Role of Spectra in Period-Doubling Instabilities of Spiral Waves

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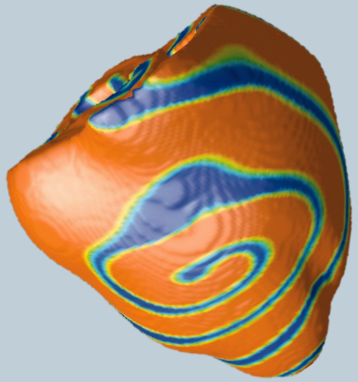
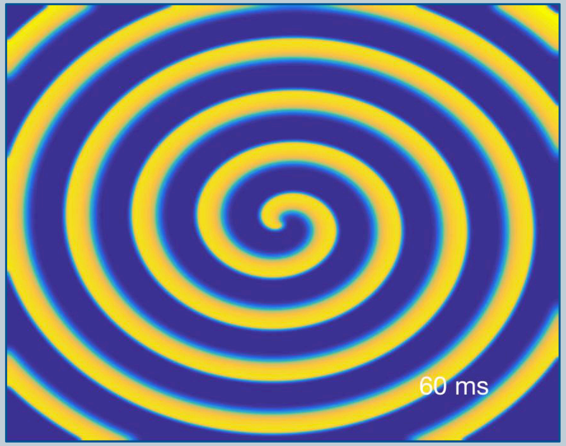
Spiral Wave Patterns

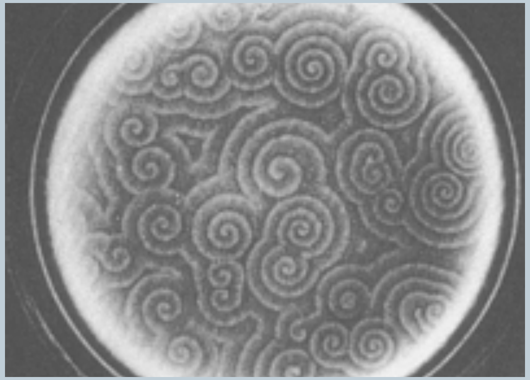
Cardiac Arrhythmias



[Cherry, 2004]



Chemical Oscillations



Dictyostelium discoideum (mold)
[Ball, 1994]

Belousov-Zhabontinsky reaction
[Yoneyama, 1995]

Goal: Understand stability of spiral wave patterns

Spiral Wave Patterns

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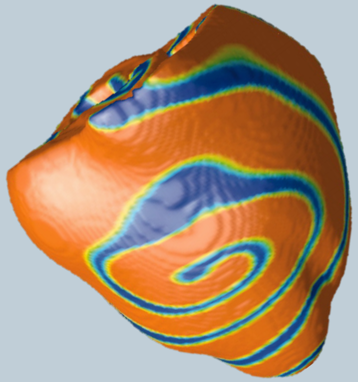

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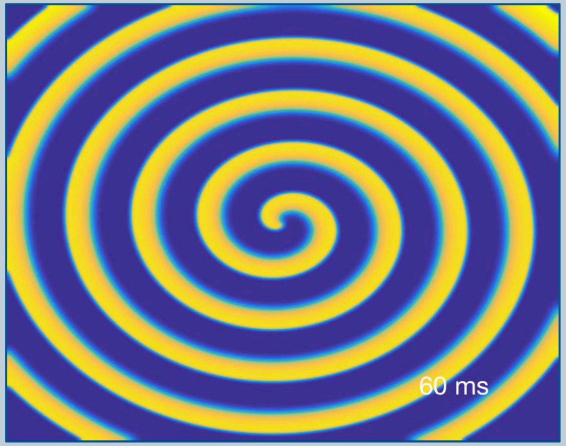
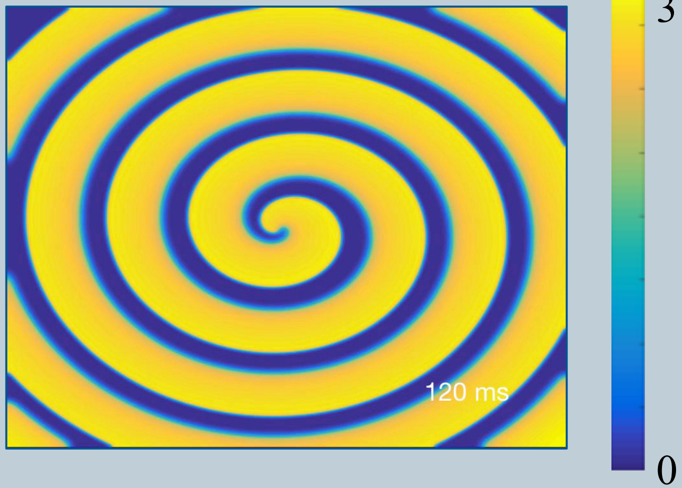


Alternans



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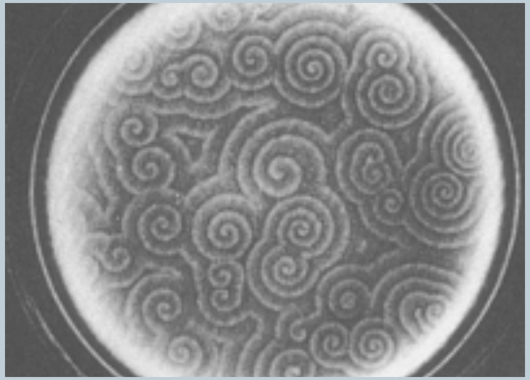

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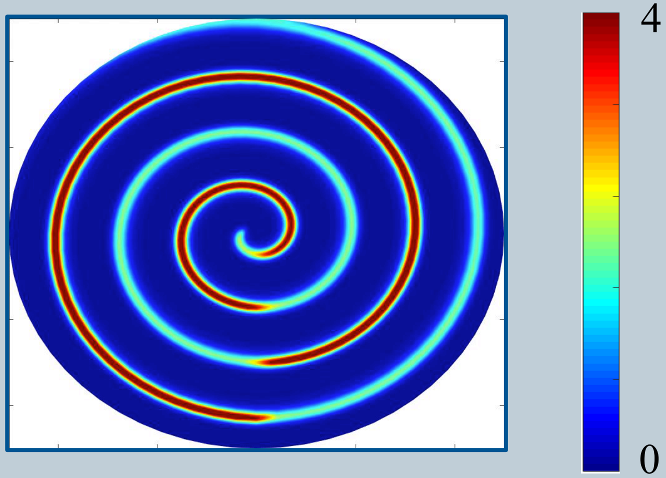
Line Defect

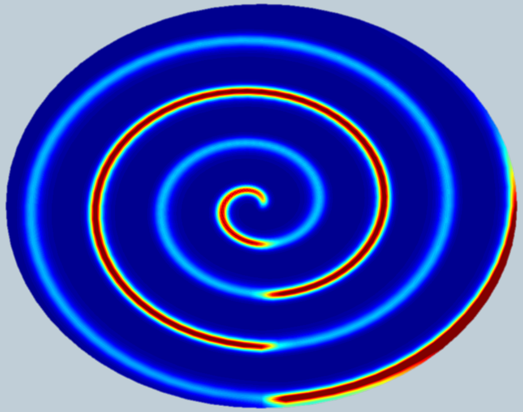


Goal: Understand stability of spiral wave patterns

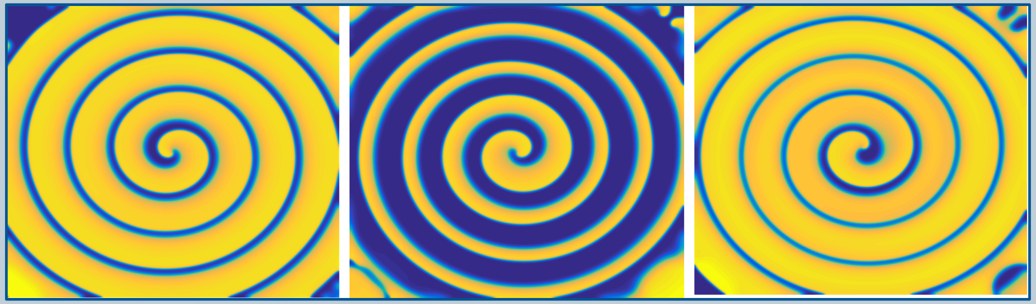
→ Mechanisms responsible for period-doubling instabilities on **bounded domains**

Period-Doubling Instabilities in Spiral Waves

Line Defects in Chemical Oscillations



Alternans in Cardiac Arrhythmias



Rössler System

$$u_t = 0.4\Delta u - v - w$$
$$v_t = 0.4\Delta v + u + 0.2v$$
$$w_t = 0.4\Delta w + uw - \mu_R w + 0.2$$

