Games vs. Optimizations:
Smoothed and Approximation Complexity

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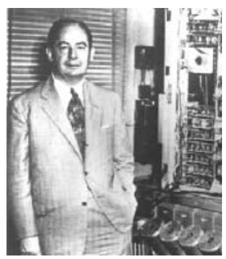
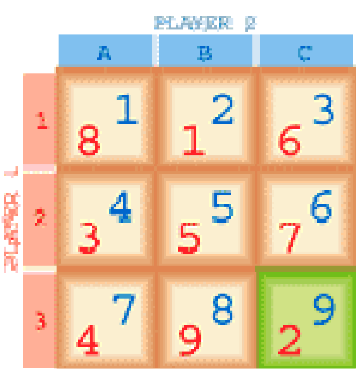
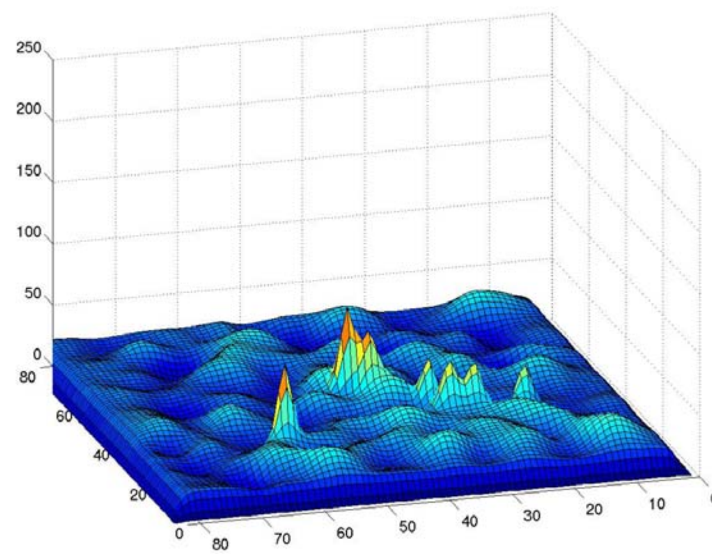
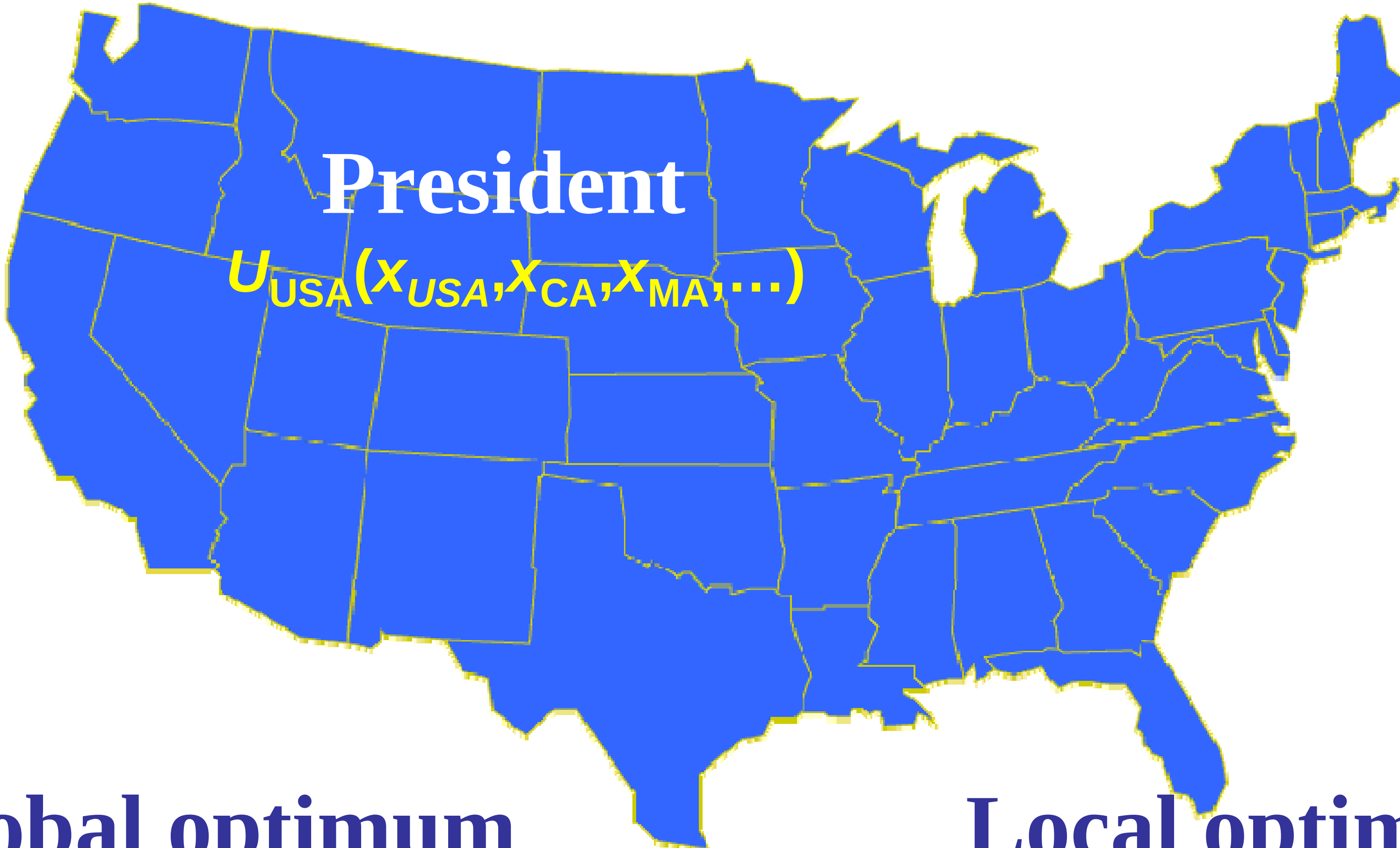






