The Primal-Dual Hybrid Gradient Method for Semiconvex Splittings

SIAM Conference on Imaging Science, Albuquerque
Minisymposium on Non-Convex Regularization Methods in Image Restoration

May 26th, 2016

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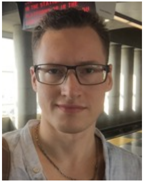
Joint work with:



Evgeny Strekalovskiy

Michael Moeller

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Motivation: First-Order Splitting Methods

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- Well-established theory in the convex setting

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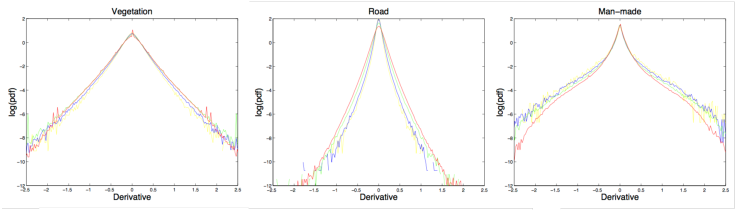

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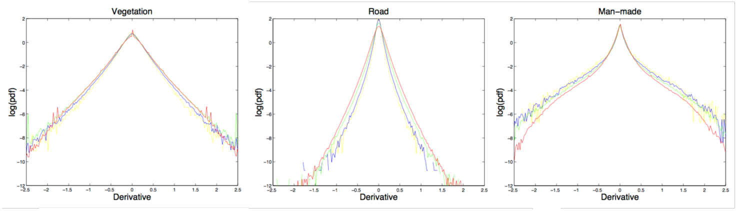

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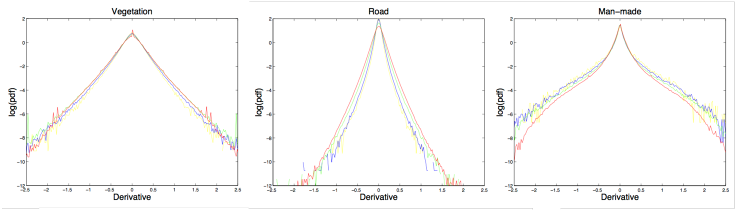

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- Piecewise smooth approximations [Blake, Zisserman '87], [Geman, Geman '84]

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- Key computation in most splitting methods is evaluation of *proximal mapping*

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- Remark: many other possibilities exist for nonsmooth nonconvex optimization

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- Can be applied to nonconvex F in a meaningful way

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- Reformulated PDHG applied to Mumford-Shah [Strekalovskiy, Cremers '14]

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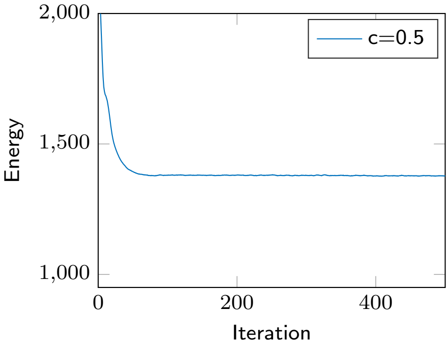

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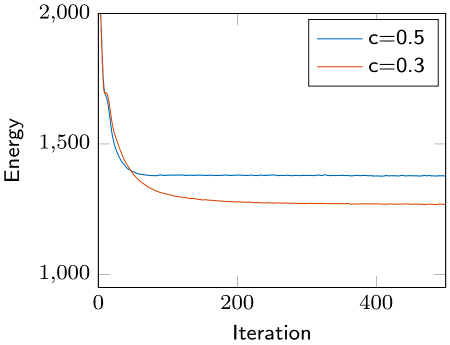

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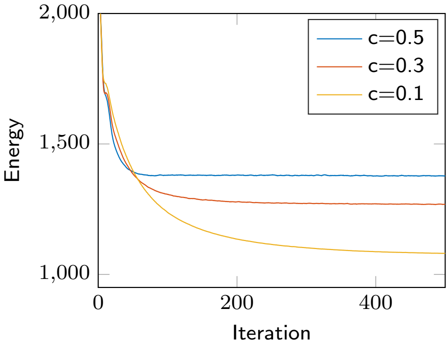