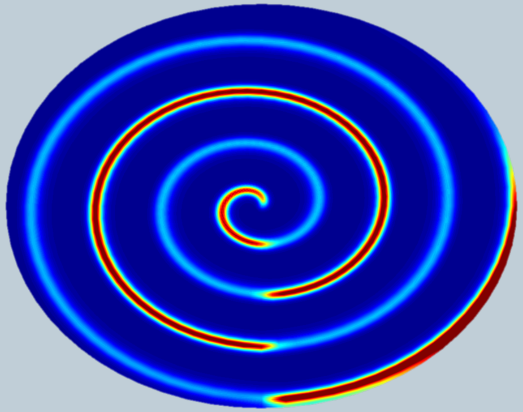
Karma Model

$$u_t = 1.1\Delta u + 400\left(-u + (1.5414 - v^4)\,(1 - \tanh(u-3))\,\frac{u^2}{2}\right)$$
$$v_t = 0.1\Delta v + 4\left(\frac{1}{1 - e^{-\mu_K}}\theta_s(u-1) - v\right)$$

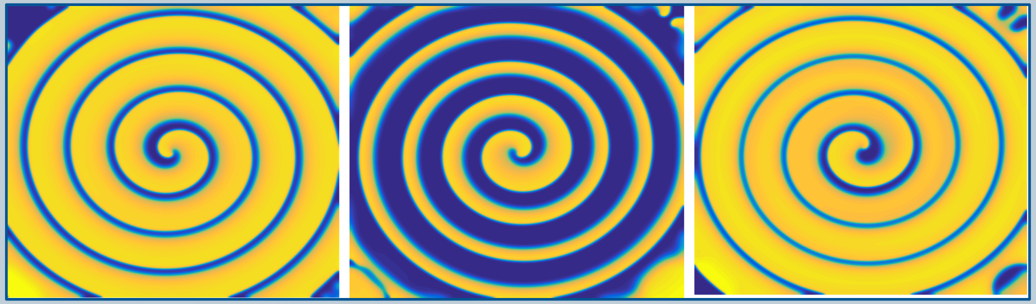
$$U_t = D\Delta U + F(U)$$

Period-Doubling Instabilities in Spiral Waves

Line Defects in Chemical Oscillations



Alternans in Cardiac Arrhythmias



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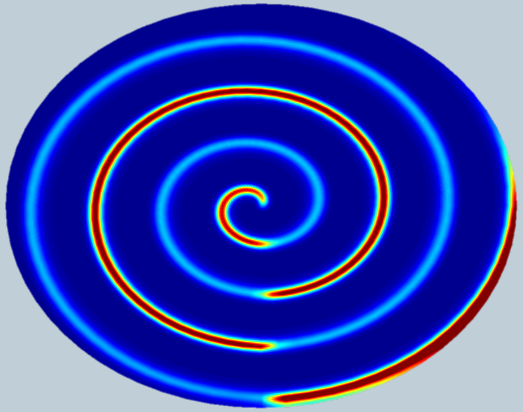

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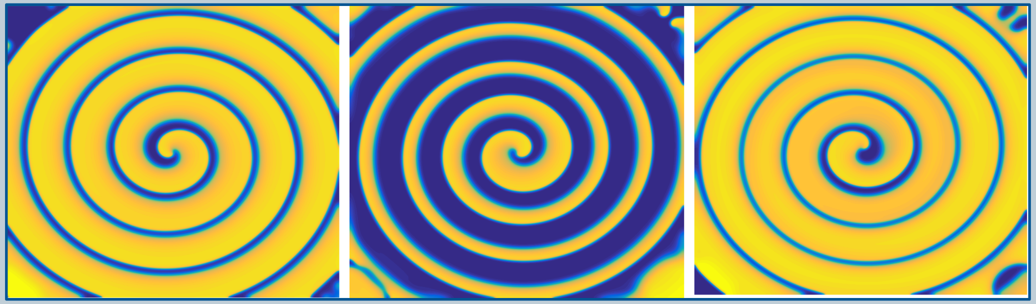


Outline:

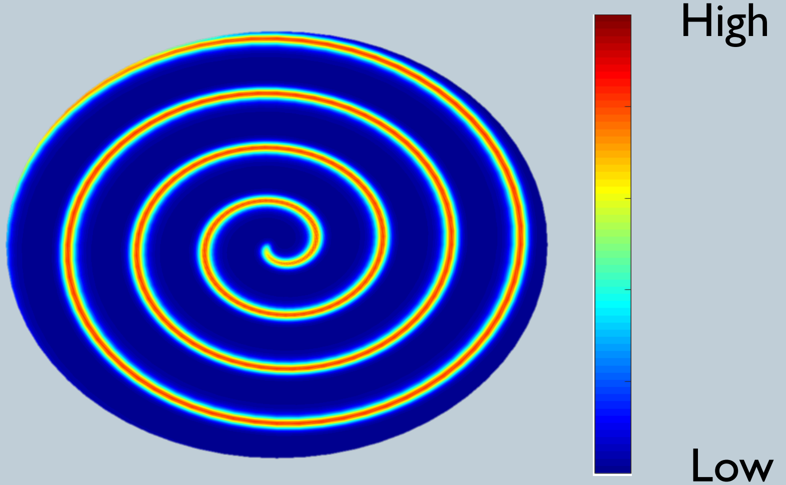
- Mathematical and spectral properties
- Drivers of instabilities
- Shape of alternans eigenfunction

Planar Spiral Waves

Rigidly-rotating spirals:

- SE(2) symmetry on the plane
- Stationary solutions in rotating, polar frame

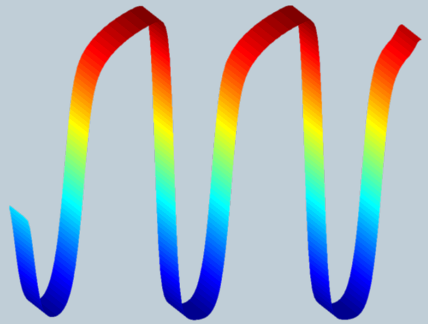
$$(r, \phi) \to (r, \psi) = (r, \phi - \omega t)$$
$$0 = D\Delta_{r,\psi} U_* + \omega \partial_\psi U_* + F(U_*)$$



- Asymptotic 1D (periodic) wave trains

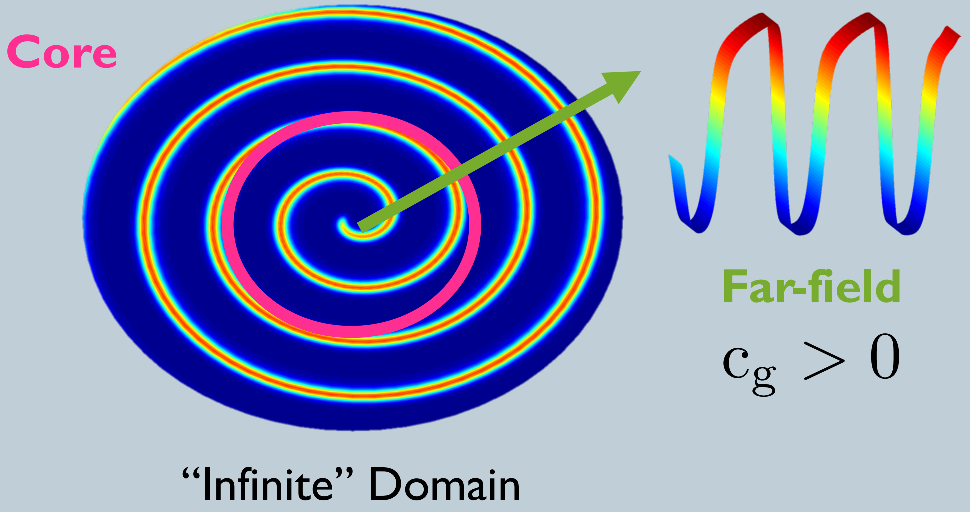
$$U_*(r, \psi) \to U_\infty(\kappa r + \psi) = U_\infty(\xi), \;\; r \to \infty$$
$$U_\infty(\xi) = U_\infty(\xi + 2\pi), \;\; \xi \in \mathbb{R}$$

$$0 = \kappa^2 D \partial_{\xi\xi} U_\infty + \omega \partial_\xi U_\infty + F(U_\infty)$$