Games and Optimization

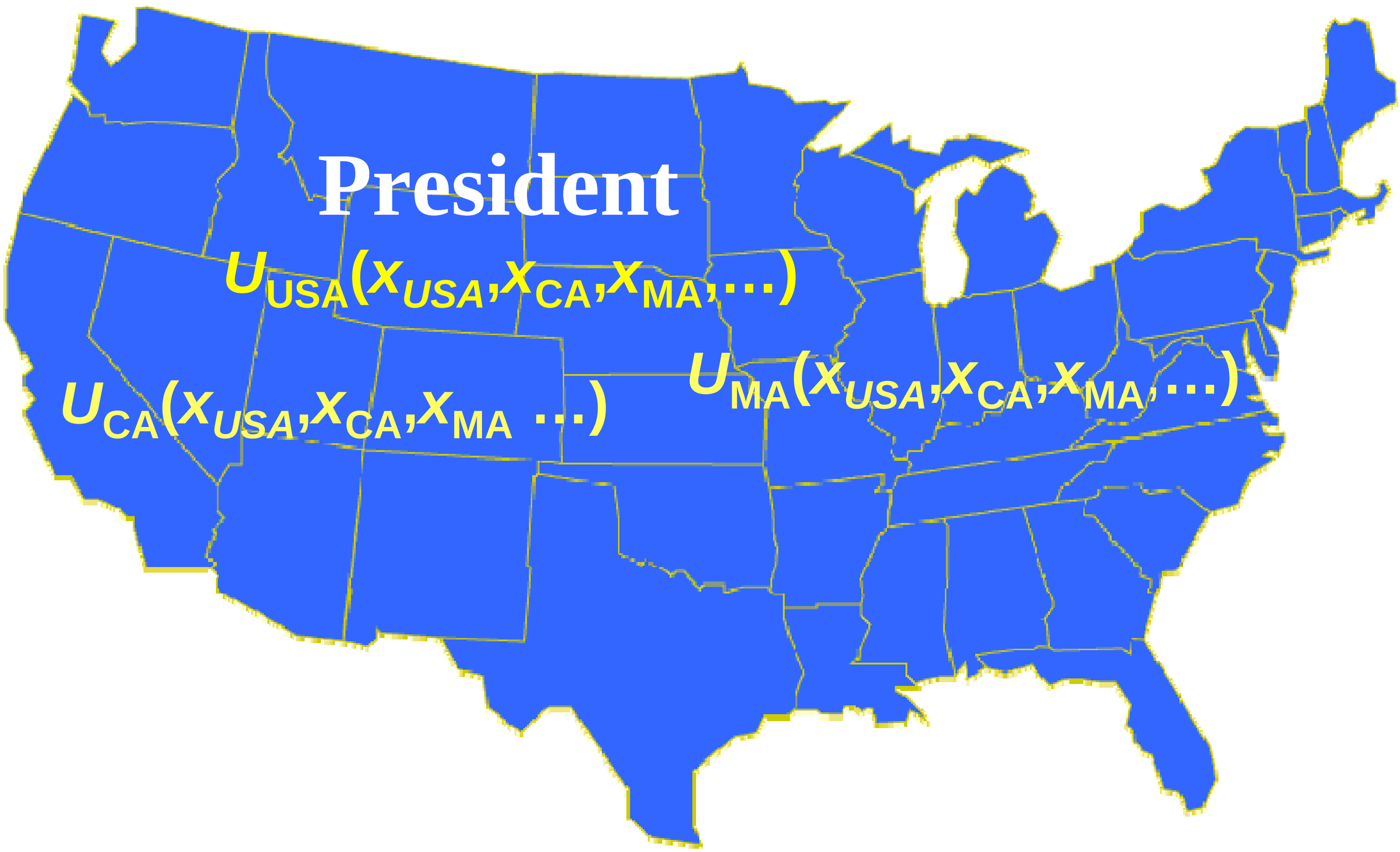
Optimization



Global optimum **Local optimum**

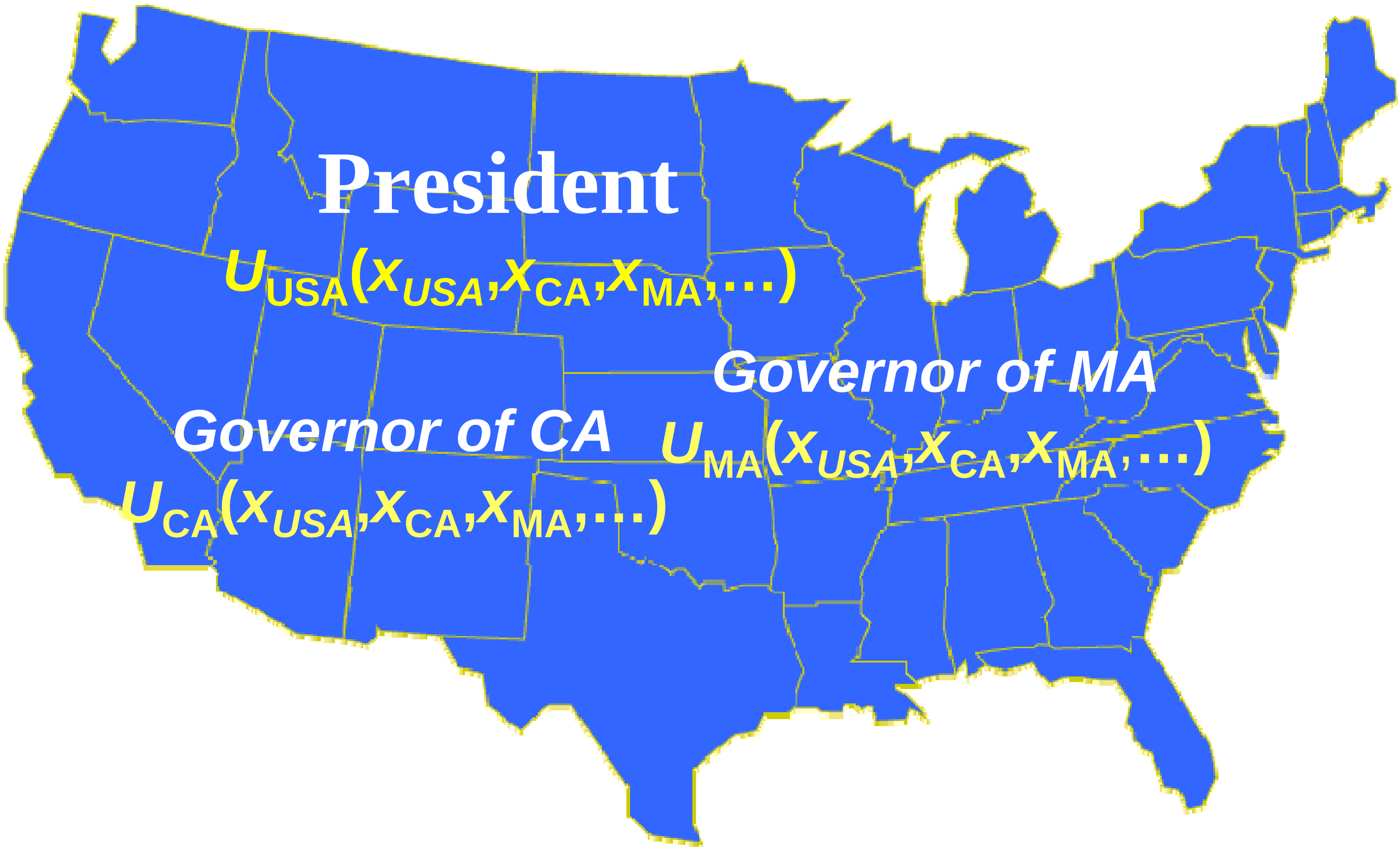
Approximation

Multi-Objective Optimization



Pareto optimum [Approximation]

Multi-Player Games

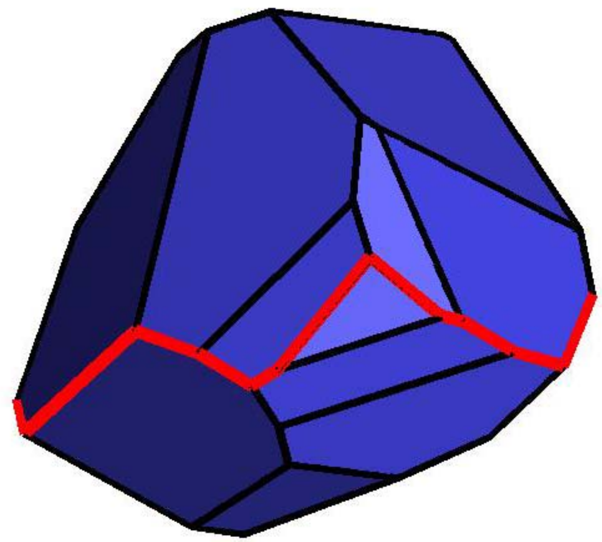


Best response **Nash equilibrium**

A classic optimization problem
and
a not so classic analysis

LP and the Simplex Method

$$\max \ \boldsymbol{c}^T \boldsymbol{x}$$
$$\text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$$



Worst-Case: exponential

Widely used in practice

Smoothed Analysis of Simplex Method

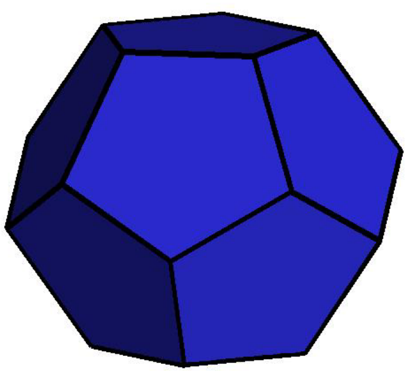
(Spielman + Teng, 2001)

max $c^T x$
s.t. $Ax \leq b$

→

max $c^T x$
s.t. $(A + \sigma \|A\| G)x \leq b$

G is Gaussian



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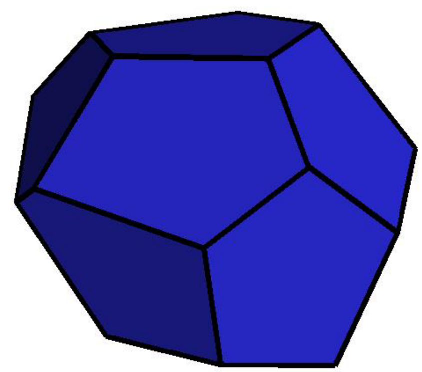


Theorem: For all A, b, c, simplex method takes expected time polynomial in $m, n, 1/\sigma$

Motivations:

Heuristics that work in practice, with no sound theoretical explanation

Exponential worst-case complexity, but works in practice

Heuristic speeds up code, with poor results in worst-case

Polynomial worst-case complexity, but much faster in practice

Smoothed Complexity

$$C(n,\sigma) = \max_{x \in R^n} \left[\mathop{\mathbf{E}}_{g \in R^n} \left[T(x + \|x\| \sigma g) \right] \right]$$

