The probabilistic convolution regularization of Zeno hybrid systems

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Outline

- Introduction.
- Hybrid Automaton.
- Zeno Phenomenon.
- Regularization By Convolution Of Zeno Hybrid Automata.
- Conclusion.

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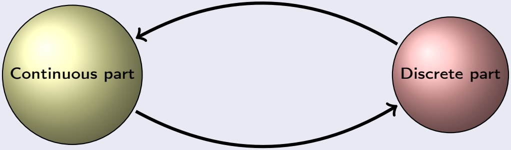
Hybrid Dynamical System (HDS)

- It contains both continuous and discrete state variables.



Hybrid Dynamical System (HDS).

- The continuous part is the physical environment in which the system evolves. The mathematical representation of the evolution of this part is mostly using ordinary differential equations (ODE).
- The discrete part is the program that controls its environment. The evolution of this part is mostly represented with an automaton (by discrete state and **instantaneous changes**).

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Hybrid Dynamical System (HDS)

Definition

A hybrid automaton is a dynamical system:

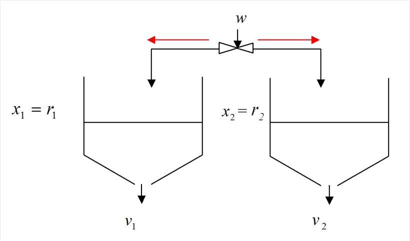
$$\mathcal{H} = \{Q, X, E, D, F, Init, G, R\}$$

where

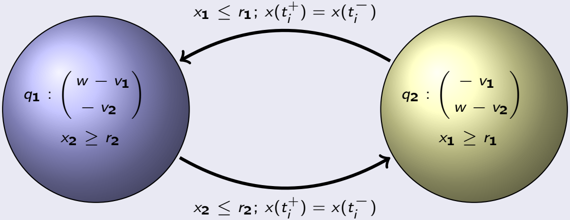
- $Q = \{q_1, \cdots, q_N\}$ is a finite set of discrete states;
- X is a finite set of continuous variables;
- $E \subseteq Q \times Q$ is a set of transitions;
- $D = \{D_q, q \in Q\}$ is the collection of domains;
- $F = \{f_q, q \in Q\}$ is the collection of vectors fields; to each discrete state is associated a dynamic system $\dot{x} = f_q(x)\ \forall\ q \in Q, f_q : D_q \to \mathbb{R}$;
- $Init \subseteq Q \times X$ is the set of initial states;
- $G = \{G_e, e \in E\}$ is the collection of the guards, $\forall e = (q, q') \in E, G_e \cap D_q \neq \emptyset$;
- $R = \{R_e, e \in E\}$ is the collection of reset functions.

We refer to $(q; x) \in Q \times X$ as a state of $\mathcal{H}$. We assume for all $e \in E, x \in G(e)$, that $R_e(x) \neq \emptyset$.

Water tank of Alur and Henzinger [Springer, 1997]



The water tank example.



Hybrid automaton of the water tank system.

Zeno execution

An execution is Zeno if the system admits an infinite number of discrete transitions in a finite time interval *i.e.*, $\sum_{i=0}^{N=\infty}(t_{i+1} - t_i) < \infty$. In this case, the time $\tau_\infty = \sum_{i=0}^{\infty}(t_{i+1} - t_i)$ is called the Zeno time.