1D Wave Train

Anatomy of a Bounded Spiral



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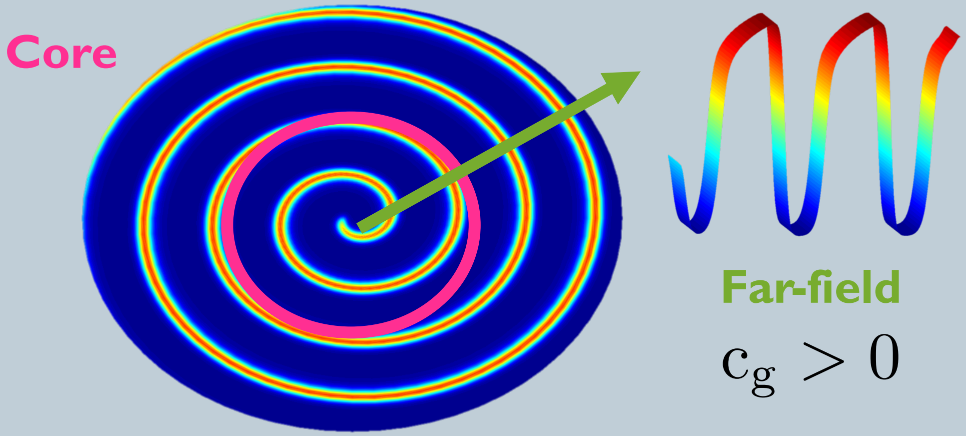
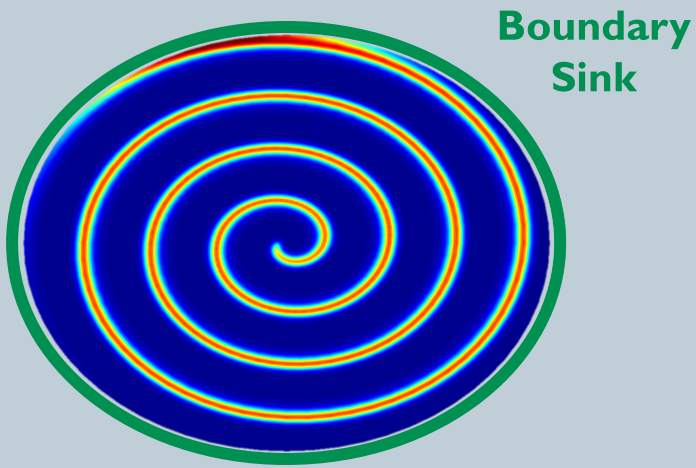
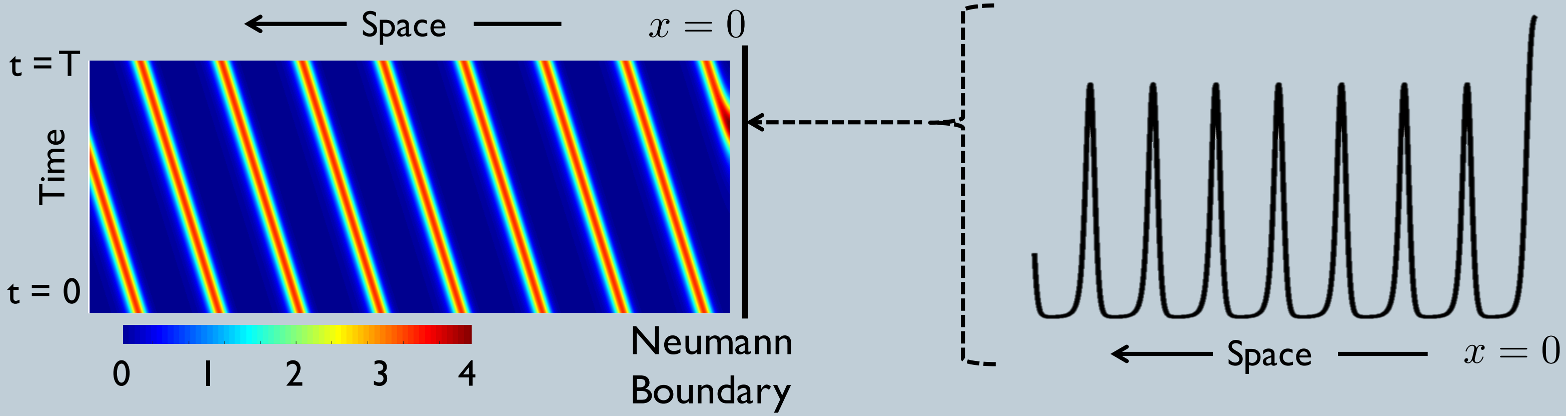


"Infinite" Domain



No-flux Boundary

Boundary Sink



Types of Spectra

Linearized Operator:

$$\mathcal{L}V = D\Delta V + \omega V_\psi + F_U(U_*)V$$

Spectrum:

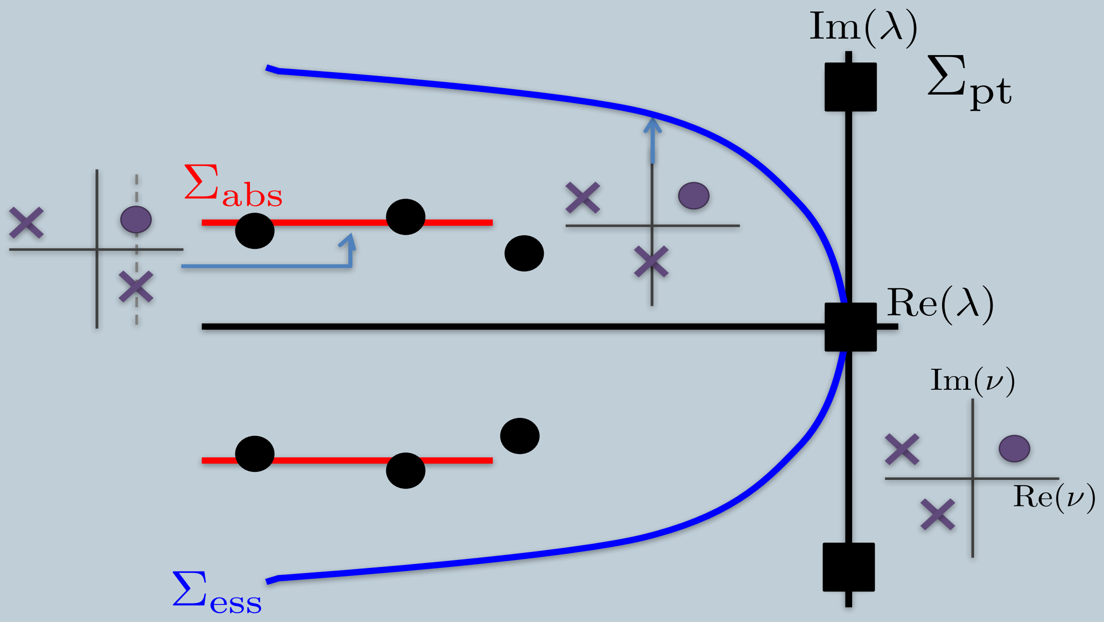
$$\Sigma(\mathcal{L}) = \{\lambda \in \mathbb{C} : \mathcal{L} - \lambda \text{ is not boundedly invertible}\}$$

Unbounded Domain:

- **Point** Spectrum (core)
- **Essential** Spectrum (far-field)
  - $(\mathcal{L} - \lambda)$ not Fredholm
  - $\nu \in i\mathbb{R}$

Bounded Domain:

- **Point** Spectrum (core)
- **Point** Spectrum (boundary)
- **Absolute** Spectrum (far-field)



Far-Field Eigenfunction: $V(r,\psi,t) = e^{\lambda t} e^{\nu r} e^{in\psi} V_\infty(\kappa r + \psi)$

λ : Time, ν : Space

[Sandstede, Scheel 2000]

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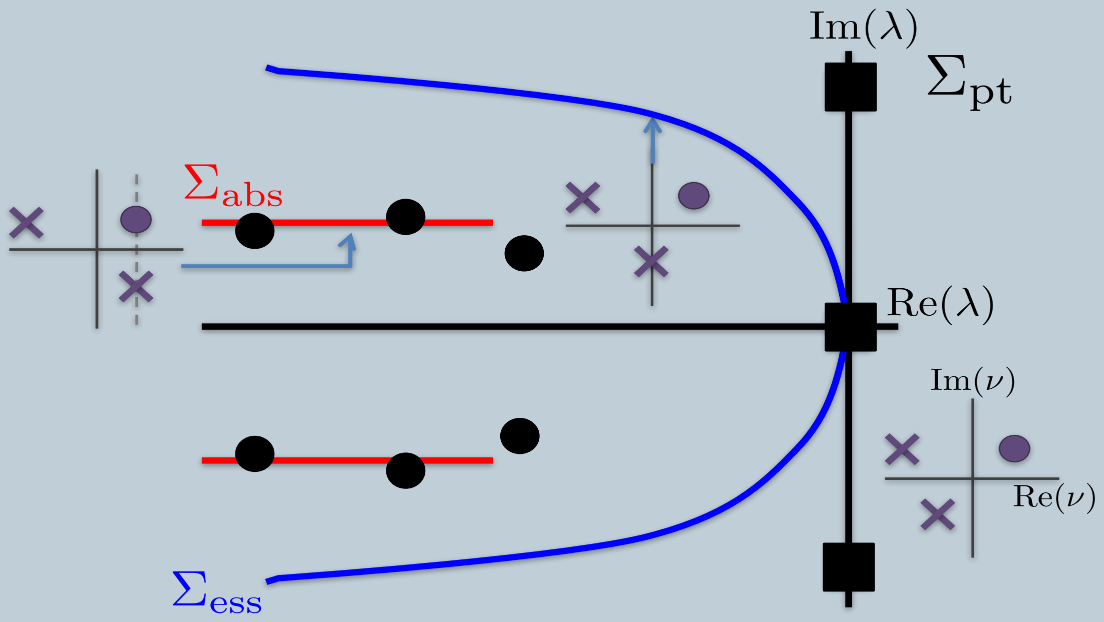