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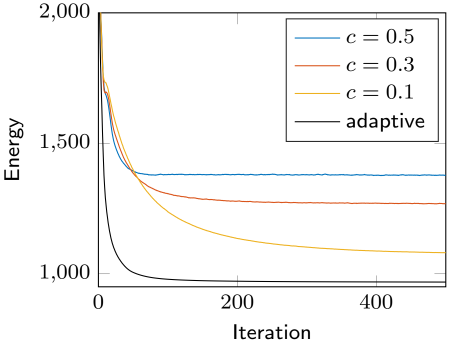

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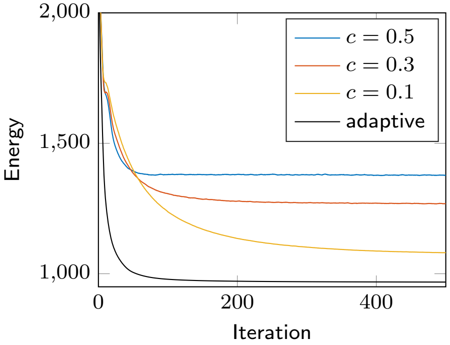
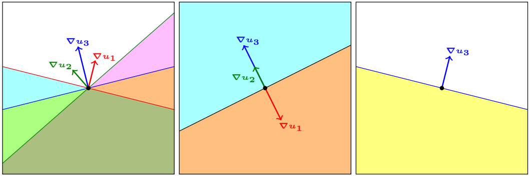


- On recent GPU: ≈ 30ms for 640×480 color image [Strekalovskiy, Cremers '14] ($\to$ application: real-time video cartooning!)

TV^q and TGV^q-like Regularization for Color Images



- For color images $u : \Omega \to \mathbb{R}^3$, consider at every pixel $i \in \Omega$ the Jacobian matrix

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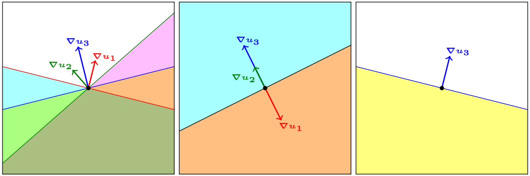

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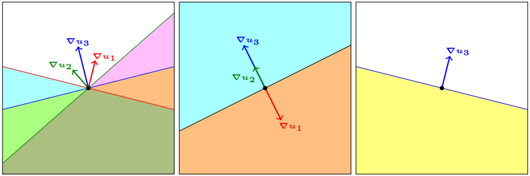

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- Nonconvex generalizations of above [M., Strekalovskiy, Moeller, Cremers '15]

$$TV_F^q(u) = \sum_{i \in \Omega} \|(\nabla u)_i\|_F^q, \; TV_{S^q}^q(u) = \sum_{i \in \Omega} \|(\nabla u)_i\|_{S^q}^q, \; q < 1$$

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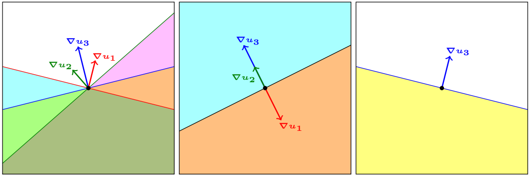

- For color images $u : \Omega \to \mathbb{R}^3$, consider at every pixel $i \in \Omega$ the Jacobian matrix

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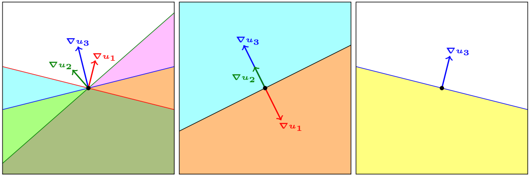

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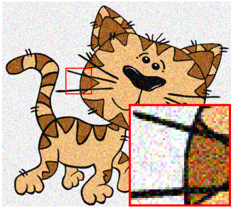
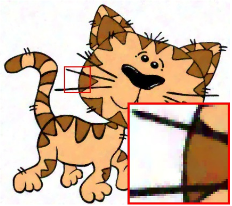
TV^q and TGV^q-like Regularization for Color Images, $q = 1/2$