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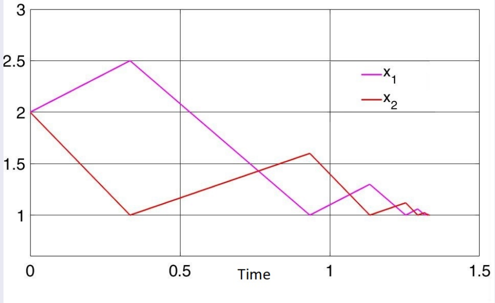
- Hybrid automaton of the water tank system can exhibit Zeno behavior.
- If $\max\{v_{\mathbf{1}}, v_{\mathbf{2}}\} < w < v_{\mathbf{1}} + v_{\mathbf{2}}$, the system has increasingly fast switches between $q_{\mathbf{1}}$ and $q_{\mathbf{2}}$ (the infinite number of switchings accumulate in finite time)
- For a simulated trajectory of this system with the conditions: $v_{\mathbf{1}} = 2.5$, $v_{\mathbf{2}} = 3$, $r_{\mathbf{1}} = r_{\mathbf{2}} = 1$, $w = 4$, $x_{\mathbf{01}} = x_{\mathbf{02}} = 2$, we can check that $\tau_{\infty} = \sum_{i=\mathbf{0}}^{\infty}(t_{i+\mathbf{1}} - t_i) = \frac{x_{\mathbf{1}}(t_{\mathbf{0}})+x_{\mathbf{2}}(t_{\mathbf{0}})-r_{\mathbf{1}}-r_{\mathbf{2}}}{v_{\mathbf{1}}+v_{\mathbf{2}}-w} = 1.33$.



Zeno behavior of the water tank system

- Simulation of such systems will generally stop or give false results after the Zeno time.
- Physical systems do not exhibit such behavior, because the switching is not in reality instantaneous, but also often because they are subject to a small amount of noise.
- The problem is how to predict the behavior of the system after the Zeno time.
- We propose in the following a new method, called the method of regularization by convolution.

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Definition

- A deterministic HDS in a discrete state, $q_i \in Q$, can (with some probability) either stay to evolve in q_i with respect to the domain of evolution $x \in D_{q_i}$, or switch to a deterministic discrete state $q_{i+\mathbf{1}} \in Q$ with respect to the domain of transition $x \in G_{(q_i, q_{i+\mathbf{1}})}$.

- Noise is present in the system because approaching the domain of transition the system (a sensor, for example) does not know with certainty whether or not $x(t) \in G_{(q_i, q_{i+\mathbf{1}})}$.
- We define the regularized vector field by convolution in each discrete state q_i by

$$\dot{x}(t) = f * \varphi_\lambda(x) = \int_{\mathbb{R}^n} f(s)\varphi_\lambda(x - s)ds$$

where, $\{\varphi_\lambda : \lambda > 0\}$ is a family of probability laws on $\mathbb{R}^n$ defined by a differentiable density, satisfying the conditions:

$$\int_{\mathbb{R}^n} x\varphi_\lambda(x)dx = 0, \text{ and}$$
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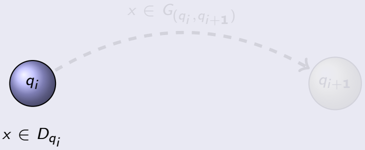

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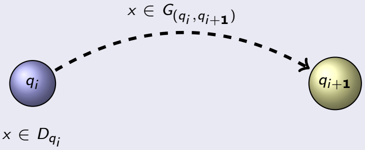

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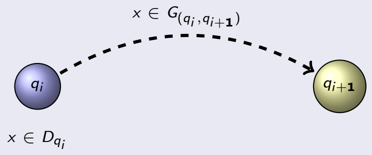

- $$\dot{x}(t) = \int_{s\in D_{q_i}} f_{q_i}(s)\varphi_\lambda(x-s)ds + \int_{s\in G_{(q_i,q_{i+\mathbf{1}})}} f_{q_{i+\mathbf{1}}}(s)\varphi_\lambda(x-s)ds.$$
- If the vector fields are piecewise constant

$$\dot{x}(t) = P_{q_i}(x \in D_{q_i})f_{q_i} + P_{q_i}(x \in G_{(q_i,q_{i+\mathbf{1}})})f_{q_{i+\mathbf{1}}}$$

with

$$P_{q_i}(x \in D_{q_i}) = \int_{s\in D_{q_i}} \varphi_\lambda(x-s)ds$$

$$P_{q_i}(x \in G_{(q_i,q_{i+\mathbf{1}})}) = \int_{s\in G_{(q_i,q_{i+\mathbf{1}})}} \varphi_\lambda(x-s)ds.$$