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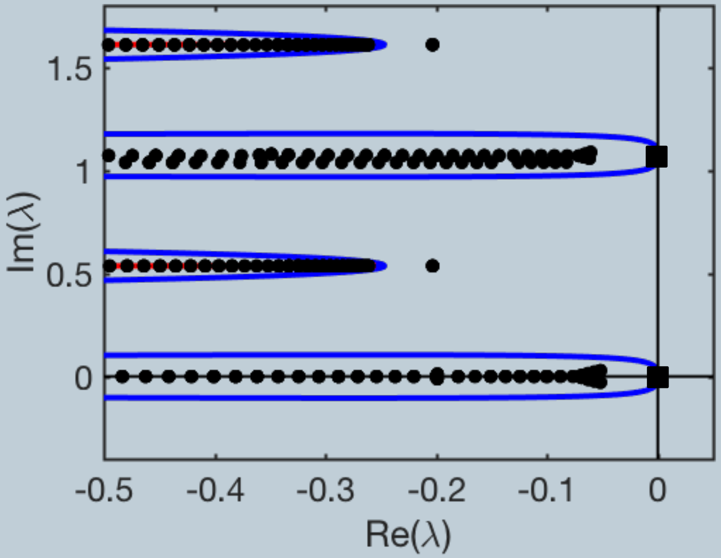
λ : Time, ν : Space

$$\Sigma(\text{Bounded Spiral}) \to \Sigma_{\text{abs}} \cup \Sigma(\text{Core}) \cup \Sigma(\text{Boundary}), \quad \text{as } R \to \infty$$

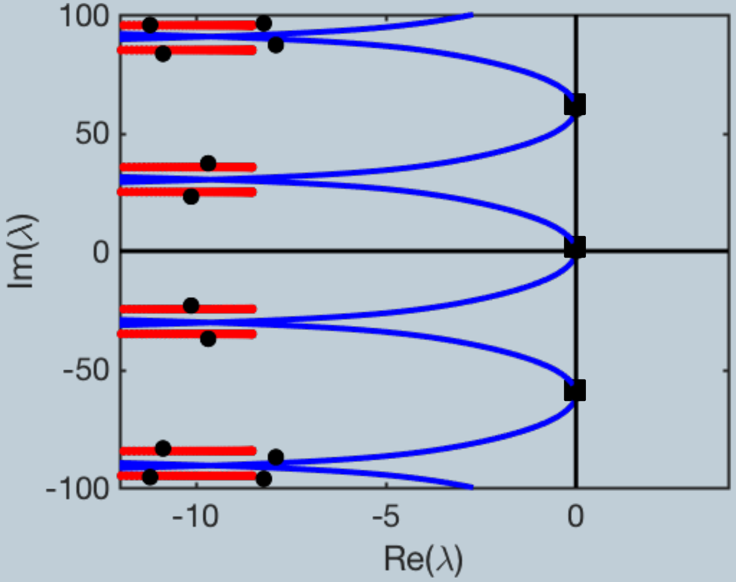
[Sandstede, Scheel 2000]

Instabilities Caused by Point Eigenvalues

Line Defects in Rössler System



Alternans in Karma Model



Essential
Absolute
● Point

Instabilities Caused by Point Eigenvalues

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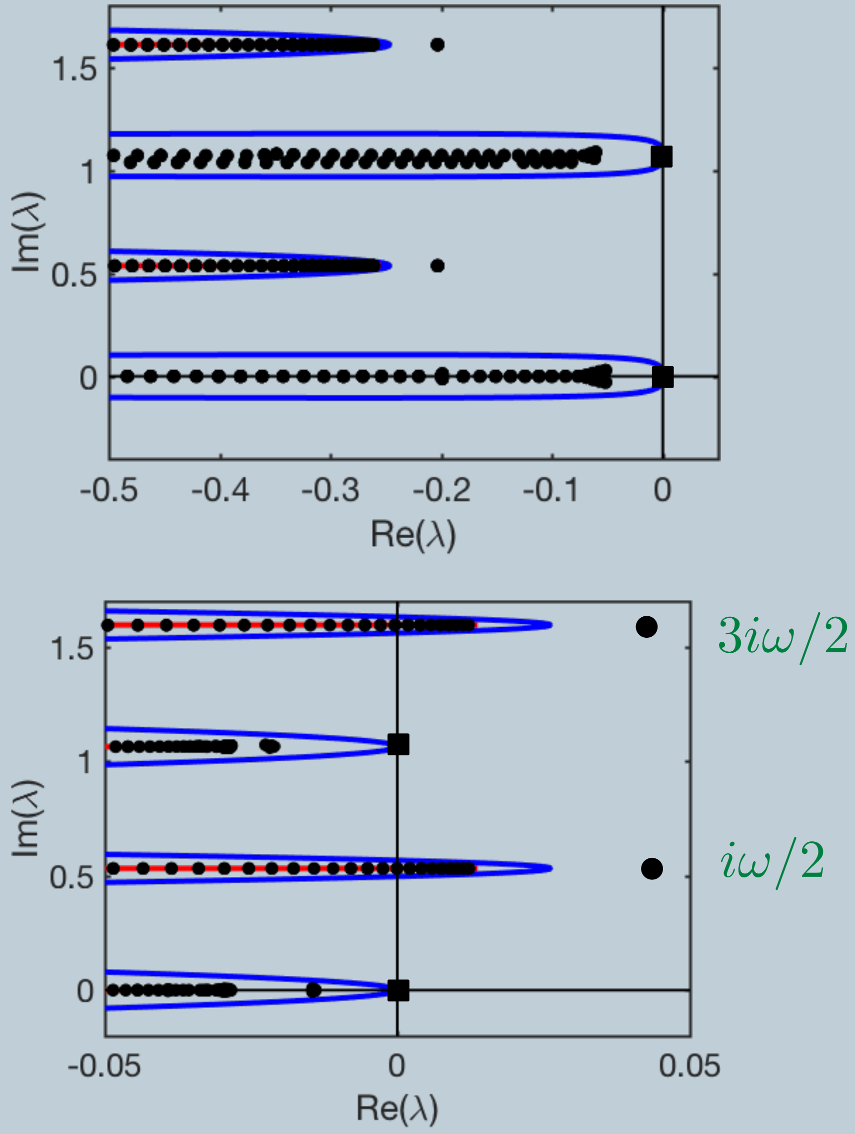
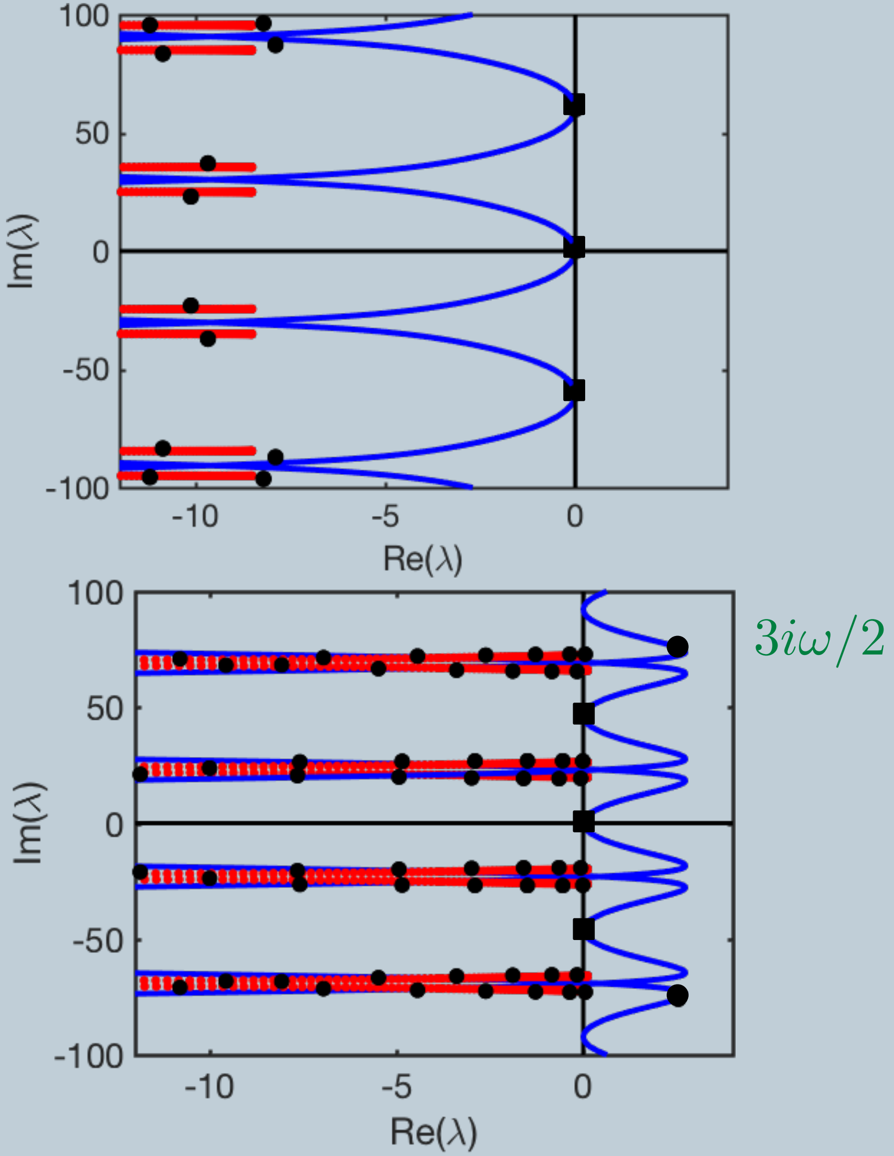


