a different formulation, as the formalism of Alibert-Bouchitté '97 was not yet available. One can however rewrite the solutions of Neustupa in the form presented here. Neustupa's solutions can be seen to be admissible, as they can be obtained e.g. from an artificial viscosity approximation.

Measure-valued solutions to compressible Euler system

- We need a slight refinement which allows us to treat sequences whose components have different growth.
- Let $(u_k, w_k)_k$ be a sequence such that (u_k) is bounded in $L^p(\Omega;\mathbb{R}^l)$ and (w_k) is bounded in $L^q(\Omega;\mathbb{R}^m)$ $(1 \le p, q < \infty)$. Define the *nonhomogeneous unit sphere*

$$\mathbb{S}^{l+m-1}_{p,q} := \{(\beta_1, \beta_2) \in \mathbb{R}^{l+m} : |\beta_1|^{2p} + |\beta_2|^{2q} = 1\}.$$

Then, there exists a a subsequence and measures

$$\nu \in L^\infty_w(\Omega; \mathcal{P}(\mathbb{R}^{l+m})), m \in \mathcal{M}^+(\bar{\Omega}),\ \ \nu^\infty \in L^\infty_w(\Omega, m; \mathcal{P}(\mathbb{S}^{l+m-1}_{p,q}))$$

such that in the sense of measures

$$f(x, u_n(x), w_n(x))dx \overset{*}{\rightharpoonup} \int_{\mathbb{R}^{l+m}} f(x, \lambda_1, \lambda_2) d\nu_x(\lambda_1, \lambda_2) dx + \int_{\mathbb{S}^{l+m-1}_{p,q}} f^\infty(x, \beta_1, \beta_2) d\nu^\infty_x(\beta_1, \beta_2) m.$$

This is valid for all integrands f whose p-q-recession function exists and is continuous on $\bar{\Omega} \times \mathbb{S}_{p,q}^{l+m-1}$. The p-q-recession function is defined as

$$f^{\infty}(x, \beta_1, \beta_2) := \lim_{\substack{x' \to x \\ (\beta_1', \beta_2') \to (\beta_1, \beta_2) \\ s \to \infty}} \frac{f(x', s^q \beta_1', s^p \beta_2')}{s^{pq}}.$$

Remark about evolutionary problems with bounded energy

If $\Omega = [0, T] \times \tilde{\Omega}$ for some measurable $\tilde{\Omega} \subset \mathbb{R}^n$ (or $\tilde{\Omega} = \mathbb{T}^n$), and if the sequence $(u_n, w_n)_n$ is bounded in $L^\infty([0, T]; L^p(\tilde{\Omega}) \times L^q(\tilde{\Omega}))$, then the corresponding concentration measure m admits a disintegration of the form

$$m = m_t(dx) \otimes dt,$$

where $t \mapsto m_t$ is bounded and measurable viewed as a map from $[0, T]$ into $\mathcal{M}^+(\overline{\tilde{\Omega}})$.

Measure-valued solutions to compressible Euler system

We say that (ν, m, ν^∞) is a measure-valued solution of CE with initial data (h_0, u_0) if for every $\tau \in [0, T]$, $\psi \in C^1([0,T] \times \mathbb{T}^n; \mathbb{R})$, $\phi \in C^1([0,T] \times \mathbb{T}^n; \mathbb{R}^n)$ it holds that

$$\int_0^\tau \int_{\mathbb{T}^n} \partial_t \psi \bar{h} + \nabla\psi \cdot \overline{hu} dxdt + \int_{\mathbb{T}^n} \psi(x,0) h_0 - \psi(x,\tau)\bar{h}(x,\tau) dx = 0,$$

$$\int_0^\tau \int_{\mathbb{T}^n} \partial_t \phi \cdot \overline{hu} + \nabla\phi : \overline{hu \otimes u} + \operatorname{div} \phi \overline{h^\gamma} - \phi \cdot \overline{hG} dxdt$$

$$+ \int_{\mathbb{T}^n} \phi(x,0) \cdot h_0 u_0 - \phi(x,\tau) \cdot \overline{hu}(x,\tau) dx = 0.$$