Interpolates between worst and average case

Considers neighborhood of every input

If low, all bad inputs are unstable

Smoothed Complexity of Integer Programming

$$\begin{array}{rc} \text{max} & c^T x \\ \text{subject to} & Ax \leq b, x \in D^n, \end{array}$$

where $A \in R^{k \times n}, b \in R^k, c \in R^n, D \subset Z$
and $|D| = \text{poly}(n)$

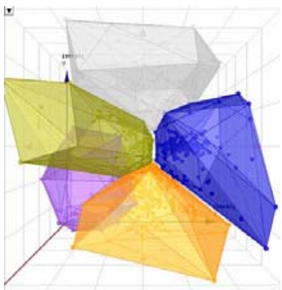
Smoothed Complexity: [Beier-Vöcking]

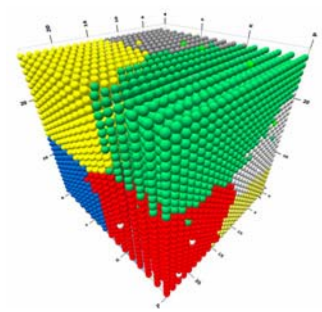
$$\text{poly}(n, k, 1/\sigma).$$

Smoothed Complexity of Local Search

***k*-Means Method:** $\mathsf{poly}(n, 1/\sigma)$

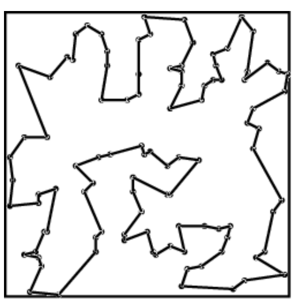
[Arthur- Röglin-Manthey]

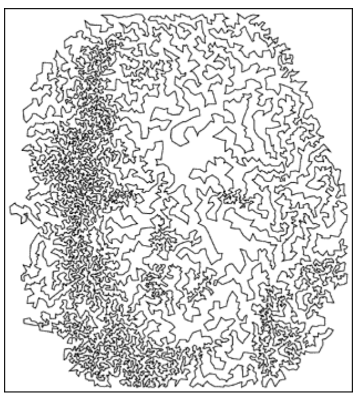




***2-opt TSP*:** $\mathsf{poly}(n, 1/\sigma)$

[Englert, Röglin, and Vöcking]





Smoothed Complexity of Multi-Objective Optimization

Röglin-Teng: The number of Pareto solutions in a binary program with a fixed number of objective functions is

$$\mathsf{poly}(n, 1/\sigma)$$

Games, Markets, and Equilibria

BIMATRIX Games

"Is the smoothed complexity of (another classic algorithm,) Lemke-Howson (algorithm) for two-player games, polynomial?"

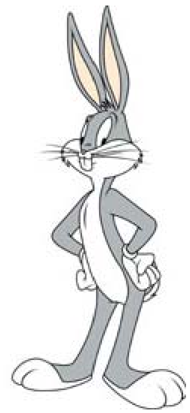
0 0	-1 1	1 -1
1 -1	0 0	-1 1
-1 1	1 -1	0 0

$$\begin{pmatrix} & A \\ B & \end{pmatrix}$$

Mixed Strategies

Nash Equilibria in Two-Player Games

A



B

Mixed equilibrium always exists:

$$(\mathbf{x}^*)^T \mathbf{A}\, \mathbf{y}^* \geq \mathbf{x}^T \mathbf{A}\, \mathbf{y}^* \quad \text{and} \quad (\mathbf{x}^*)^T \mathbf{B}\, \mathbf{y}^* \geq (\mathbf{x}^*)^T \mathbf{B}\, \mathbf{y}.$$

$$(\mathbf{x}^*)^T \mathbf{A}\, \mathbf{y}^* \geq \mathbf{x}^T \mathbf{A}\, \mathbf{y}^* - \epsilon \quad \text{and} \quad (\mathbf{x}^*)^T \mathbf{B}\, \mathbf{y}^* \geq (\mathbf{x}^*)^T \mathbf{B}\, \mathbf{y} - \epsilon.$$

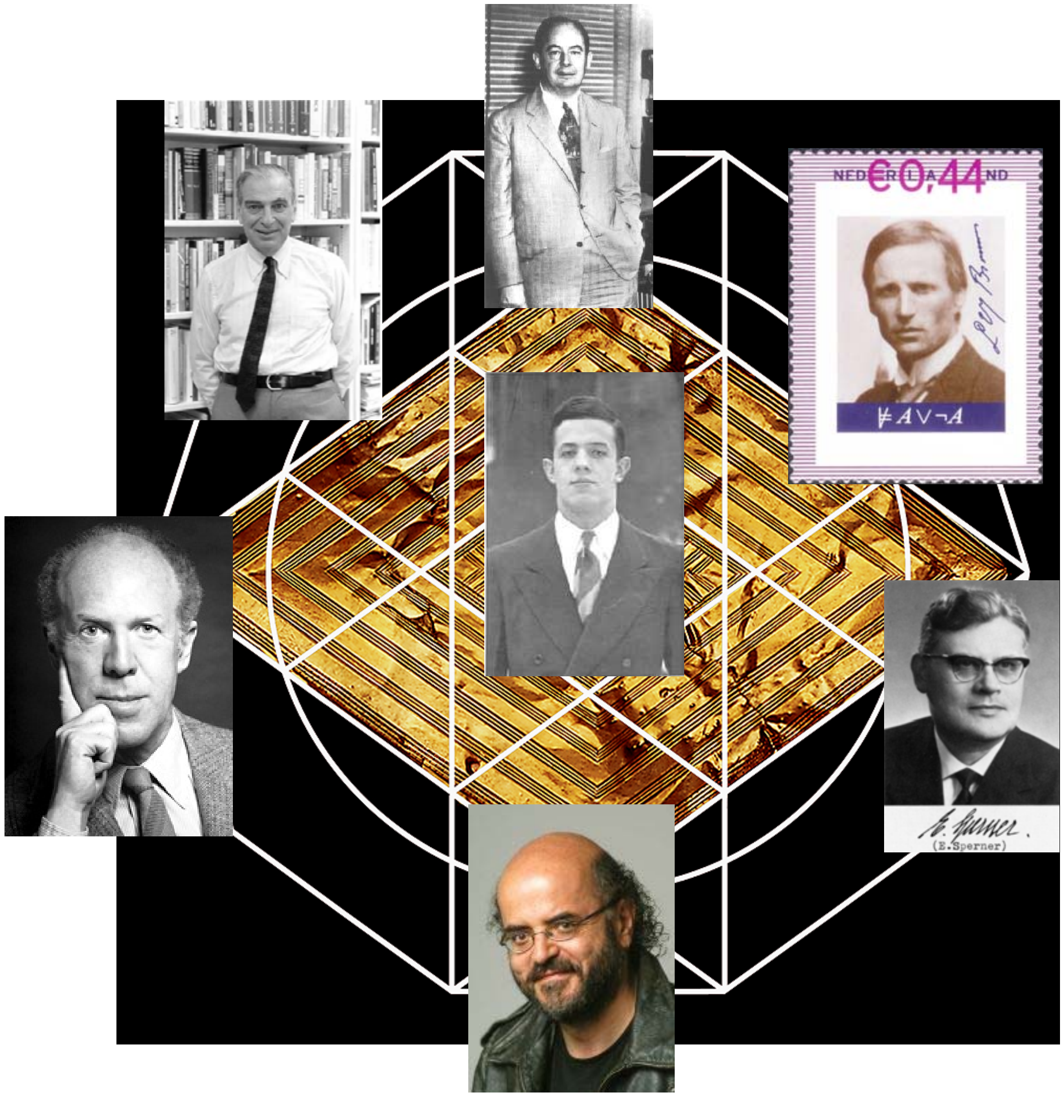
Search Problem: Find an equilibrium

Smoothed Model

$$(\bar{\mathbf{A}}, \bar{\mathbf{B}}) \rightarrow ([\bar{a}_{i,j} \pm \epsilon], [\bar{b}_{i,j} \pm \epsilon]) \rightarrow (\mathbf{A}, \mathbf{B})$$

Many-Player Games

Many-Person Games



Exchange Economies

- Traders
- Goods
- Initial Endowments: $\mathbf{E} = (\ \mathbf{e}_i\ )$
- Utilities: $\mathrm{U} = (\ \mathbf{u}_i\ )$