Increasing μ

Essential
Absolute
Point

Alternans in Karma Model



Instabilities Caused by Point Eigenvalues

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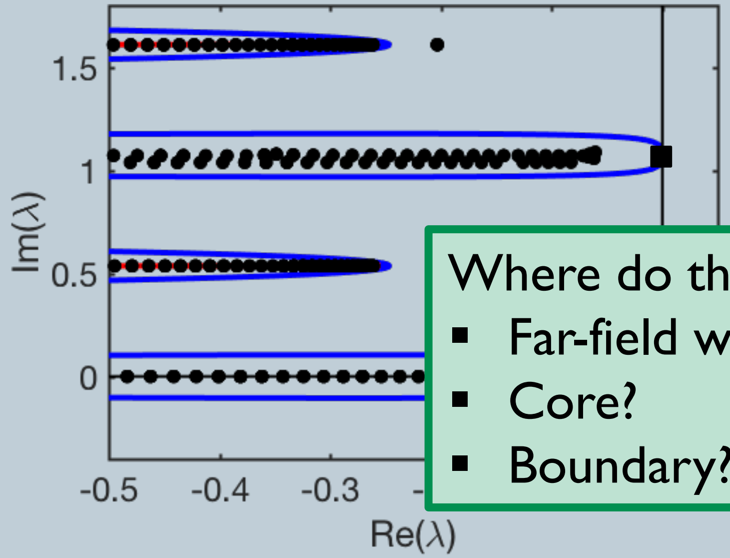

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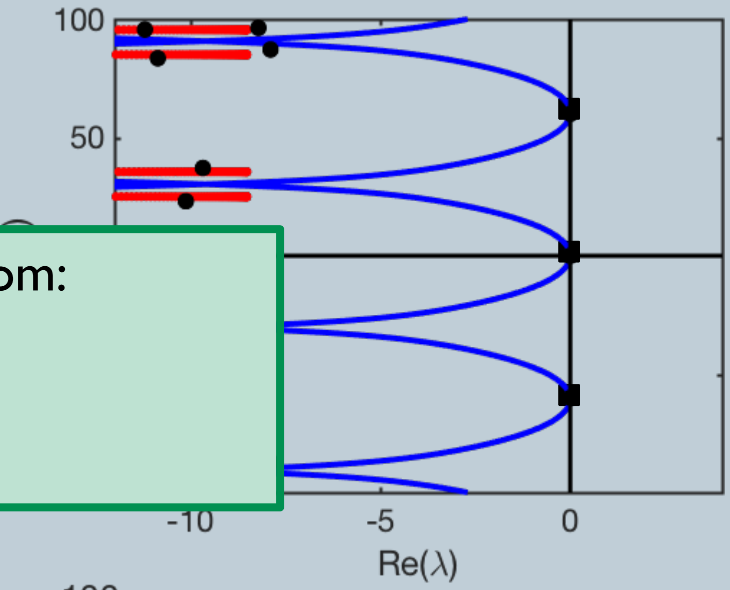
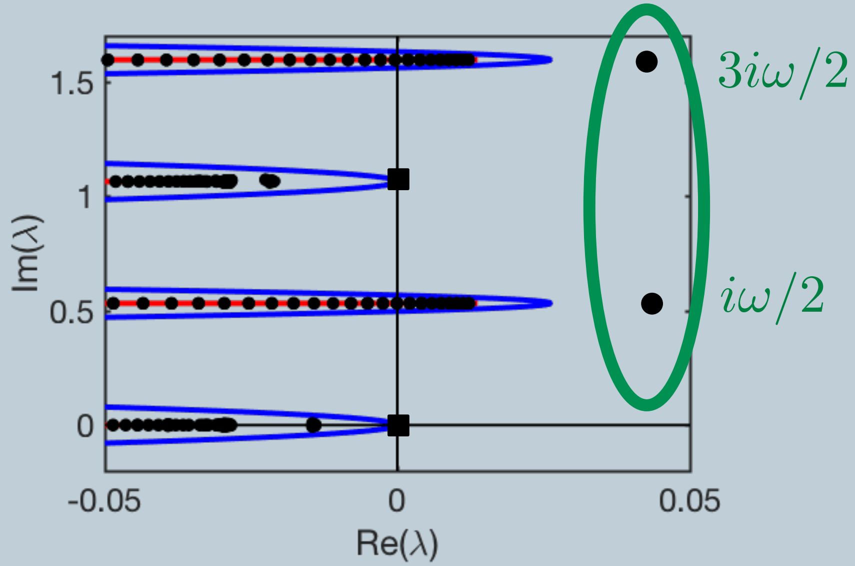
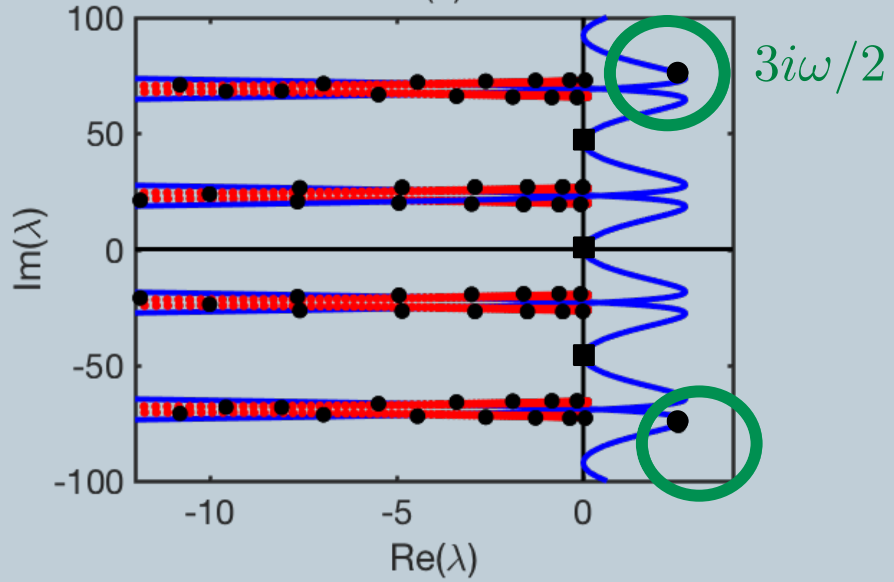


Where do these eigenvalues come from:

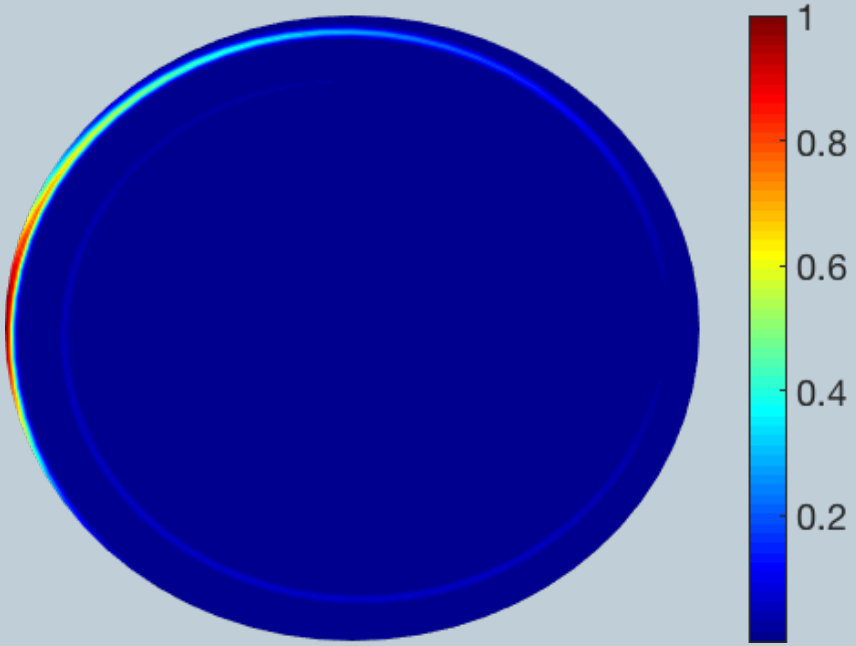
- Far-field wave trains?
- Core?
- Boundary?