Admissibility of measure-valued solutions to compressible Euler system

Let us set

$$E_{mvs}(t) := \int_{\mathbb{T}^n} \frac{1}{2}\overline{h|u|^2}(t,x) + \frac{1}{\gamma-1}\overline{h^\gamma}(t,x)dx$$

for almost every t, and

$$E_0 := \int_{\mathbb{T}^n} \frac{1}{2}h_0|u_0|^2(x) + \frac{1}{\gamma-1}h_0^\gamma(x)dx.$$

We then say that a measure-valued solution is admissible if

$$E_{mvs}(t) \leq E_0 + \int_0^t \int_{\mathbb{T}^n} \overline{hG\cdot u}(s,x)dxds$$

for almost all t.

Where

$$\bar{h} = \langle \lambda_1, \nu \rangle$$
$$\overline{h^\gamma} = \langle \lambda_1^\gamma, \nu \rangle + \langle \beta_1^\gamma, \nu^\infty \rangle m$$
$$\overline{hu} = \langle \sqrt{\lambda_1} \lambda', \nu \rangle$$
$$\overline{hu \otimes u} = \langle \lambda' \otimes \lambda', \nu \rangle + \langle \beta' \otimes \beta', \nu^\infty \rangle m$$
$$\overline{h|u|^2} = \langle |\lambda'|^2, \nu \rangle + \langle |\beta'|^2, \nu^\infty \rangle m$$
$$\overline{hG} = \langle \lambda_1 G, \nu \rangle = \bar{h} G.$$

If the solution is generated by some approximation sequences, then the black terms on right-hand side correspond to the biting limit of sequences whereas the blue ones corespond to concentration measure

Note that:

$$\begin{aligned} \|\langle \beta' \otimes \beta', \nu^\infty \rangle m\|_{\mathrm{TV}} &\leq C \|\langle \mathrm{tr}(\beta' \otimes \beta'), \nu^\infty \rangle m\|_{\mathrm{TV}} \\ &= C \|\langle |\beta'|^2, \nu^\infty \rangle m\|_{\mathrm{TV}} \end{aligned}$$

Savage-Hutter system

Existence of measure-valued solutions to S-H system - G. '05

Result is similar but slightly more complicated: Be careful! Young measures + multivalued function

Weak-Strong Uniqueness

Theorem (G., Świerczewska-Gwiazda, Wiedemann, 2015)

Let $G \in L^\infty([0,T]; L^2(\mathbb{T}^n))$ *and suppose* $H \in W^{1,\infty}([0,T]\times\mathbb{T}^n)$, $U \in C^1([0,T]\times\mathbb{T}^n)$ *is a solution of CE with initial data* $h_0 \geq c > 0$, $h_0 \in L^\gamma(\mathbb{T}^n)$, $h_0|u_0|^2 \in L^1(\mathbb{T}^n)$, *and* $H(x,t) \geq c > 0$ *for some constant* c *and all* $(t,x) \in [0,T]\times\mathbb{T}^n$. *If* (ν, m, ν^∞) *is an admissible measure-valued solution with the same initial data, then*

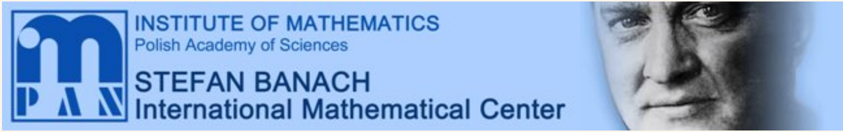
$$\nu_{t,x} = \delta_{(H(t,x),\sqrt{H(t,x)}U(t,x))} \text{ for a.e. } t,x, \text{ and } m = 0.$$

