Analysis of Pattern Emergence in Turing Systems with Inhomogeneity in Reaction Term

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Introduction: Classical Turing's model of pattern formation

Reaction-diffusion system:

$$\begin{aligned} \partial_t u &= d_1 \Delta u + f(u,v) \\ \partial_t v &= d_2 \Delta v + g(u,v) \end{aligned} \quad \text{in } (0,\infty)\times\Omega,$$

with Neumann boundary conditions.

Let (u^*, v^*) be stationary, spatially homogeneous solution (ground state) which is

- stable without diffusion,
- unstable with diffusion.

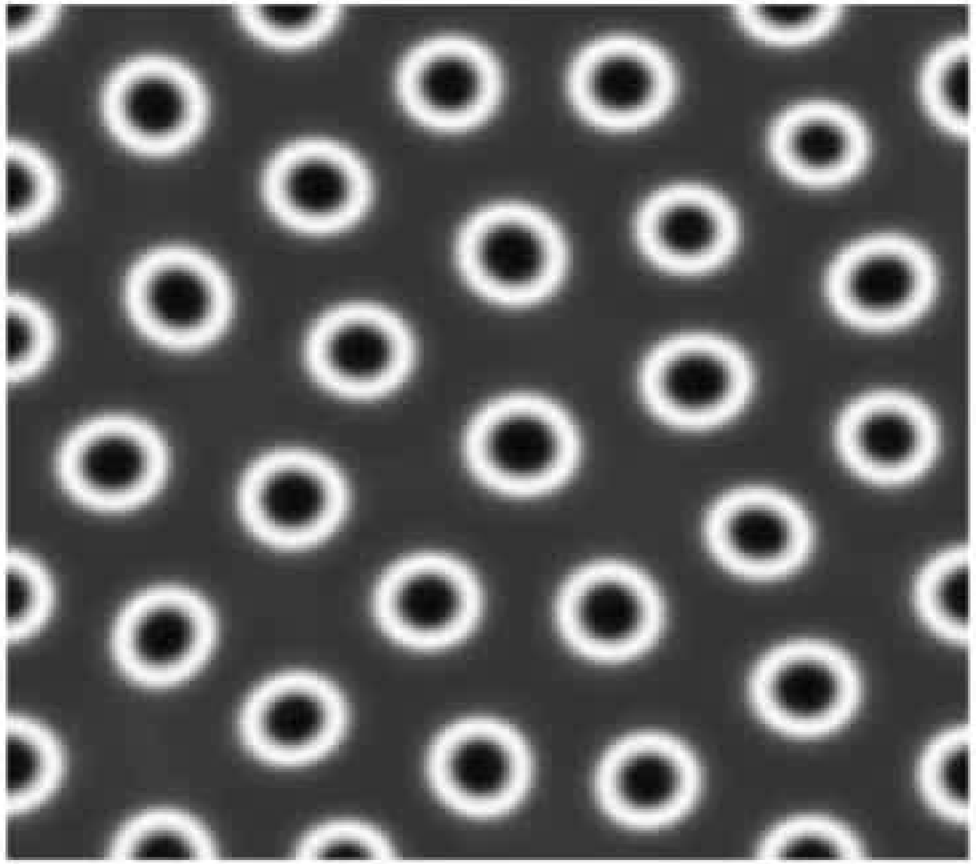
Evolution of perturbations around (u^*, v^*):

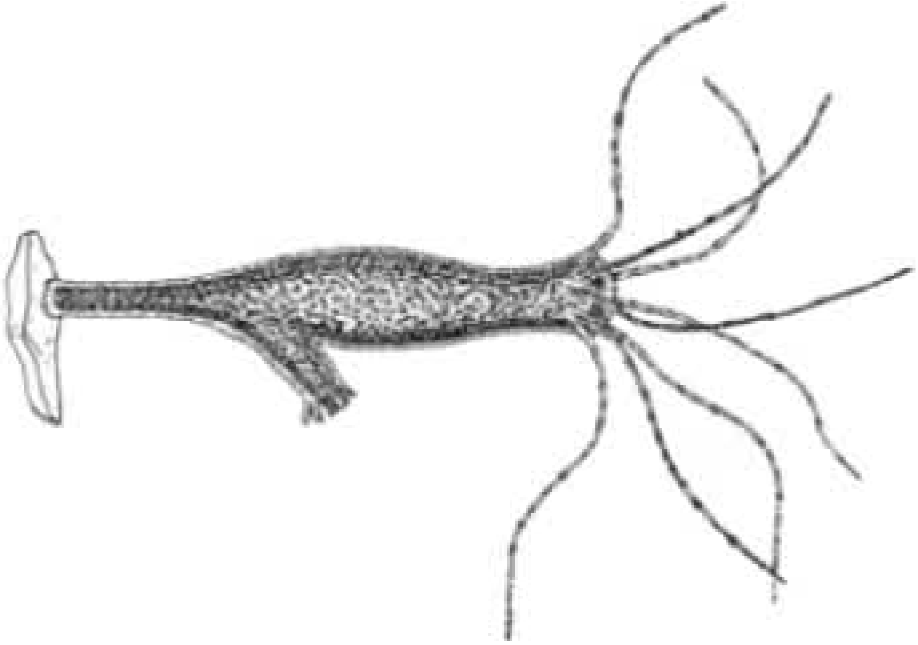
$$\begin{aligned} \partial_t \widetilde{u} &= d_1 \Delta \widetilde{u} + b_{11}\widetilde{u} + b_{12}\widetilde{v} + n_1(\widetilde{u},\widetilde{v}), \\ \partial_t \widetilde{v} &= d_2 \Delta \widetilde{v} + b_{21}\widetilde{u} + b_{22}\widetilde{v} + n_2(\widetilde{u},\widetilde{v}). \end{aligned}$$