Essential
Absolute
Point

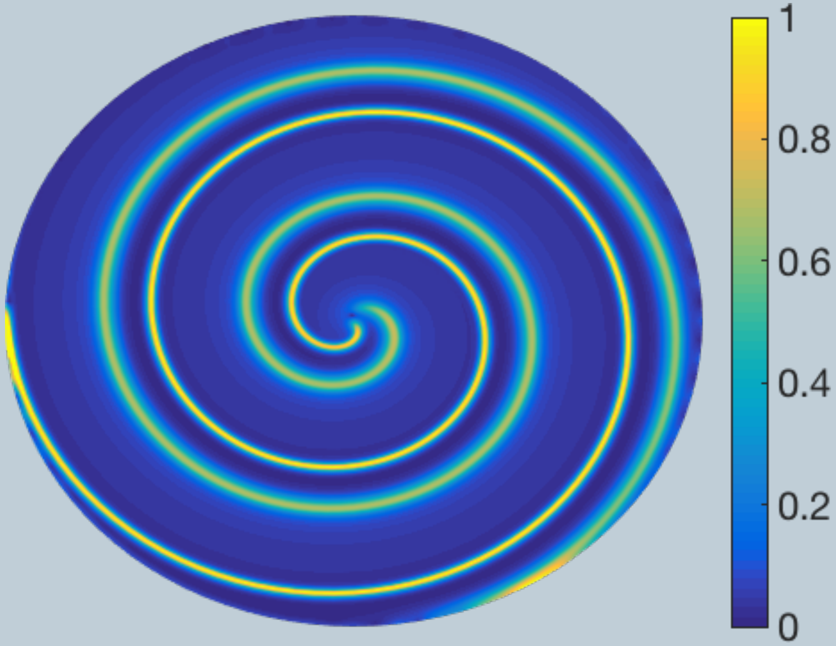


Unstable Point Eigenfunctions



Line defect in Rössler

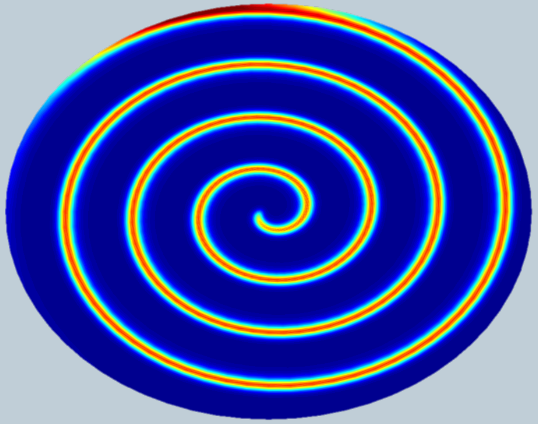
- Localized at boundary



Alternans in Karma

- Slight growth toward boundary

Methodology to Determine Origin of Point Eigenvalues



Bounded Disk

$$\mathcal{L}_{*,R}V = D\Delta_{r,\psi}V + \omega V_\psi + F_U(U_*)V$$
$$\left\{V(r,\psi) \in H^1\left([0,R] \times S^1\right) : V_r(R,\cdot) = 0\right\}$$

- ✓ Far-field
- ✓ Boundary conditions
- ✓ Core

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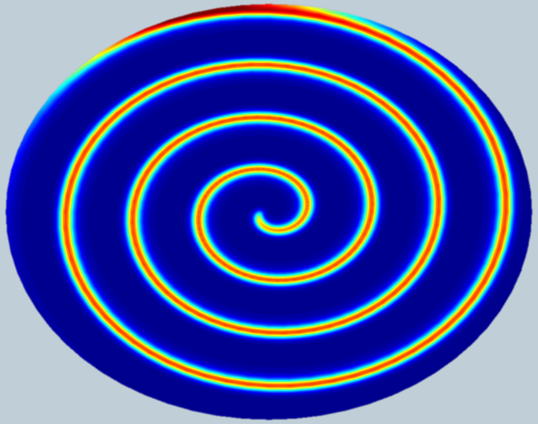

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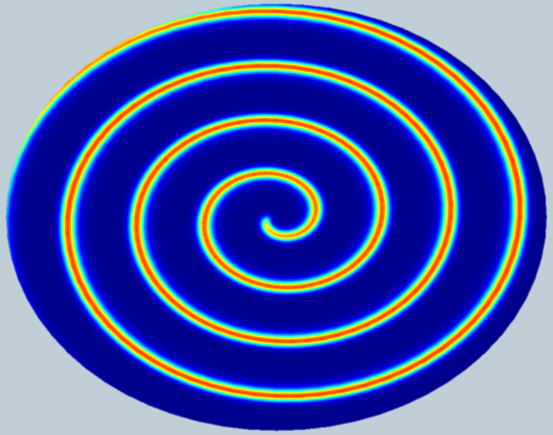
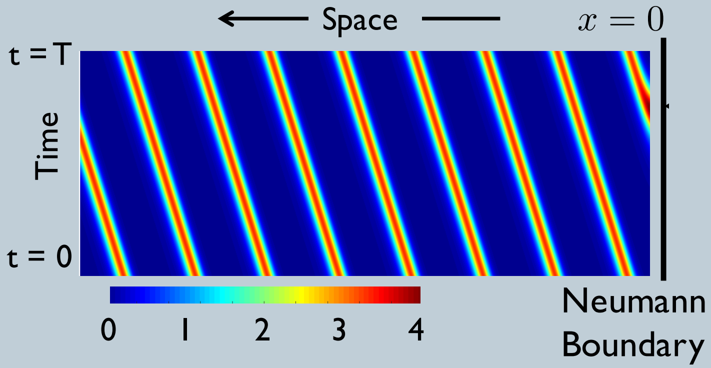


Non-Reflecting Boundary

$$\mathcal{L}_{\mathrm{R,nr}}V = D\Delta_{r,\psi}V + \omega V_\psi + F_U(U_{\mathrm{nr}})V$$
$$\left\{V(r,\psi) \in H^1\left([0,R] \times S^1\right) : V_r(R,\psi) = \kappa V_\psi(R,\psi)\right\}$$

- ✓ Far-field
- ✗ Boundary conditions
- ✓ Core



Boundary Sink

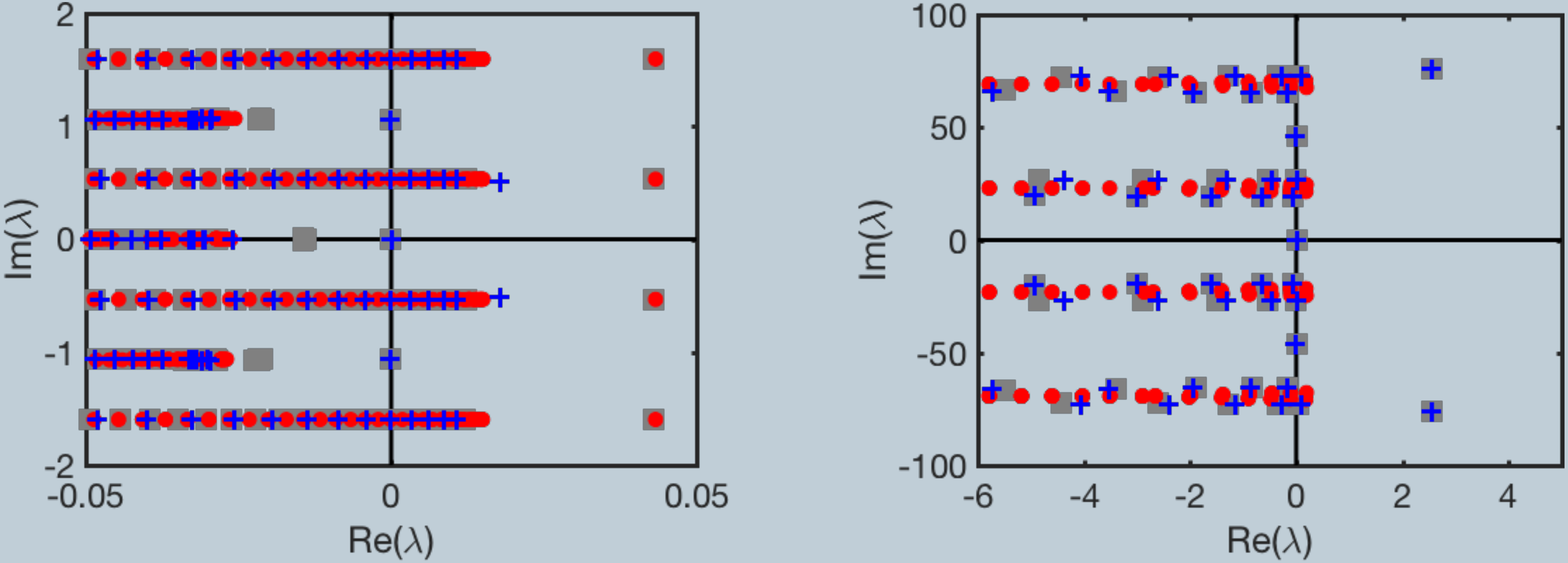
$$\mathcal{L}_{\mathrm{bdry}}V = -V_t + DV_{xx} + F_U\left(U_{\mathrm{bdry}}\right)$$
$$\left\{V(x,t) \in H^1\left([-L,0] \times S^1\right) : V_x(0,\cdot) = 0\right\}$$