Savage-Hutter

An analogue result holds for the Savage-Hutter system

References:

- P. G. *An existence result for a model of granular material with non-constant density*, Asympt. Anal. 30 (2002), no. 1, 43-60
- P. G. *On measure valued solution to 2d gravity driven avalanche flow model*. Math. Methods Appl. Sci. 28, No. 18, (2005)
- P. G, A. Świerczewska-Gwiazda, E. Wiedemann *Weak-Strong Uniqueness for Measure-Valued Solutions of Some Compressible Fluid Models*, Nonlinearity, 2015
- E. Feireisl, P. G., A. Swierczewska-Gwiazda *On weak solutions to the 2D Savage-Hutter model of the motion of a gravity driven avalanche flow*, to appear in Comm. PDE, arXiv:1502.06223



SIMONS SEMESTER in BANACH CENTER **CrossFields PDEs**, WARSAW, 1.12.2016 – 31.03.2017

Opening event: **Winter School, 5 - 9.12.2016, Będlewo**

Organizers

- Eduard Feireisl
- Piotr Gwiazda
- Piotr Mucha
- Agnieszka Świerczewska-Gwiazda

Thank you for your attention

Appendix I

Application of the method of convex integration to S-H system

$$\partial_t h + \operatorname{div}(h\mathbf{u}) = 0,$$

$$\partial_t(h\mathbf{u}) + \operatorname{div}(h\mathbf{u}\otimes\mathbf{u}) + \nabla(ah^2) = h\left(-\gamma\frac{\mathbf{u}}{|\mathbf{u}|} + \mathbf{f}\right),$$

- $h_0 \in C^2(\Omega)$, $\mathbf{u}_0 \in C^2(\Omega; R^2)$, $h_0 > 0$ in Ω.
- using the standard Helmholtz decomposition, we may write

$$h_0\mathbf{u}_0 = \mathbf{v}_0 + \mathbf{V}_0 + \nabla\Psi_0,\ \operatorname{div}\mathbf{v}_0 = 0,\ \int_\Omega \Psi_0 \,\mathrm{d}x = 0,\ \int_\Omega \mathbf{v}_0 \,\mathrm{d}x = 0,$$

- We look for solutions in the form

$$h\mathbf{u} = \mathbf{v} + \mathbf{V} + \nabla\Psi,$$

where

$$\operatorname{div}\mathbf{v} = 0,\ \int_\Omega \Psi(t,\cdot) = 0,\ \int_\Omega \mathbf{v}(t,\cdot) = 0,\ \mathbf{V} = \mathbf{V}(t) \in \mathbb{R}^2.$$

Continuity equation

$$\partial_t h + \Delta_x \Psi = 0 \text{ in } (0,T)\times\Omega,\ h(0,\cdot) = h_0,\ \Psi(0,\cdot) = \Psi_0.$$

We can *choose* $h = h(t,x) \in C^2([0,T]\times\Omega)$ such that

$$h(0,\cdot) = h_0,\ \partial_t h(0,\cdot) = -\Delta_x \Psi_0,\ h(t,\cdot) > 0,$$

$$\int_\Omega h(t,\cdot)\,\mathrm{d}x = \int_\Omega h_0\,\mathrm{d}x \text{ for all } t \in [0,T],$$

and compute

$$-\Delta_x \Psi(t,\cdot) = \partial_t h(t,\cdot),\ \int_\Omega \Psi(t,\cdot)\,\mathrm{d}x = 0.$$