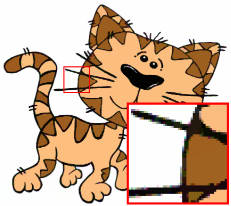
Noisy

$\sigma = 0.15$



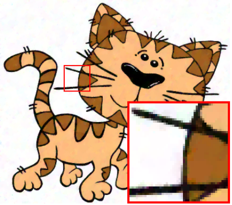
TV_F

PSNR=26.9



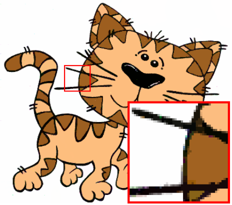
TV_F^q

PSNR=28.4



TV_{S^1}

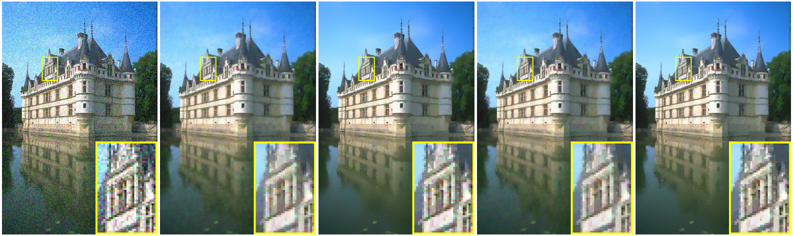
PSNR=27.9



$TV_{S^q}^q$

PSNR=**29.8**

TV^q and TGV^q-like Regularization for Color Images, $q = 3/4$



Noisy	TGV_F	TGV_F^q	TGV_{S^1}	$TGV_{S^q}^q$
$\sigma = 0.1$	PSNR=28.5	PSNR=28.9	PSNR=29.0	PSNR=**29.4**

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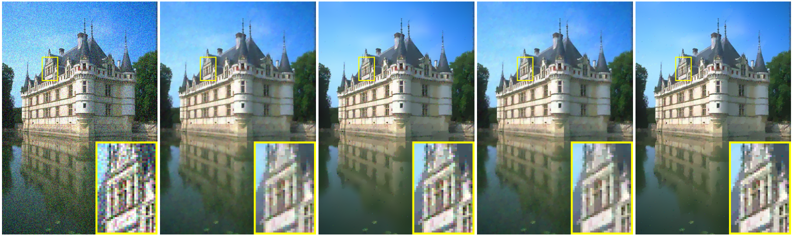

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Algorithm works well in practice. Theoretical convergence properties?

Convergence Analysis of Nonconvex PDHG

Theorem ([M., Strekalovskiy, Moeller, Cremers '15])

Let $G - \frac{c}{2}\|\cdot\|^2$ *and* $F + \frac{\omega}{2}\|\cdot\|^2$ *be convex with* $c > \omega\|K\|^2$*. Then the (ergodic) iterates* (X^k) *produced by the PDHG converge to the (unique) global minimizer*

$$\widehat{x} = \arg\min_x \; G(x) + F(Kx),$$

with $\|X^k - \widehat{x}\|^2 = \mathcal{O}(1/k)$ *for* $0 < \sigma = 2\omega, \; \tau\sigma\|K\|^2 \leq 1$*, and any* $\theta \in [0,1]$*.*

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- Notice that for our proof, σ has to be twice as big as to make the proximal minimization subproblem in z convex:

$$\text{prox}_{1/\sigma, F}(\tilde{z}) = \arg\min_{z} \; F(z) + \frac{\sigma}{2}\|z - \tilde{z}\|^2$$

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Sharpness of the Step-Size Restriction and Consequences

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- Application: convex non-convex (CNC) models [Parekh, Selesnick '15], [Lanza, Morigi, Sgallari '16]

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- For $K \neq I$, $\theta = 1$, PDHG can be seen as inexact ADMM on the *dual problem* (hasn't been studied for nonconvex F to best of our knowledge)
- Local convergence result for nonlinear K [Valkonen '13], can also be used to do nonconvex regularization [Shekhovtsov, Reinbacher, Graber, Pock '16]

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Thank you for your attention!

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References

1. T. Möllenhoff, E. Strekalovskiy, M. Moeller, D. Cremers. The primal-dual hybrid-gradient method for semiconvex splittings, SIAM Journal on Imaging Science, 2015.
2. T. Möllenhoff, E. Strekalovskiy, M. Moeller, D. Cremers. Low rank priors for color image regularization, Energy Minimization Methods in Computer Vision and Pattern Recognition, 2015.
3. E. Strekalovskiy, D. Cremers. Real-time minimization of the piecewise smooth Mumford-Shah functional, European Conference on Computer Vision, 2014.
4. T. Pock, D. Cremers, H. Bischof, A. Chambolle. An algorithm for minimizing the Mumford-Shah functional, International Conference on Computer Vision, 2009.