- ✓ Far-field
- ✓ Boundary conditions
- ✗ Core

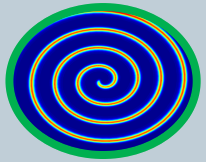
Line Defects from Boundary, Alternans from Core

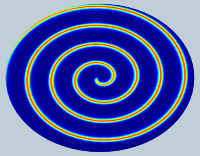
Rössler Model

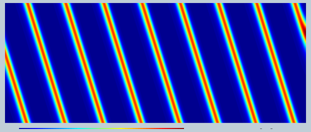
Karma Model



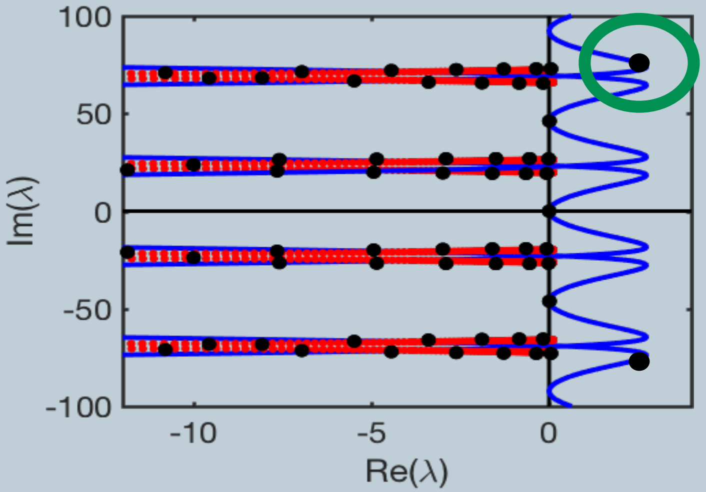
■ Neumann Spiral ($\mathcal{L}_{*,R}$) + Non-Reflecting Spiral($\mathcal{L}_{*,\mathrm{nr}}$) ● Boundary Sink ($\mathcal{L}_{\mathrm{bdry}}$)







Alternans from Interaction of Point Eigenvalue and Essential Spectrum



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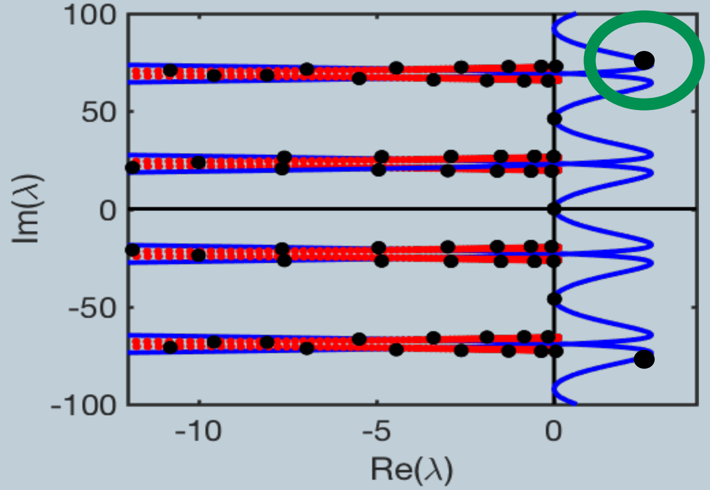
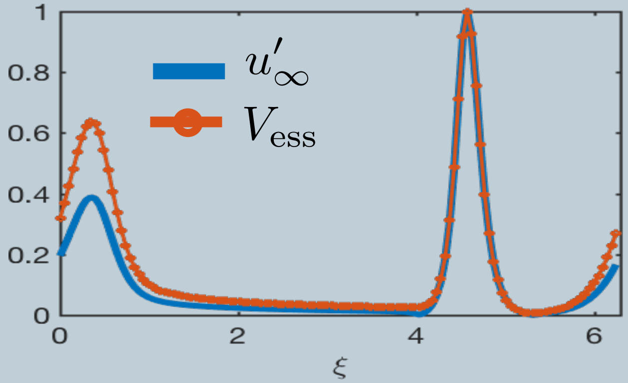


Leading order spiral eigenfunction

$$V(r,\psi) = \underbrace{e^{i\gamma r}}_{\text{Radial growth}} \underbrace{V_{\text{ess}}\left(\kappa r - \psi\right)}_{\text{Shape}}$$

Radial growth

Shape



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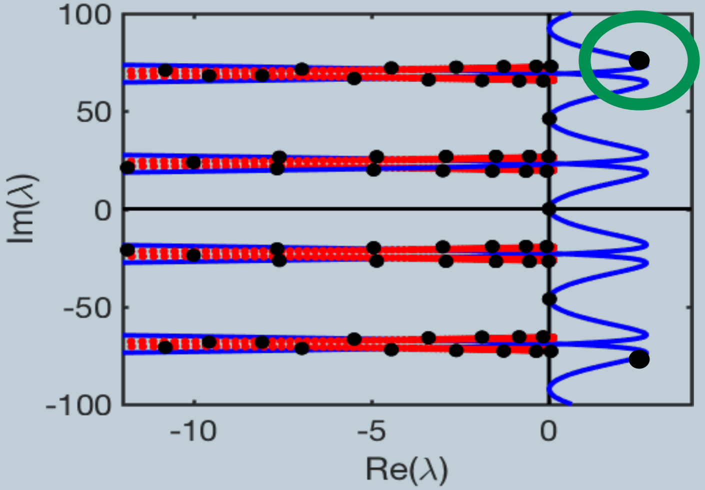

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Radial growth

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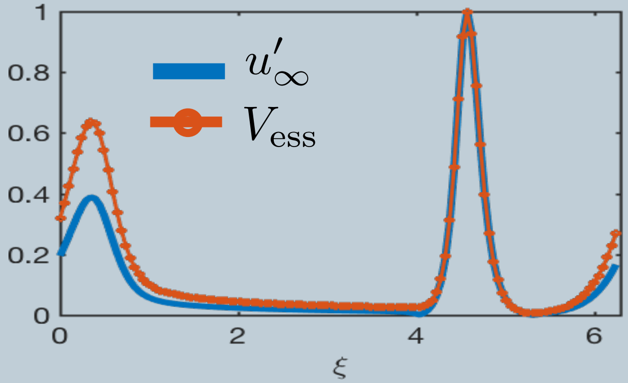
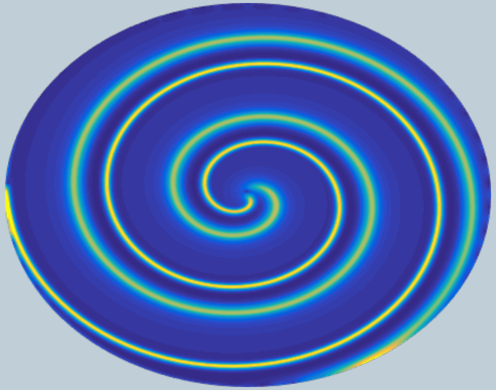
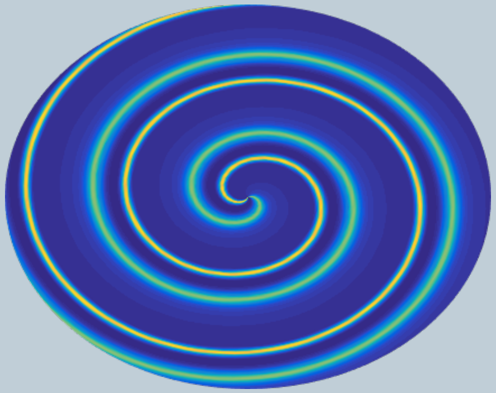
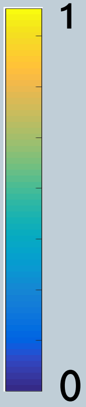