Consequently, the original problem reduces to finding the functions $\mathbf{v}$, $\mathbf{V}$ satisfying (weakly)

$$\partial_t \mathbf{v} + \partial_t \mathbf{V} + \operatorname{div}\left(\frac{(\mathbf{v} + \mathbf{V} + \nabla\Psi) \otimes (\mathbf{v} + \mathbf{V} + \nabla\Psi)}{h} + \left(ah^2 + \partial_t \Psi\right) \mathbb{I} \right)$$

$$= h\left(-\gamma \frac{\mathbf{v} + \mathbf{V} + \nabla\Psi}{|\mathbf{v} + \mathbf{V} + \nabla\Psi|} + \mathbf{f} \right),$$

$$\operatorname{div} \mathbf{v} = 0, \ \int_\Omega \mathbf{v}(t, \cdot) \, \mathrm{d}x = 0,$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{V}(0) = \mathbf{V}_0.$$

Kinetic energy

We denote

$$E = \frac{1}{2} \frac{|\mathbf{v} + \mathbf{V} + \nabla \Psi|^2}{h}$$

the kinetic energy density associated with the Savage-Hutter system.
Analogously, we rewrite the system in the form

$$\partial_t \mathbf{v} + \partial_t \mathbf{V} + \mathrm{div} \left(\frac{(\mathbf{v} + \mathbf{V} + \nabla \Psi) \otimes (\mathbf{v} + \mathbf{V} + \nabla \Psi)}{h} - \frac{1}{2} \frac{|\mathbf{v} + \mathbf{V} + \nabla \Psi|^2}{h} \mathbb{I} \right)$$

$$+ \nabla \left(E - \Lambda + a h^2 + \partial_t \Psi \right) = -\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V} + \nabla \Psi) + h \mathbf{f},$$

where $\Lambda = \Lambda(t)$ is a spatially homogeneous function to be determined below. Finally, for

$$E = \Lambda - a h^2 - \partial_t \Psi,$$

the above equation reduces.

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where $\Lambda = \Lambda(t)$ is a spatially homogeneous function to be determined below. Finally, for

$$E = \Lambda - ah^2 - \partial_t \Psi,$$

the above equation reduces.

- Determine $\mathbf{V}$ as the unique solution of the ODE

$$\partial_t \mathbf{V} - \left[\frac{1}{|\Omega|}\int_\Omega \gamma\left(\frac{h}{2E}\right)^{1/2} \mathrm{d}x\right]\mathbf{V}$$

$$= \frac{1}{|\Omega|}\int_\Omega \left[\gamma\left(\frac{h}{2E}\right)^{1/2}(\mathbf{v}+\nabla\Psi) + h\mathbf{f}\right] \mathrm{d}x, \ \mathbf{V}(0) = \mathbf{V}_0.$$

- Finally, we find a tensor $\mathbb{M} = \mathbb{M}[\mathbf{v}]$ such that $\mathbb{M}(t,x) \in R^{2\times 2}_{\mathrm{sym},0}$, for any t, x, and

$$\operatorname{div} \mathbb{M} = -\gamma\left(\frac{h}{2E}\right)^{1/2}(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla\Psi)$$

$$+\frac{1}{|\Omega|}\int_\Omega \gamma\left(\frac{h}{2E}\right)^{1/2}(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla\Psi)\ \mathrm{d}x + h\mathbf{f} - \frac{1}{|\Omega|}\int_\Omega h\mathbf{f}\ \mathrm{d}x,$$

Finally we write S-H system in a concise form

$$\partial_t \mathbf{v} + \mathrm{div}\left[\frac{(\mathbf{v}+\mathbf{V}[\mathbf{v}]+\nabla\Psi)\otimes(\mathbf{v}+\mathbf{V}[\mathbf{v}]+\nabla\Psi)}{h} - \frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}[\mathbf{v}]+\nabla\Psi|^2}{h}\mathbb{I} - \mathbf{M}[\mathbf{v}]\right] = 0$$

$$\mathrm{div} v = 0$$

and proceed with the method of convex integration.