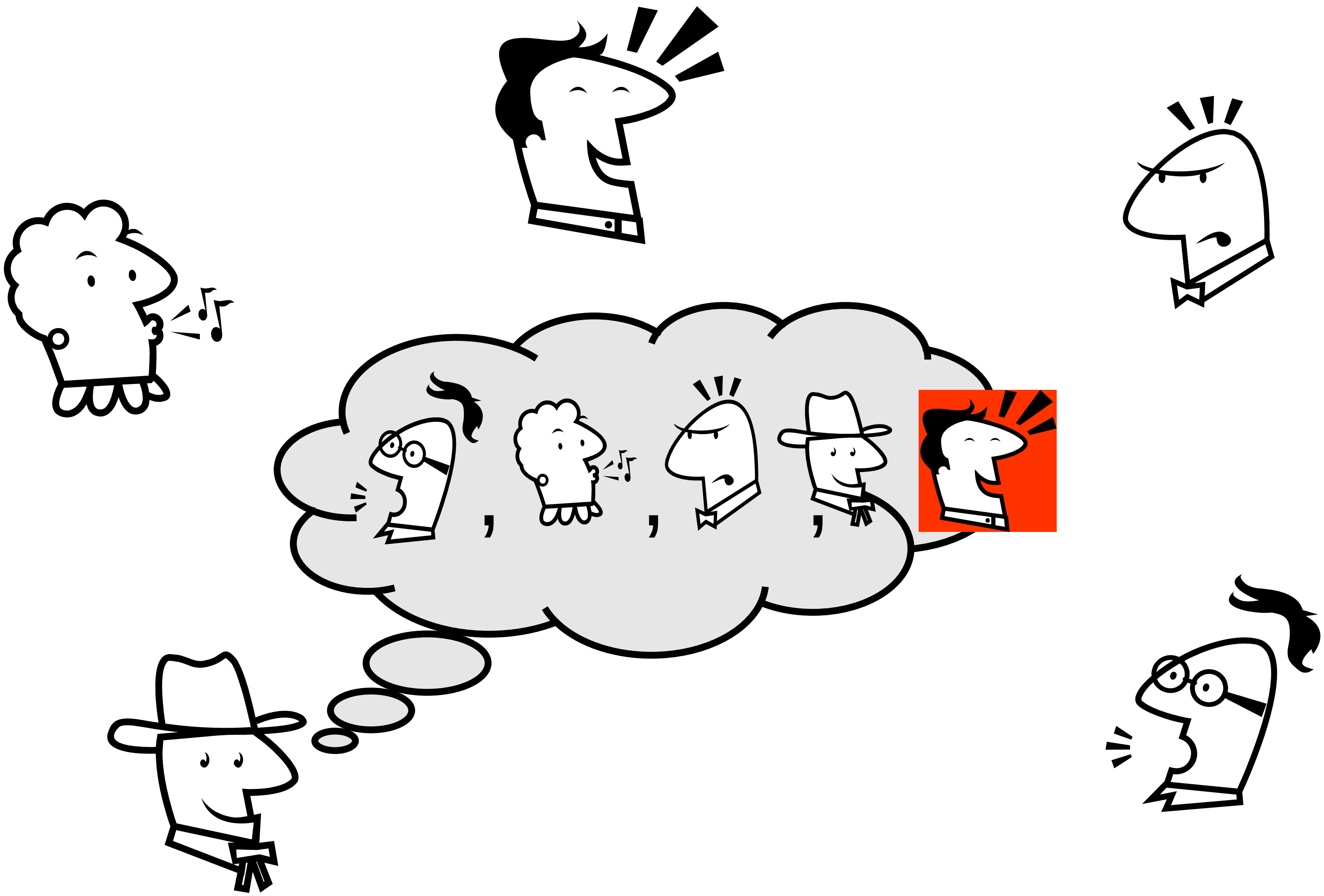

Arrow-Debreu Equilibrium Price

A price vector

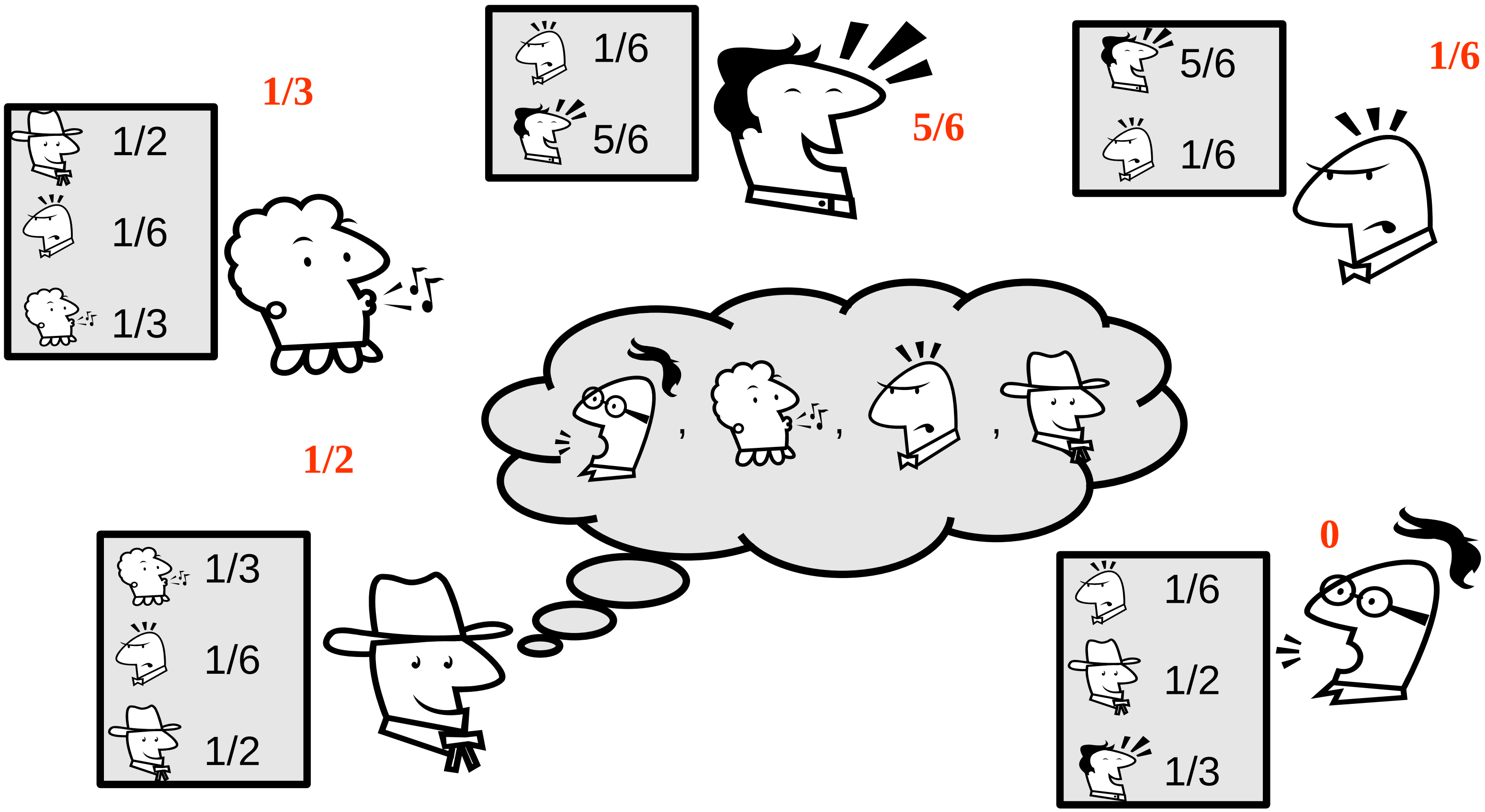
Distributed Exchange

- Every Trader:
  - Sells the initial endowment to “market”: (to get a budget)
  - Buys from the “market” to optimize her individual utilities
- Market Clearing Price

The Preference Game
(the blogsphere game)



How Much Blog to Write? (best response and equilibrium)



Mathematical Questions

- Is there an equilibrium?

Complexity Questions

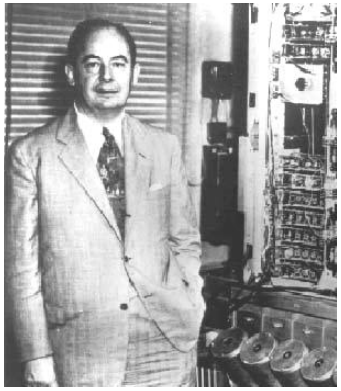
- Polynomial time algorithm for equilibria?
- Smoothed polynomial-time algorithm for equilibria?