Alternans Eigenfunction

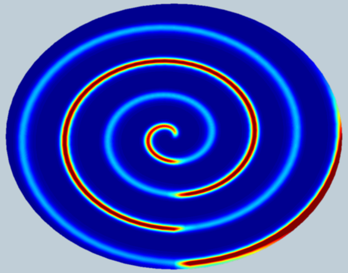


Constructed Eigenfunction

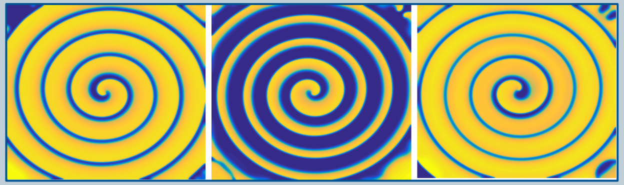


Summary:

- **Different mechanisms responsible for instabilities**
- **Findings on one domain may not be relevant for all others**



Boundary conditions



- Spiral core
- Point eigenvalue and essential spectrum

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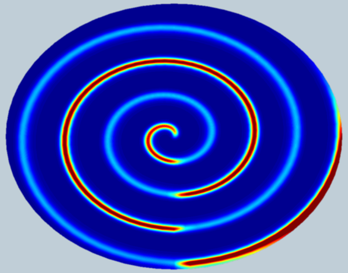

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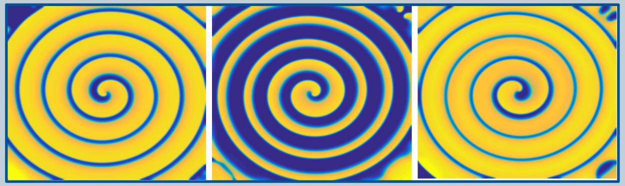

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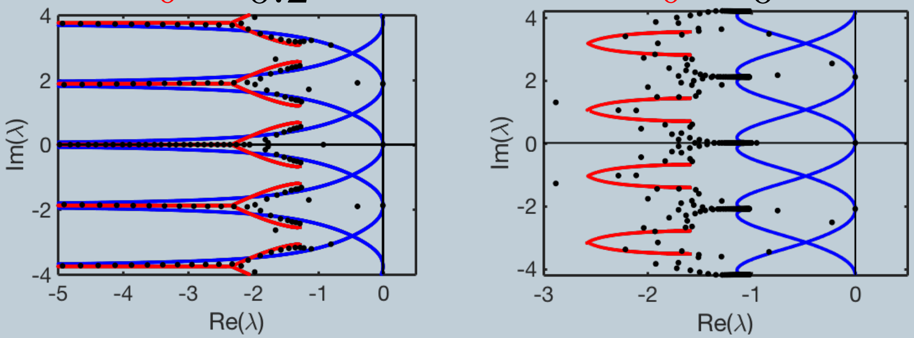
Spectra with Rank-Deficient Diffusion Matrix

$\delta = 0.2$ $\qquad$ $\delta = 0$



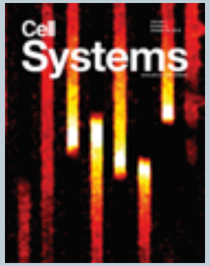
$$u_t = \Delta u + f(u,v)$$
$$v_t = \delta \Delta v + g(u,v)$$

Theorem:

$$\Sigma^0_{\mathrm{ess}} \neq \lim_{\delta \to 0} \Sigma^\delta_{\mathrm{ess}}$$

Geometry-Dependent Arrhythmias in Bioengineered Tissue

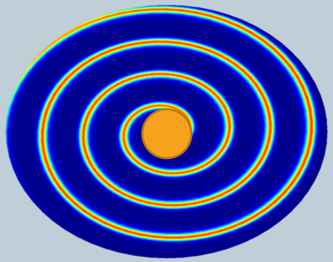
McNamara, H. M., **Dodson, S.**, et. al. *Cell Systems* (2018).



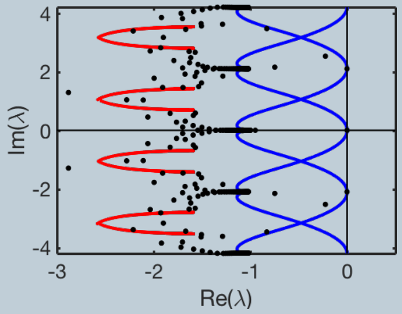
Collaboration with Cohen Lab at Harvard

Extensions

- Alternans: Sub- or supercritical Hopf bifrucation?



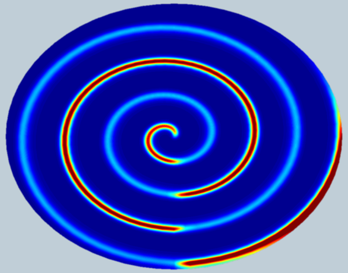
Pinned Spirals



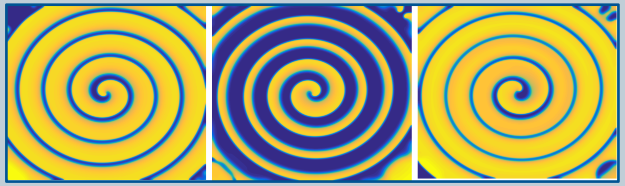
Point Spectrum

Summary:

- Different mechanisms responsible for instabilities
- Findings on one domain may not be relevant for all others

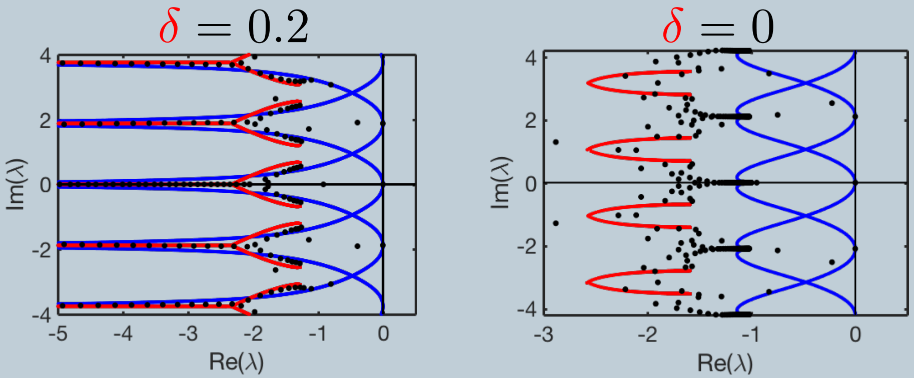


Boundary conditions



- Spiral core
- Point eigenvalue and essential spectrum

Spectra with Rank-Deficient Diffusion Matrix



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$$v_t = \delta\Delta v + g(u,v)$$

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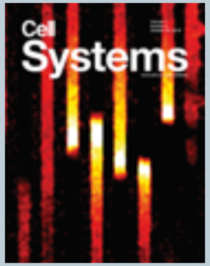
Thank you!
Questions?

Geometry-Dependent A
Bioengineered T

McNamara, H. M., **Dodson, S.**, et. al. *Cell Systems* (2018).



Collaboration with Cohen Lab at Harvard

Extensions

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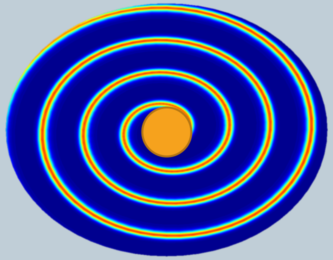

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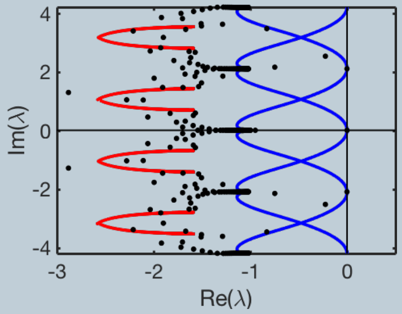

Point Spectrum

EXTRA SLIDES

Additional Spectral Details

Planar linear operator

$$\mathcal{L}_* V = D\Delta_{r,\psi} V + \omega V_\psi + F_U(U_*)V, \ (r,\psi) \in [0,\infty) \times S^1$$

Linear operator for bounded disk

$$\mathcal{L}_{*,R} V = D\Delta_{r,\psi} V + \omega V_\psi + F_U(U_{*,R})V, \ (r,\psi) \in [0,R) \times S^1$$

$$V_r(R,\psi) = 0, \ \psi \in S^1$$

Weighted space

$$L^2_\eta(\mathbb{R}^2) := \left\{ u \in L^2_{loc} : |u|_{L^2_\eta} < \infty \right\}, \ |u|^2_{L^2_\eta} := \int_{\mathbb{R}^2} \left| u(x) e^{\eta|x|} \right|^2 dx, \ \eta \in \mathbb{R}$$

Extended point spectrum

$$\Sigma_{\text{expt}}(\mathcal{L}_*) := \left\{ \lambda \in \mathbb{C} : \mathcal{L}_* - \lambda \text{ is not boundedly invertible in } L^2_\eta(\mathbb{R}^2) \right\}$$