Short summary of these results (incompressible Euler)

- De Lellis, Székelyhidi 2010 showed that weak solutions need not to be unique
- Wiedemann 2011: for any initial data $v_0 \in L^2$ there exists infinitely many weak solutions
- Székelyhidi, Wiedemann 2012: if we require the energy to be non-increasing, then a global existence and non-uniqueness result is known for an L^2-dense subset of initial data
- Weak-strong uniqueness

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- $h_0 \in C^2(\Omega)$, $\mathbf{u}_0 \in C^2(\Omega; R^2)$, $h_0 > 0$ in Ω.
- using the standard Helmholtz decomposition, we may write

$$h_0\mathbf{u}_0 = \mathbf{v}_0 + \mathbf{V}_0 + \nabla\Psi_0,\ \mathrm{div}\,\mathbf{v}_0 = 0,\ \int_\Omega \Psi_0\,\mathrm{d}x = 0,\ \int_\Omega \mathbf{v}_0\,\mathrm{d}x = 0,$$

- We look for solutions in the form

$$h\mathbf{u} = \mathbf{v} + \mathbf{V} + \nabla\Psi,$$

where

$$\mathrm{div}\,\mathbf{v} = 0,\ \int_\Omega \Psi(t,\cdot) = 0,\ \int_\Omega \mathbf{v}(t,\cdot) = 0,\ \mathbf{V} = \mathbf{V}(t) \in \mathbb{R}^2.$$

Continuity equation

$$\partial_t h + \Delta_x \Psi = 0 \text{ in } (0, T) \times \Omega, \ h(0, \cdot) = h_0, \ \Psi(0, \cdot) = \Psi_0.$$

We can *choose* $h = h(t, x) \in C^2([0, T] \times \Omega)$ such that

$$h(0, \cdot) = h_0, \ \partial_t h(0, \cdot) = -\Delta_x \Psi_0, \ h(t, \cdot) > 0,$$

$$\int_\Omega h(t, \cdot) \, \mathrm{d}x = \int_\Omega h_0 \, \mathrm{d}x \text{ for all } t \in [0, T],$$

and compute

$$-\Delta_x \Psi(t, \cdot) = \partial_t h(t, \cdot), \ \int_\Omega \Psi(t, \cdot) \, \mathrm{d}x = 0.$$

Consequently, the original problem reduces to finding the functions $\mathbf{v}$, $\mathbf{V}$ satisfying (weakly)

$$\partial_t \mathbf{v} + \partial_t \mathbf{V} + \operatorname{div}\left(\frac{(\mathbf{v} + \mathbf{V} + \nabla\Psi) \otimes (\mathbf{v} + \mathbf{V} + \nabla\Psi)}{h} + \left(ah^2 + \partial_t \Psi\right) \mathbb{I} \right)$$

$$= h \left(-\gamma \frac{\mathbf{v} + \mathbf{V} + \nabla\Psi}{|\mathbf{v} + \mathbf{V} + \nabla\Psi|} + \mathbf{f} \right),$$

$$\operatorname{div} \mathbf{v} = 0, \ \int_\Omega \mathbf{v}(t, \cdot) \, \mathrm{d}x = 0,$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{V}(0) = \mathbf{V}_0.$$

Kinetic energy

We denote

$$E = \frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}+\nabla\Psi|^2}{h}$$

the kinetic energy density associated with the Savage-Hutter system.

Analogously, we rewrite the system in the form

$$\partial_t \mathbf{v} + \partial_t \mathbf{V} + \operatorname{div}\left(\frac{(\mathbf{v}+\mathbf{V}+\nabla\Psi)\otimes(\mathbf{v}+\mathbf{V}+\nabla\Psi)}{h} - \frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}+\nabla\Psi|^2}{h}\mathbb{I} \right)$$

$$+\nabla\left(E - \Lambda + ah^2 + \partial_t\Psi\right) = -\gamma\left(\frac{h}{2E}\right)^{1/2}(\mathbf{v}+\mathbf{V}+\nabla\Psi) + h\mathbf{f},$$

where $\Lambda = \Lambda(t)$ is a spatially homogeneous function to be determined below. Finally, for

$$E = \Lambda - ah^2 - \partial_t\Psi,$$

the above equation reduces.