Conditions for Turing instability:

$$\begin{aligned} \operatorname{tr} B &< 0, & b_{11}d_2 + b_{22}d_1 &> 0, \\ \det B &> 0, & (b_{11}d_2 + b_{22}d_1)^2 &> 4 d_1 d_2 \det B. \end{aligned}$$

Introduction: Biological motivation





Linear kinetics: Summary of numerical experiments

- $T1^L \wedge T1^R \wedge T2^L \wedge T2^R$:

	$T3^R \wedge T4^R$	$\neg T3^R \wedge T4^R$	$T3^R \wedge \neg T4^R$	$\neg T3^R \wedge \neg T4^R$
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- $\neg(T1^L \wedge T1^R \wedge T2^L \wedge T2^R)$:

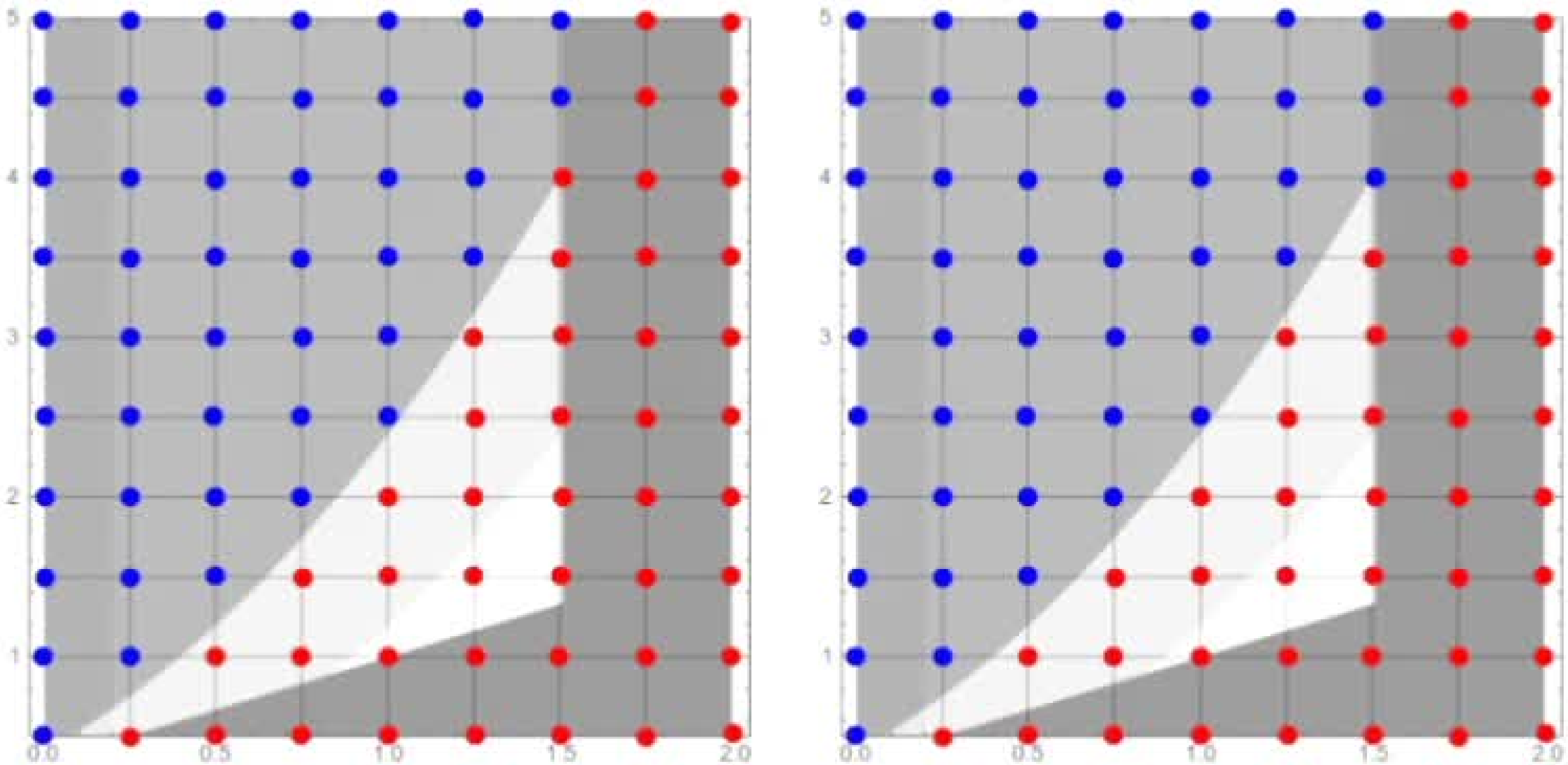
Crucial observations:

- the results are constant in each cell (except at the boundary),
- results of both methods correspond.

Linear kinetics: Ilustration of numerical results

Parameters:

$$d_1 = 1, \quad d_2 = 10, \quad b_{21} = -3, \quad b_{22} = -2, \quad s = 0.5, \quad L = 100, \quad \xi = 30.$$



spectrum	:	evolution
one positive real part		unbounded
all real parts negative		no pattern

Non-linear kinetics: Conditions and numerical verification

Hypothetical distinction of pattern types:

$$T1^L \wedge T1^R \wedge T2^L \wedge T2^R \bigwedge$$

$$\bigwedge \ (T3^L \wedge T4^L) \wedge \ (T3^R \wedge T4^R) \quad \text{pattern on both sides,}$$

$$\bigwedge \neg(T3^L \wedge T4^L) \wedge \ (T3^R \wedge T4^R) \quad \text{pattern on the right side,}$$

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Verifiable by numerical simulations of evolution problem with Schnakenberg's kinetics

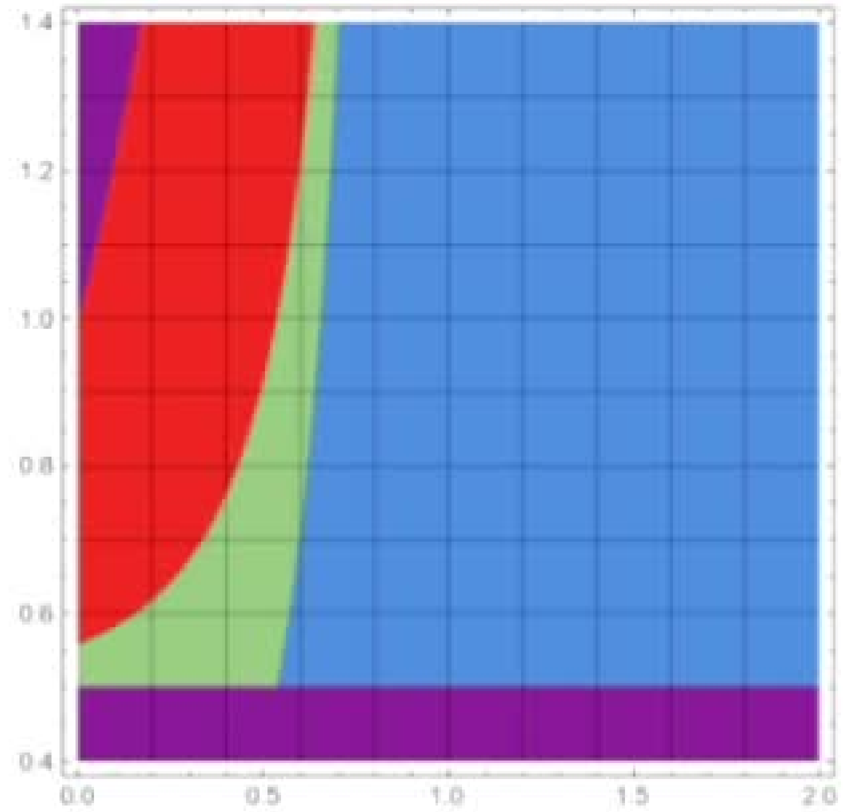
$$f(u,v) = a - u + u^2 v \qquad g(u,v) = b - u^2 v.$$

and Gierer-Meinhardt's kinetics

$$f(u,v) = a - bu + \frac{u^2}{v} \quad g(u,v) = u^2 - v,$$

with a, b positive constants.