- Is a 2-player Nash equilibrium is easier to compute than a 3-player Nash equilibrium or a 51-player Nash equilibrium, or market equilibria?

- Is an approximate equilibrium easier to compute than an "exact" equilibrium?

Zero-Sum Two-Player Games
Linear Programming
(John von Neumann)

$$A + B = 0$$



Min-Max Theorem

Linear Programming Duality

Two Natural Questions: Learning from History

- Ellipsoid Method
  Interior Point Method
  Polynomial

- Simplex Method
  Smoothed Polynomial

- BIMATRIX
  in P?

- BIMATRIX
  in Smoothed P?

  $\mathsf{poly}\left(n, 1/\sigma\right)$

Path Following: Lemke-Howson

Does Lemke-Howson have polynomial *Smoothed* Complexity?

Smoothed Complexity & Approximation

$$(\bar{\mathbf{A}}, \bar{\mathbf{B}}) \rightarrow ([\bar{a}_{i,j} \pm \epsilon], [\bar{b}_{i,j} \pm \epsilon]) \rightarrow (\mathbf{A}, \mathbf{B})$$

Each Nash equilibrium of $(\mathbf{A}, \mathbf{B})$ is an

2ϵ–approximate Nash equilibrium of $(\bar{\mathbf{A}}, \bar{\mathbf{B}})$

A Unified Question

Does BIMATRIX have a Fully-Polynomial-Time Approximation Scheme?

ε-approximate Nash equilibrium in poly(n,1/ε) time?

log(n)-bits of an equilibrium in poly(n) time?

The Tale of Two Types of Economies

Linear Exchange Economies:

Piece-wise Linear Exchange Economies:

Equilibrium in Linear Exchange Economies

Polynomial Time Computable

- [Nenakov-Primak 83]
- [Devanur-Papadimitriou-Saberi-Vazirani 02]
- [Jain-Mahdian-Saberi 03]
- [Garg-Kapoor 04]
- [Jain 04]
- [Ye 04]

Complexity Results: Multi-players

[Daskalakis-Goldberg-Papadimitriou, 2005]

- **For any constant k ⩾ 4, every polynomial-time algorithm for k-player Nash equilibria can be used to design a polynomial-time algorithm for (k+1)-player Nash equilibria.**

- **If the computation of a 4-player Nash equilibrium is in P, then the computation of a general Arrow-Debreu equilibrium as well as well as the computation of a fixed point of a general Brouwer function is in P.**

[Chen-Deng; Daskalakis-Papadimitriou, 2005]

- **For any constant k ⩾ 3, …**

Complexity Results: Two-Players

[Chen-Deng, 2005]

- **If the computation of a 2-player Nash equilibrium is in P, then the computation of**
  - **a 3-player Nash equilibrium,**
  - **a general Arrow-Debreu equilibrium,**
  - **a fixed point of a general Brouwer function**

  is in P.

[Chen-Deng-Teng, 2006]

- **If the computation of an approximate, 2-player Nash equilibrium is in P, then …**

[Huang-Teng, 2006]

- **If the computation of an approximate equilibrium of a Leontief exchange economy is in P, then …**

  Build upon [Codenotti-Saberi-Varadarajan-Ye; Chen-Deng-Teng]

[Chen-Dai-Du-Teng 2009]

- **Extended to additively separable piece-wise linear markets**

Smoothed Complexity of Equilibria

[Chen-Deng-Teng, 2006]

- **NO Smoothed Polynomial-Time Complexity for Lemke-Howson or any BIMATRIX algorithm, unless computation of game and market equilibria and Brouwer fixed points is in randomized P!**

[Huang-Teng, 2006]

- **Computation of Arrow-Debreu equilibria in Leontief Exchange Economies is not in Smoothed P, unless …**

Sperner's Lemma

(any legal coloring has a tri-chromatic triangle)