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$$= -\gamma \left(\frac{h}{2E}\right)^{1/2} (\mathbf{v} + \mathbf{V} + \nabla\Psi) + h\mathbf{f},$$

where $\Lambda = \Lambda(t)$ is a spatially homogeneous function to be determined below. Finally, for

$$E = \Lambda - ah^2 - \partial_t\Psi,$$

the above equation reduces.

- Determine $\mathbf{V}$ as the unique solution of the ODE

$$\partial_t \mathbf{V} - \left[\frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} \, \mathrm{d}x \right] \mathbf{V}$$

$$= \frac{1}{|\Omega|} \int_\Omega \left[\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \nabla \Psi) + h\mathbf{f} \right] \, \mathrm{d}x, \;\; \mathbf{V}(0) = \mathbf{V}_0.$$

- Finally, we find a tensor $\mathbb{M} = \mathbb{M}[\mathbf{v}]$ such that $\mathbb{M}(t,x) \in R^{2\times 2}_{\mathrm{sym},0}$, for any t, x, and

$$\operatorname{div} \mathbb{M} = -\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla \Psi)$$

$$+ \frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla \Psi) \, \mathrm{d}x + h\mathbf{f} - \frac{1}{|\Omega|} \int_\Omega h\mathbf{f} \, \mathrm{d}x,$$

Finally we write S-H system in a concise form

$$\partial_t \mathbf{v} + \mathrm{div}\left[\frac{(\mathbf{v}+\mathbf{V}[\mathbf{v}]+\nabla\Psi)\otimes(\mathbf{v}+\mathbf{V}[\mathbf{v}]+\nabla\Psi)}{h} - \frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}[\mathbf{v}]+\nabla\Psi|^2}{h}\mathbb{I} - \mathbf{M}[\mathbf{v}]\right] = 0$$

$$\mathrm{div} v = 0$$

and proceed with the method of convex integration.

Subsolutions

The set of subsolutions

X_0 - the set of velocity fields such that

$$\partial_t \mathbf{w} + \operatorname{div} \mathbb{F} = 0 \text{ in } (0,T) \times \Omega \text{ for some } \mathbb{F}$$

$$\lambda_{\max}\left[\frac{(\mathbf{w} + \mathbf{V}[\mathbf{w}] + \nabla\Psi) \otimes (\mathbf{w} + \mathbf{V}[\mathbf{w}] + \nabla\Psi)}{h} - \mathbb{F} - \mathbb{M}[\mathbf{w}]\right] < E - \delta$$

$$\text{in } (0,T) \times \Omega \text{ for some } \delta > 0,$$

where E is the kinetic energy introduced before.

Key steps

1. Recall that
$$E = \Lambda - ah^2 - \partial_t \Psi.$$
The first observation is that the set X_0 is non-empty provided
$$\Lambda(t) \geq \Lambda_0 > 0 \text{ in } [0, T]$$
2. Take the closure $\overline{X}_0$ of the set of subsolutions X_0.
3. Show that X_0 has infinite cardinality.

1. The first observation is that the set X_0 is non-empty provided

$$\Lambda(t) \geq \Lambda_0 > 0 \text{ in } [0, T]$$

and Λ_0 is large enough. Here "large enough" means in terms of the initial data, $\mathbf{f}$, and the time T.

$$\lambda_{\max}\left[\frac{(\mathbf{v}_0 + \mathbf{V}[\mathbf{v}_0] + \nabla\Psi) \otimes (\mathbf{v}_0 + \mathbf{V}[\mathbf{v}_0] + \nabla\Psi)}{h} - \mathbb{M}[\mathbf{v}_0]\right] < E - \delta$$

$$= \Lambda - ah^2 - \partial_t\Psi - \delta.$$

Infinitely many solutions

We introduce the functional

$$I[\mathbf{v}] = \int_0^T \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla \Psi|^2}{h} - E \right] \, \mathrm{d}x \, \mathrm{d}t$$

$I : \overline{X}_0 \to (-\infty, 0]$ is a lower semi-continuous functional with respect to the topology of the space $C_{\mathrm{weak}}([0, T]; L^2(\Omega; R^2))$. Consequently, by virtue of Baire category argument, the set of points of continuity of I in $\overline{X}_0$ has infinite cardinality.