Gierer-Meinhardt's kinetics with s=0.5



$L = 400,\ \xi = 120,$
$d_1 = 1,\ d_2 = 100,$
$s = 0.5,$

axis: (a, b),

both-side pattern
right-side pattern
left-side pattern
no pattern
unknown

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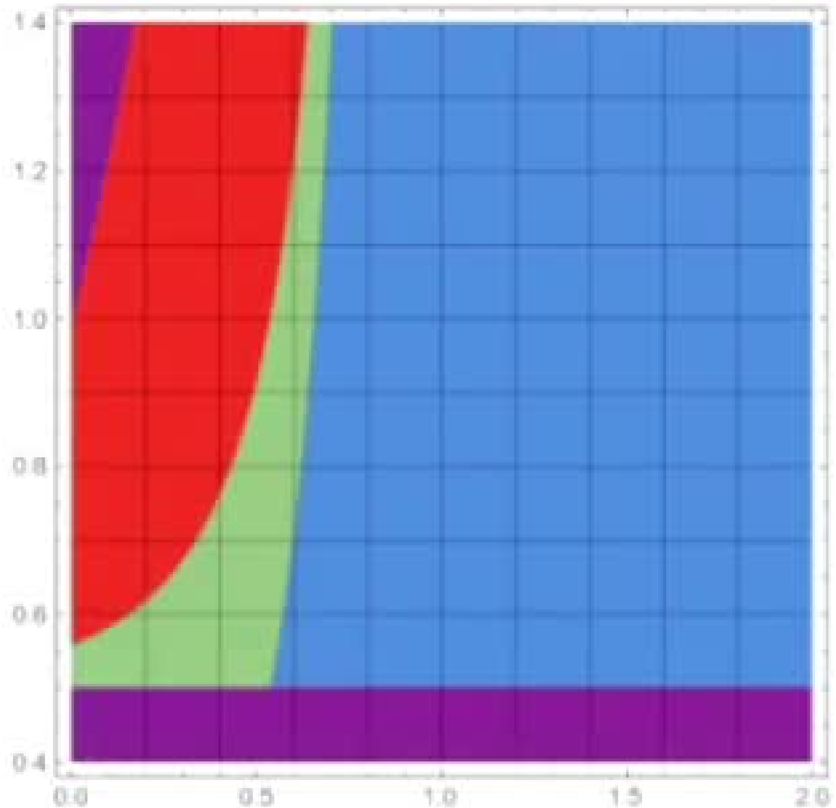

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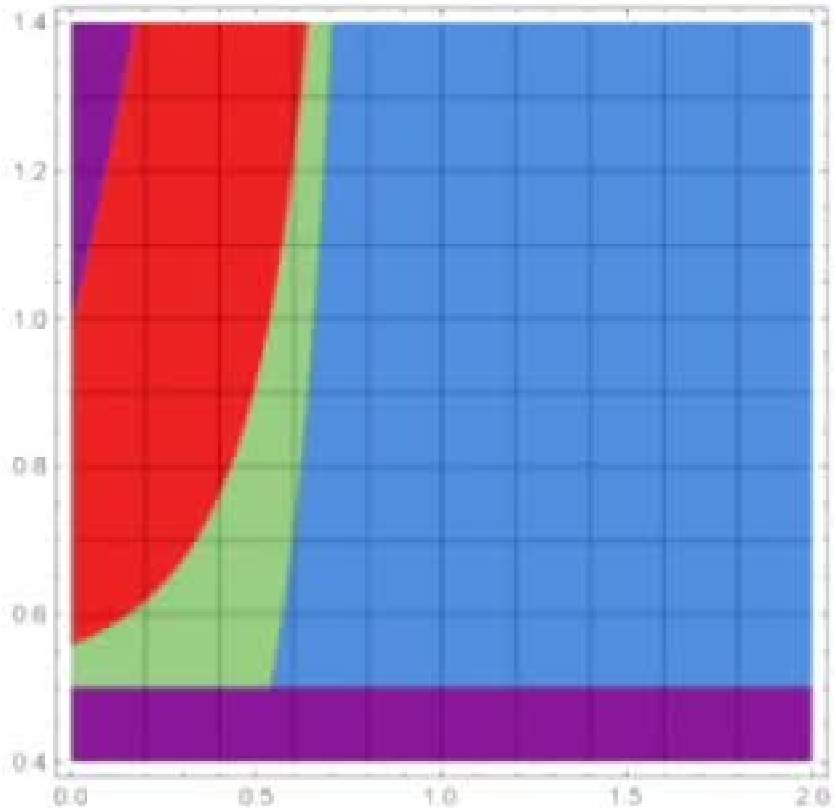