Nash Equilibria

↓

Kakutani's fixed-points

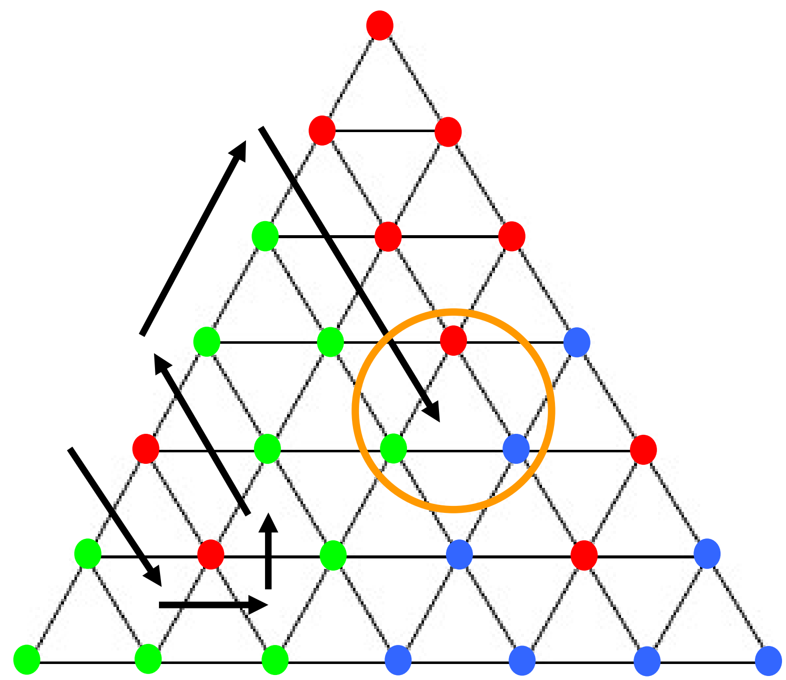
↓

Brouwer's Fixed Points

↓

Sperner's Lemma

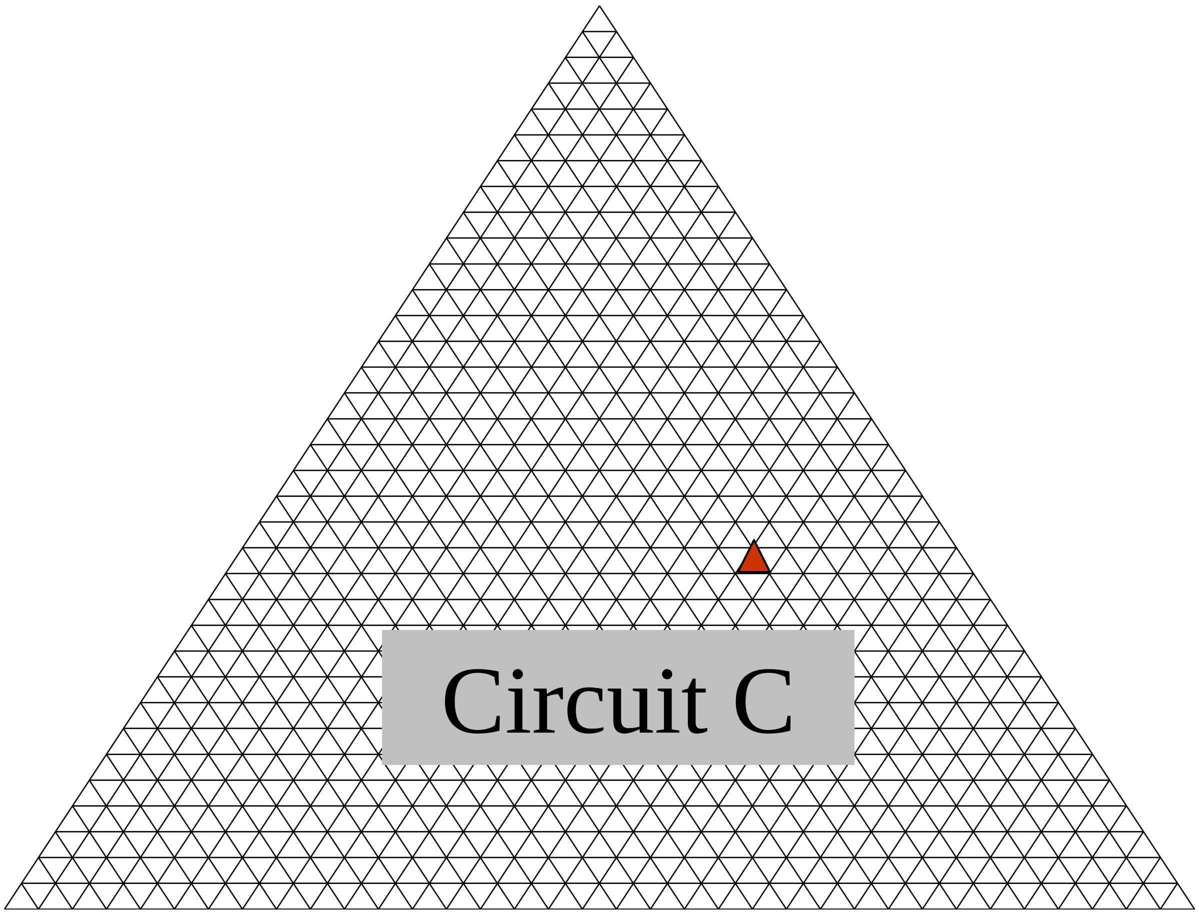
↓



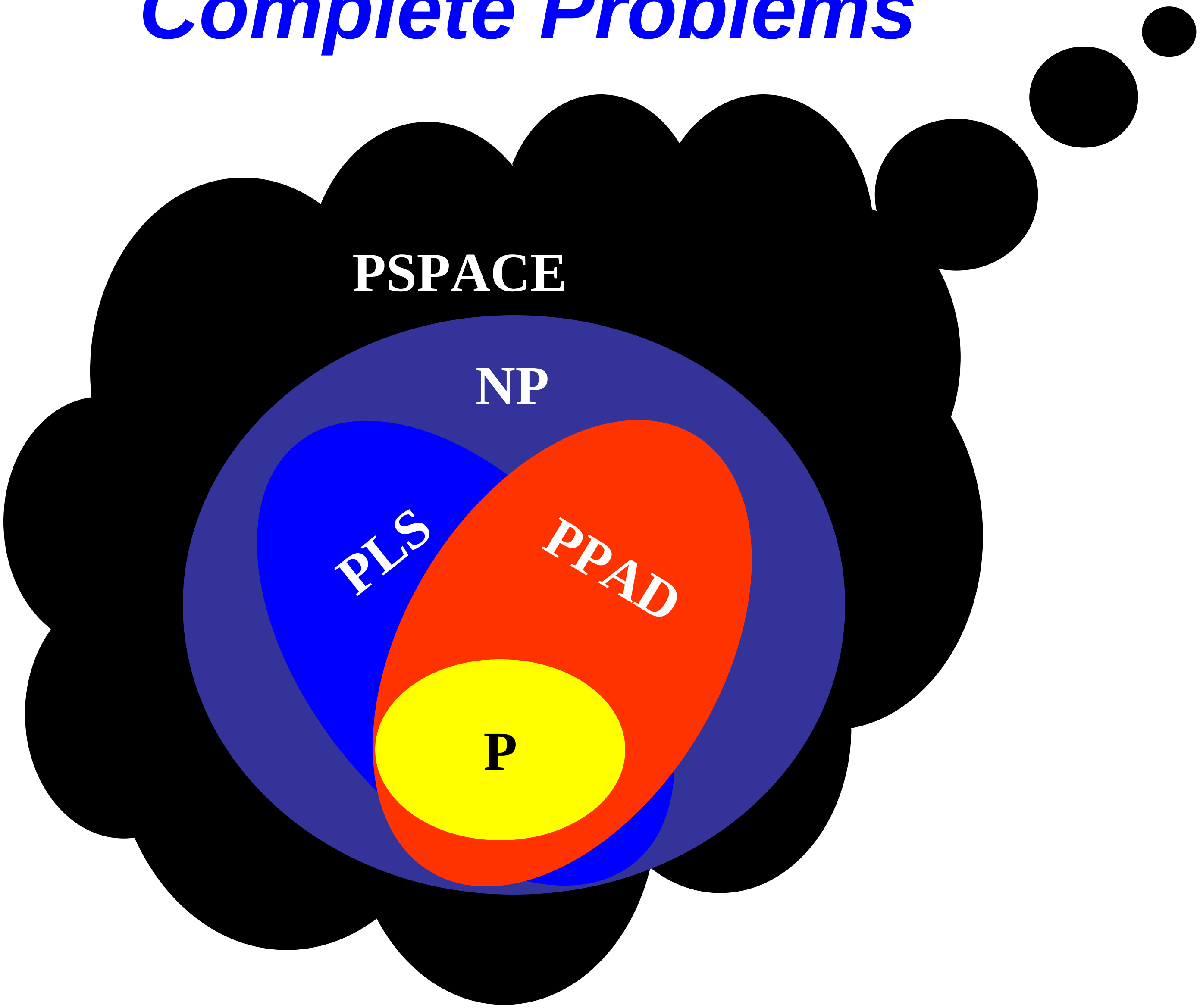
Polynomial Parity Argument (Directed Version)

Think Large and Think Exponential:

2^n-Barycentric Sperner



Complexity Classes and Complete Problems



Tale of Two Types of Equilibria

Local Search

(Potential Games)

- Linear Programming
  - P
- Simplex Method
  - Smoothed P
- PLS
  - FPTAS

- Intuitive

Fixed-Point Computation

(Matrix Games)

- 2-Player Nash equilibrium
  - Unknown
- Lemke-Howson Algorithm
  - If in P, then NASH in RP
- PPAD
  - FPTAS, then NASH in RP

- Intuitive to some

A Basic Question

Is fixed point computation fundamentally harder than local search?

Random Separation of Local Search and Fixed Point Computation

Aldous (1983):

- **Randomization helps local search**

Chen & Teng (2007):

- **Randomization doesn't help Fixed-Point-Computation!!!**

… in the black-box query model

Query Model

- Oracle
- Query point
- Deterministic

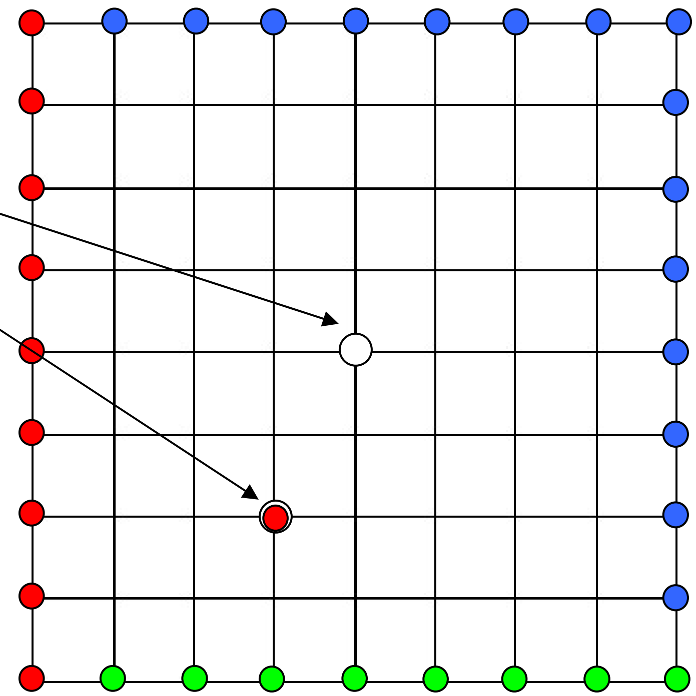
$$\mathsf{DQ}^d_{FP}(n)$$

- Randomized

$$\mathsf{RQ}^d_{FP}(n)$$

- Quantum

$$\mathsf{QQ}^d_{FP}(n)$$



Deterministic Query Complexity

- [Hirsch, Papadimitriou and Vavasis 89]

$$\mathsf{DQ}_{FP}^{d}(n) = \Omega(n^{d-2})$$

- [Chen & Deng 05]

$$\mathsf{DQ}_{FP}^{d}(n) = \Theta(n^{d-1})$$

Local search over [1:n]^d

- Find a *local minimum*

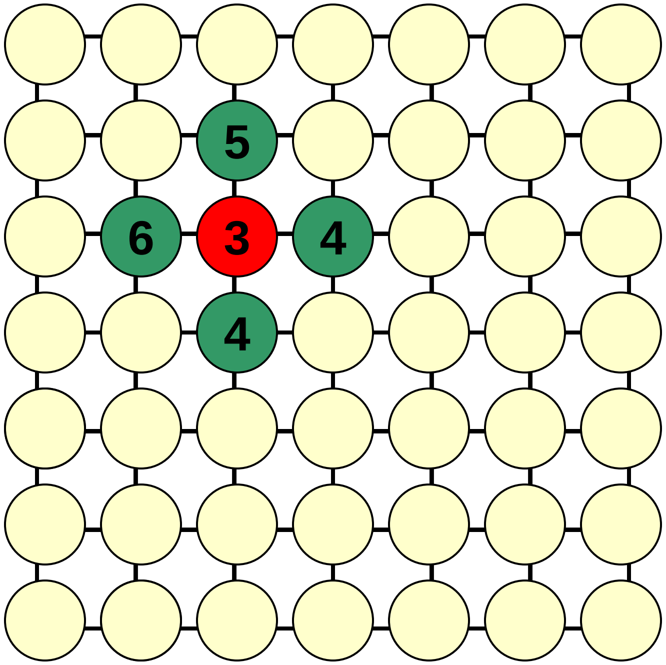
$$f : [1 : n]^d \rightarrow \mathbb{N}$$