Lemma

Let $U \subset R \times R^N$, $N = 2, 3$ *be a bounded open set. Suppose that*

$\mathbf{g} \in C(U; R^N)$, $\mathbb{W} \in C(U; R^{N\times N}_{\mathrm{sym},0})$, $e, r \in C(U)$, $r > 0$, $e \leq \overline{e}$ *in* U

are given such that

$$\frac{N}{2}\lambda_{\max}\left[\frac{\mathbf{g}\otimes\mathbf{g}}{r} - \mathbb{W}\right] < e \text{ in } U.$$

Then there exist sequences

$$\mathbf{w}_n \in C_c^\infty(U; R^N),\ \mathbb{G}_n \in C_c^\infty(U; R^{N\times N}_{\mathrm{sym},0}),\ n = 0, 1, \ldots$$

such that $\mathbf{w}_n \to 0$ *weakly in* $L^2(U; R^N)$ *and*

$$\partial_t \mathbf{w}_n + \operatorname{div} \mathbb{W}_n = 0,\ \operatorname{div} \mathbf{w}_n = 0 \text{ in } R^N,$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{g}+\mathbf{w}_n)\otimes(\mathbf{g}+\mathbf{w}_n)}{r} - (\mathbb{W}+\mathbb{G}_n)\right] < e \text{ in } U,$$

Appendix II

Origins of the system

Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way. Highly developed social organization: insects (ants, bees ...), fish, birds, micro-organisms.

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Basic particle model

$$\frac{dx_i}{dt} = v_i\,,$$
$$\frac{dv_i}{dt} = \underbrace{v_i - \alpha v_i |v_i|^2}_{\text{propulsion}-\text{friction}} \underbrace{- \sum_{j \neq i} \nabla_x K(x_i - x_j)}_{\text{attraction}-\text{repulsion}} + \underbrace{\sum_i \psi(x_i - x_j)(v_j - v_i)}_{\text{alignment}}$$

where $i \in \{1, \ldots, n\}$ and $\alpha > 0$.

- Consider the limit of the empirical measure associated to the above system

$$\mu_n(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{(x_i(t),v_i(t))}$$

defined as a probability measure in phase space $(x, v) \in \mathbb{R}^N \times \mathbb{R}^N$.

- Under certain assumptions on K and ψ, it is proven that $\mu_n(t)$ converges as $n \to \infty$ to a solution of a Vlasov-like kinetic equation

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \left(F[f]f + \left(1 - \alpha|v|^2\right) vf \right) = 0\,,$$

with $F[f]$ being the nonlocal force field given by

$$F[f] = -\nabla K * \varrho + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \psi(x-y)(w-v)f(t,x,w)dw\,dy$$

and

$$\varrho(t,x) := \int_{\mathbb{R}^N} f(t,x,v)dv\,.$$

By imposing the so-called hydrodynamic or mono-kinetic ansatz, i.e., looking for distributional solutions to

$$\partial_t f + v \cdot \nabla_x f + \mathrm{div}_v \left(F[f]f + \left(1 - \alpha |v|^2\right) vf \right) = 0\,,$$

of the form

$$f_h(t,x,v) = \varrho(t,x)\delta(v - \mathbf{u}(t,x))\,,$$

one deduces that the pair $(\varrho(t,x), \mathbf{u}(t,x))$ must satisfy the hydrodynamic equations with $H(s) = \alpha s^2$