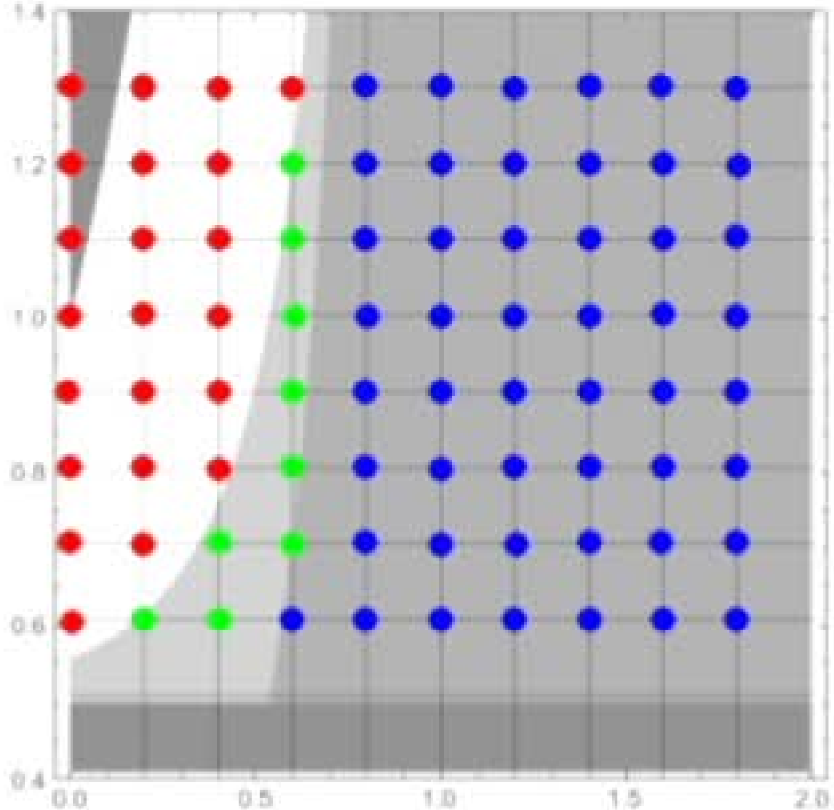

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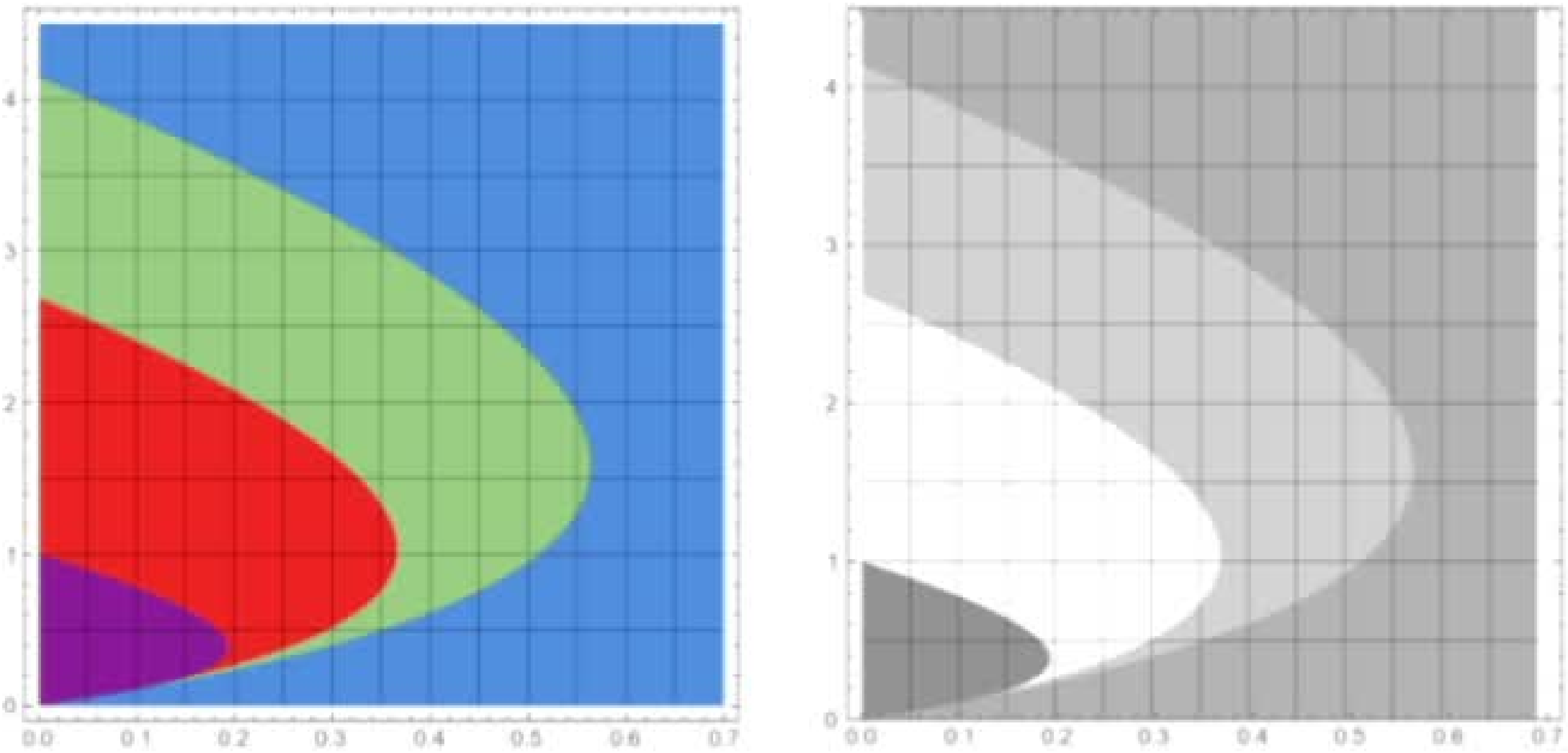
Schnakenberg's kinetics with s=0.25 and s=-0.25