- Deterministic

$$\mathsf{DQ}^d_{LS}(n) = \Omega(n^{d-1})$$

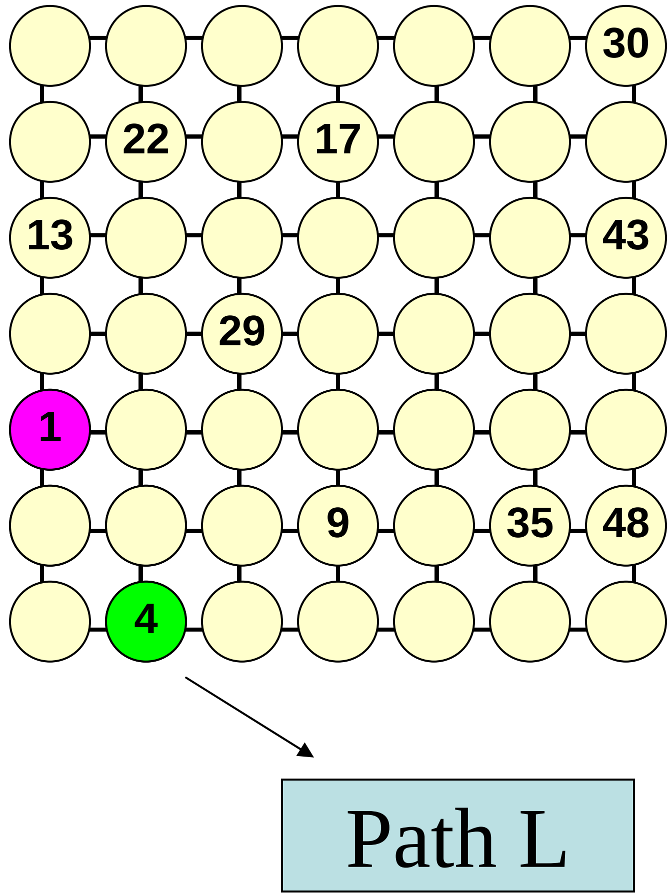
- [Aldous 83]

$$\mathsf{RQ}^d_{LS}(n) = O(n^{d/2})$$



Aldous's Algorithm

- Query $n^{d/2}$ points uniformly at random
- v: f(v) is smallest
- Follow v to a local minimum by using steepest descent
- # Query = $n^{d/2} + |L|$
- W.H.P, $|L| = O(n^{d/2})$

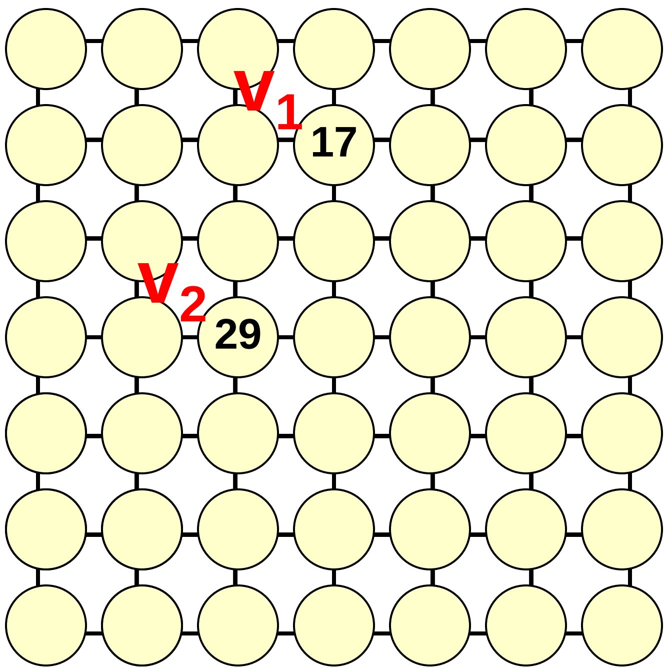


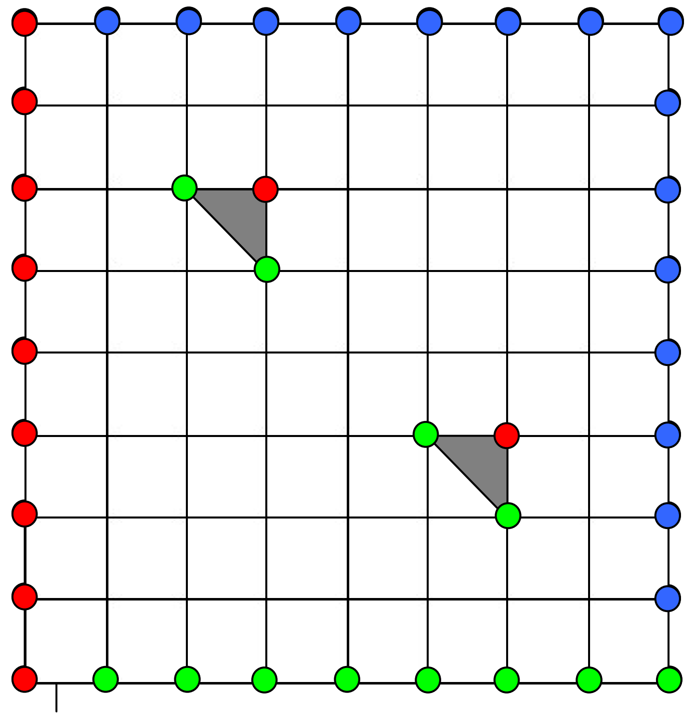
Aldous's Bound is Tight

- Scott Aaronson
- Shenyu Zhang
- Xiaoming Sun and Andy Yao

Structural Differences

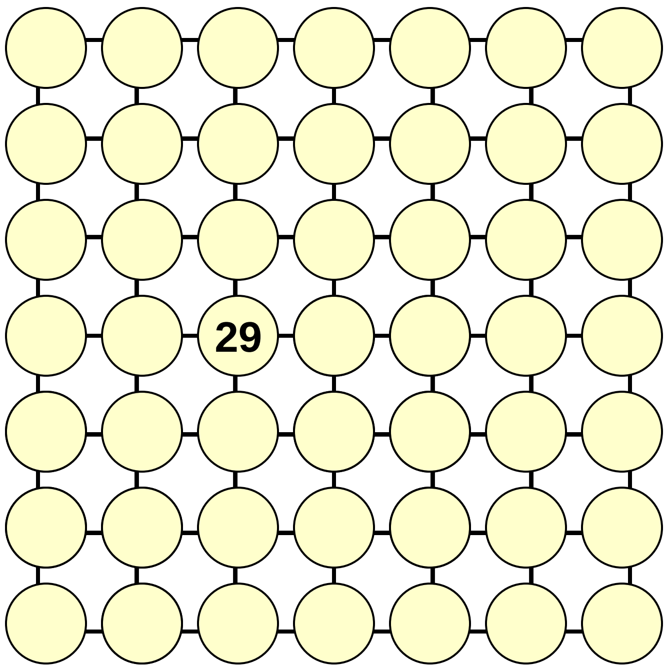
- Measure-of-Progress

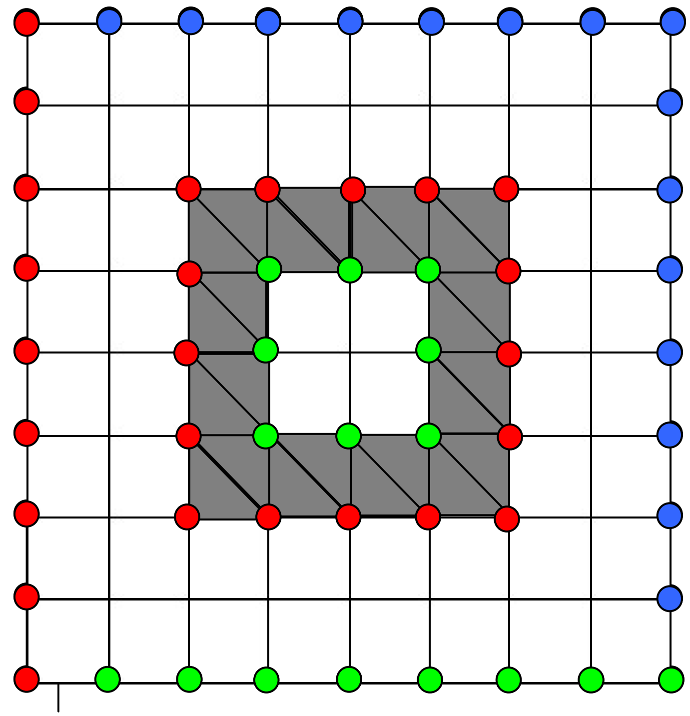




Structural Differences

- Graph Structure





Randomized Lower Bound
(Chen-Teng)

- Randomization doesn't help

$$\mathsf{RQ}^d_{FP}(n) = \Omega(n^{d-1})$$

- The significant gap between fixed point computation and local search is revealed in the randomized model.

Implication

- In the randomized query model over grids:
  - Global optimization is harder than fixed-point computation
  - Fixed-point computation is harder than local search

Complexity for Equilibrium Computation and Approximation

Fixed Points and Equilibria

Topology and Combinatorics

Existence Proof and Algorithmic Proofs

Mathematical Theorems and Algorithms

Brouwer, Sperner, von Neumann, Nash, Arrow, Debreu, Scarf, Papadimitriou …

Open Questions

- Polynomial-time approximation scheme?
- Nature condition for “easy games” and “easy markets”?
- How hard is PPAD?

Randomized Simplex Method (Kelner-Spielman)

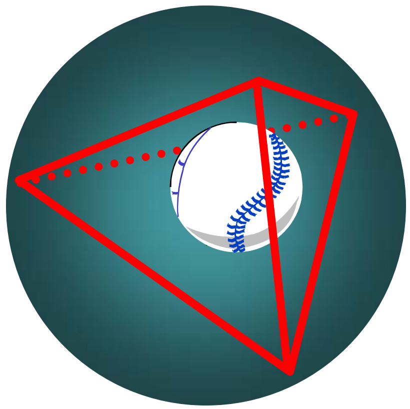
$$\max \; c^T x \quad \text{subject to } Ax \leq b.$$

$\Downarrow$

$$\text{Is } \widehat{A}x \leq \widehat{b} \text{ bounded?}$$

Boundedness does no dependent on the righthand side

Shadow of Perturbed of Rounded Polytope

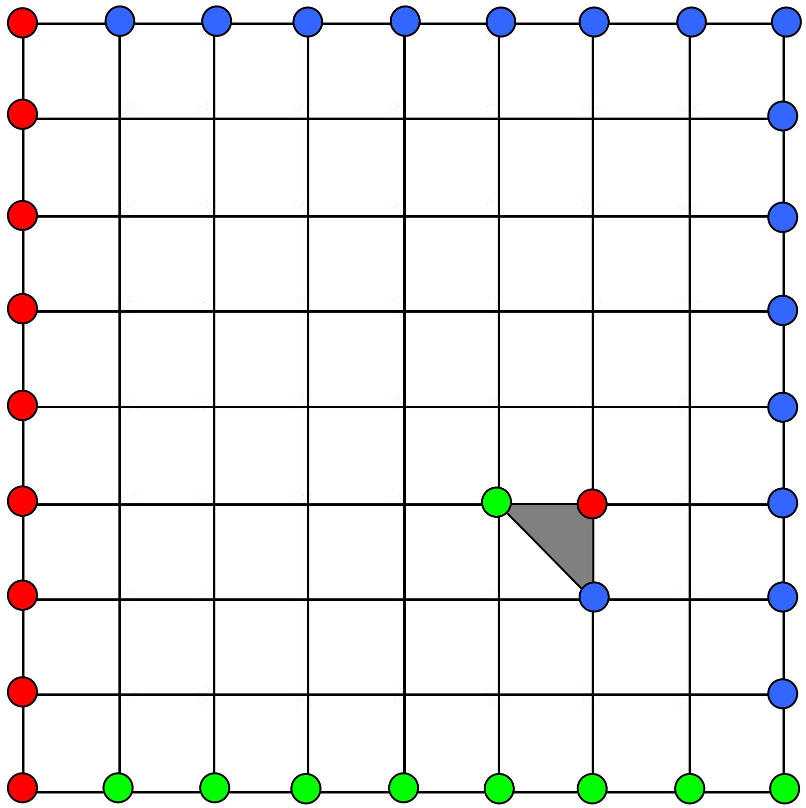


$$\hat{A}x \leq \vec{\mathbf{1}} + r$$

Kelner-Spielman: Boundedness of a rounded polytope can be tested in random polynomial time.

If the testing algorithm fails to determine the boundedness in polynomial time, “scale” to make it more round
Generalized simplex step

Discrete Brouwer's Fixed-Point Theorem

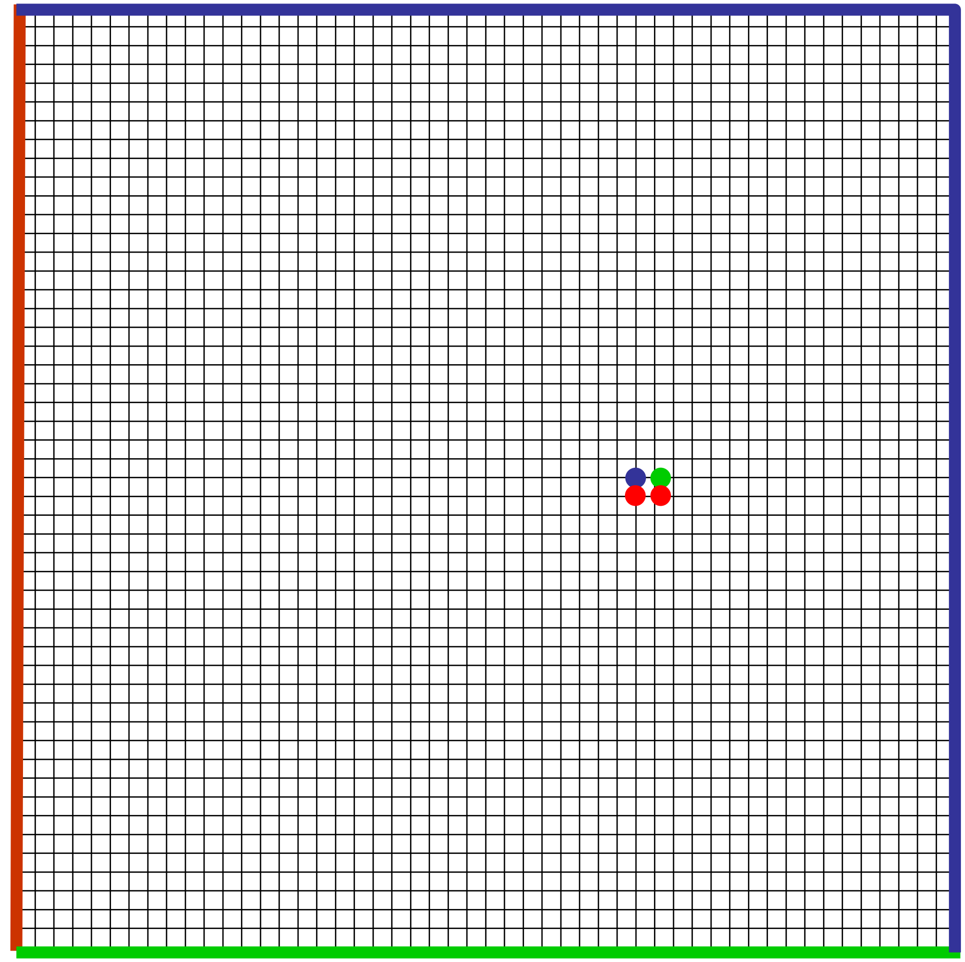


Given a valid *3*-coloring of a *2*D grid: [*1,2,…N*] × [*1,2,…N*],

there exists a unit size and tri-chromatic triangle.

2D Brouwer is PPAD-complete

(think large:
think exponential
$N = 2^n$)



- *2*D (Chen-Deng)
- *3*D (Daskalakis-Goldberg-Papadimitrou)