$s = 0.25$



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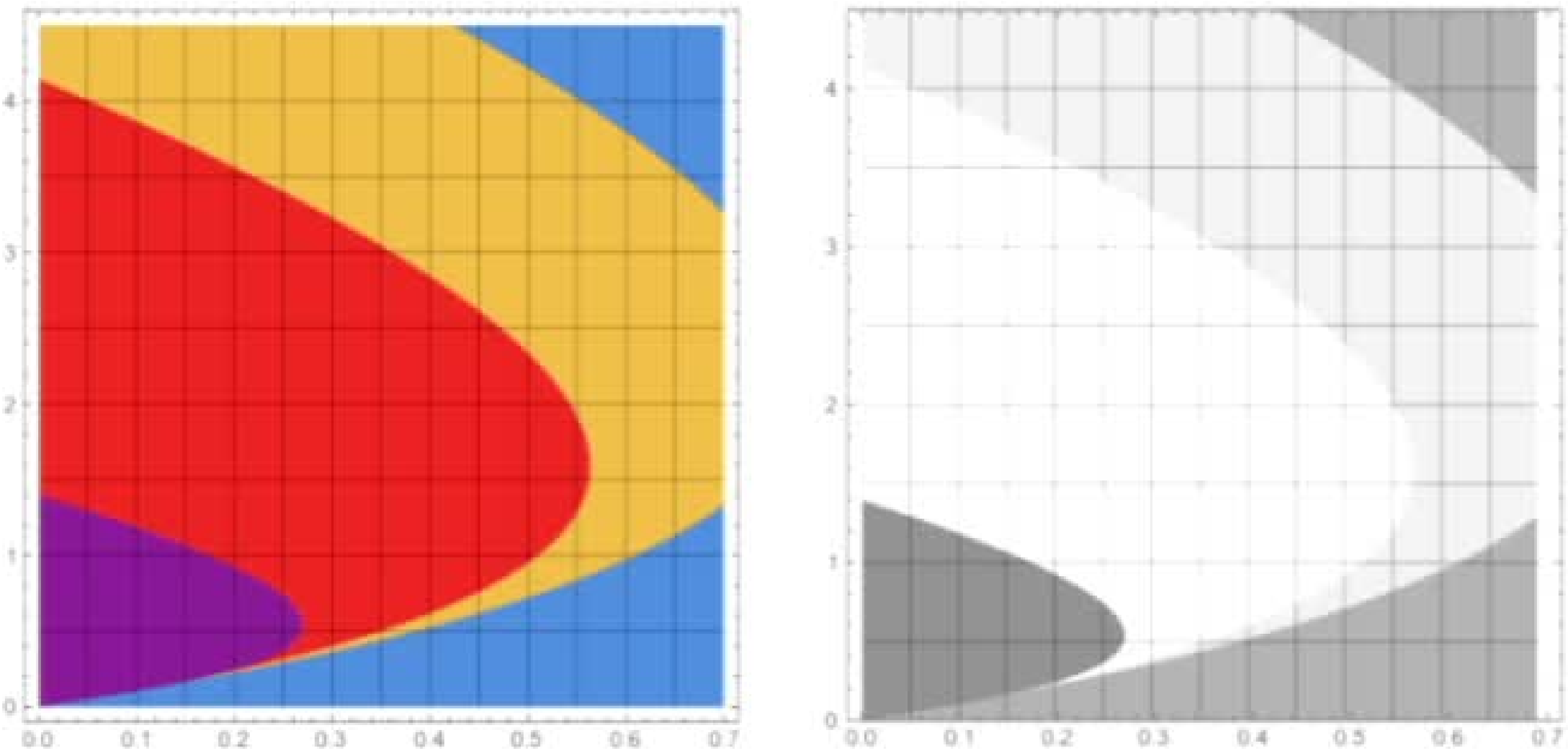

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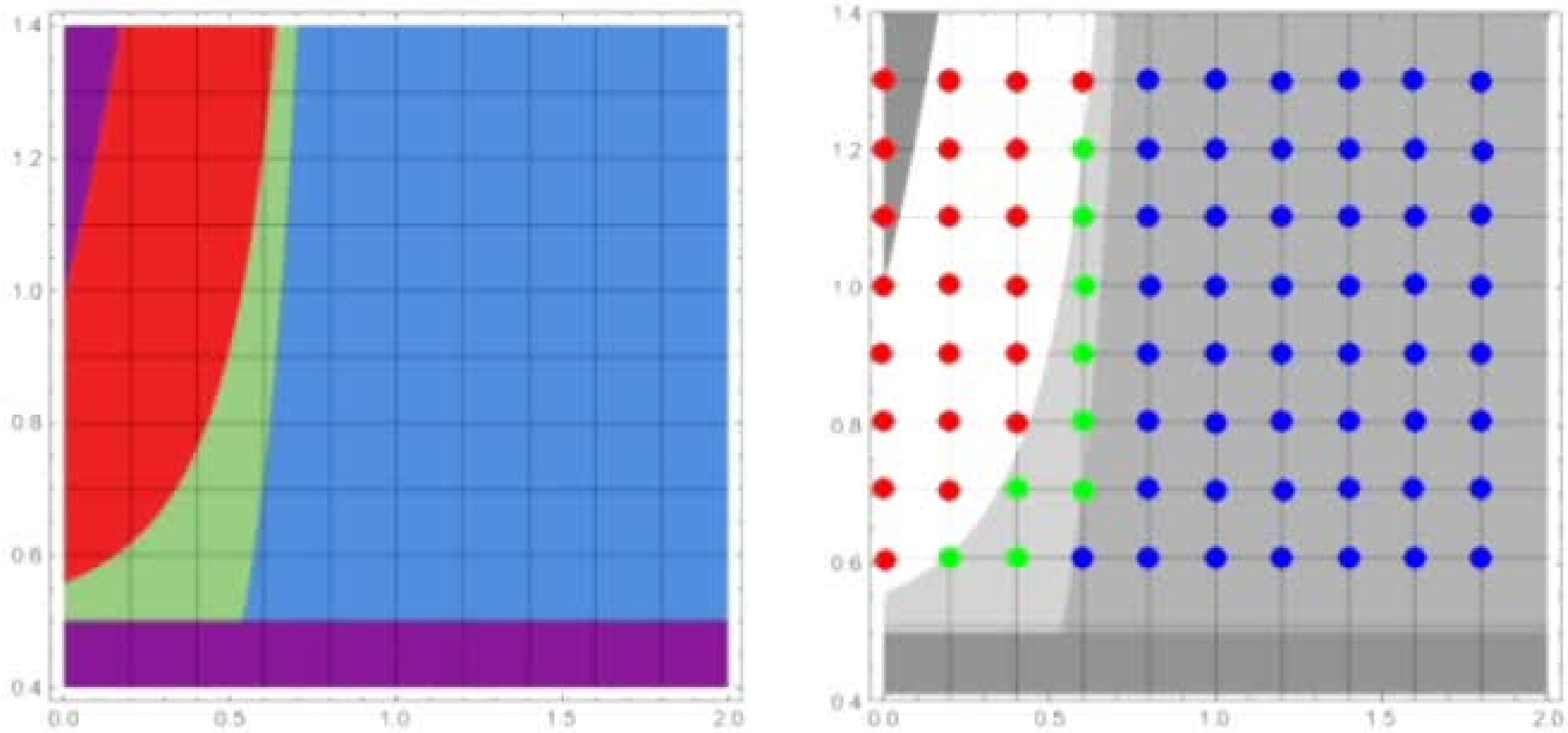

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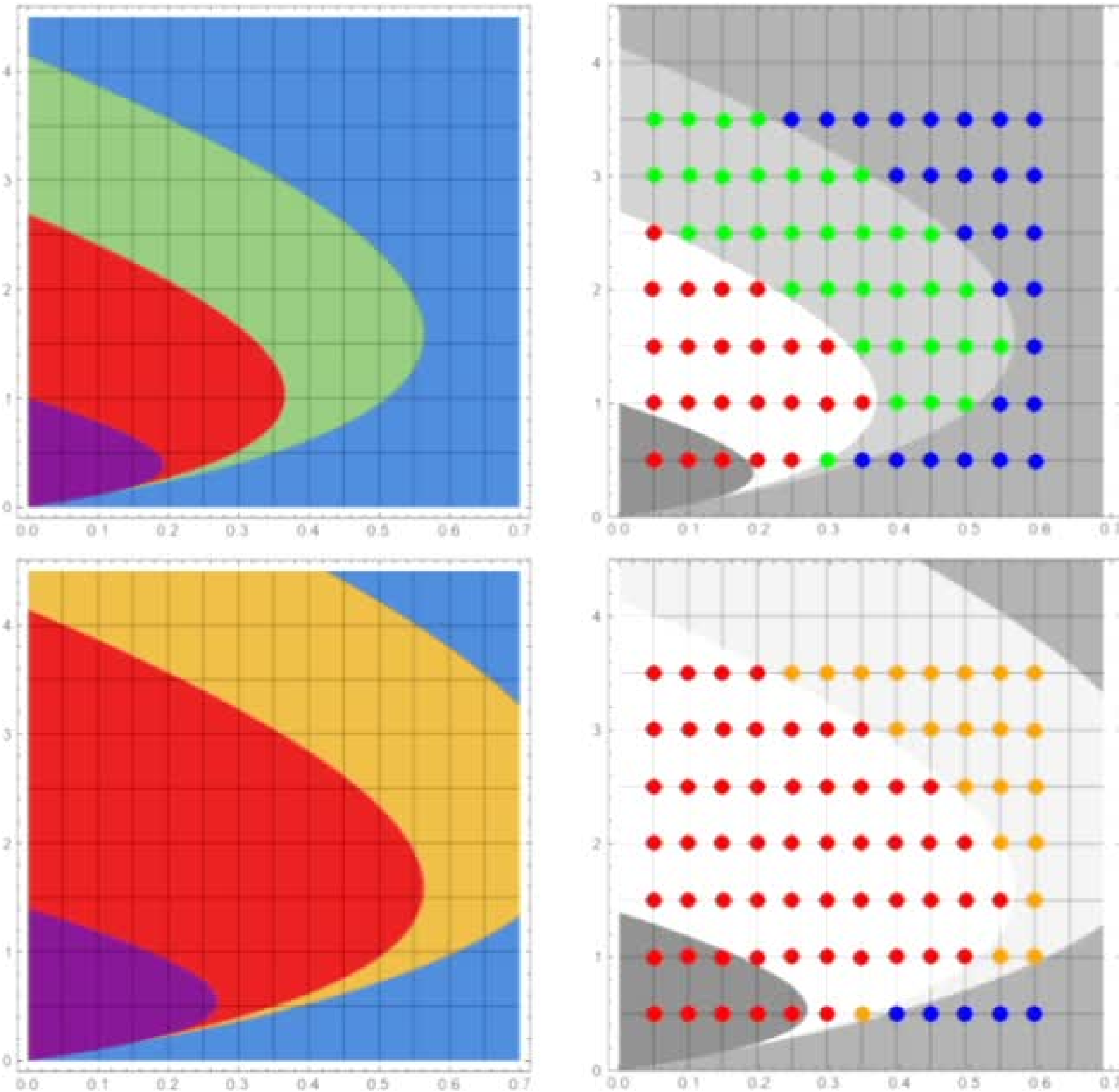

Outline and conclussions

Summary

- effect of small spatial dependence of coefficient was analysed,
- pattern with different frequencies emerges,
- Turing's idea was extended to this case,
- conditions to distinguish patterns in general case was stated,
- and verified by an analytical-numerical approach.

Remarks

- positive: helpful conditions,
- negative: accuracy of the conditions; only analytical approach?
- should work for any linear coefficient,
- should work for N steps.

Thank you for your attention.

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