- If $G_{(q_i,q_{i+\mathbf{1}})} \cap D_{q_i}$ has zero area ($\int_{s\in(G_{(q_i,q_{i+\mathbf{1}})}\cap D_{q_i})} \varphi_\lambda(x-s)ds = 0$), then

$$P_{q_i}(x \in G_{(q_i,q_{i+\mathbf{1}})}) = 1 - P_{q_i}(x \in D_{q_i})$$

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Regularization By Convolution Of Zeno Hybrid automaton of the water tank system

- We return to the water tank system (**the example given above**).
- We take a family of Gaussians of standard deviation λ. The joint probability density of two such random variables is given by

$$\varphi_\lambda(x_1, x_2) = \frac{1}{2\pi\lambda^2} \exp\left\{-\frac{1}{2}\left(\frac{x_1^2}{\lambda^2} + \frac{x_2^2}{\lambda^2}\right)\right\}.$$

- The regularized vector fields are

$$\begin{aligned} f^r_{q_1}(x) &= f * \varphi_\lambda(x(t)) \\ &= P^+_{q_1}(x_2 - r_2) f_{q_1}(x) + P^-_{q_1}(x_2 - r_2) f_{q_2}(x) \\ &= \frac{1}{2}\mathrm{erfc}\left(\frac{r_2 - x_2}{\lambda\sqrt{2}}\right) f_{q_1}(x) + \frac{1}{2}\mathrm{erfc}\left(\frac{x_2 - r_2}{\lambda\sqrt{2}}\right) f_{q_2}(x), \\ f^r_{q_2}(x) &= P^+_{q_2}(x_1 - r_1) f_{q_2}(x) + P^-_{q_2}(x_1 - r_1) f_{q_1}(x) \\ &= \frac{1}{2}\mathrm{erfc}\left(\frac{r_1 - x_1}{\lambda\sqrt{2}}\right) f_{q_2}(x) + \frac{1}{2}\mathrm{erfc}\left(\frac{x_1 - r_1}{\lambda\sqrt{2}}\right) f_{q_1}(x). \end{aligned}$$

where,

$$P^+_{q_1}(x_2 - r_2) = \iint_{s_2 > r_2} \varphi_\lambda(s_1 - x_1, s_2 - x_2) ds_1 ds_2,$$

$$P^+_{q_2}(x_1 - r_1) = \iint_{s_1 > r_1} \varphi_\lambda(s_1 - x_1, s_2 - x_2) ds_1 ds_2$$

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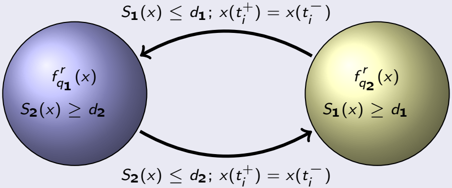
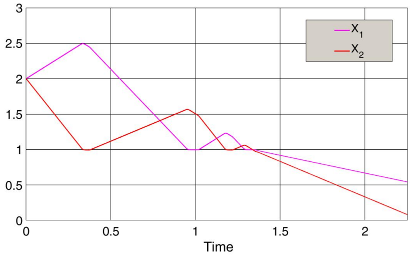
- The convolution regularization of the water tank automaton.



- $S_1(x) = x_1 - r_1,\ S_2(x) = x_2 - r_2$
- d_1 and d_2 are defined respectively such that $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$.

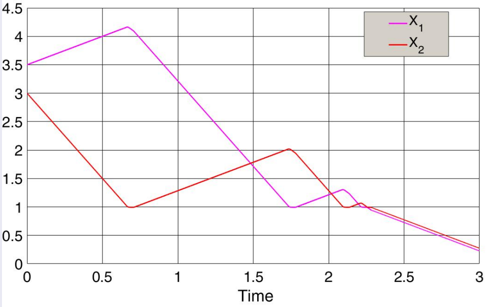
The regularized automaton of the water tank system

- Taking a small noise amplitude $\lambda = 0.01$ and the same values of parameters taken above ($v_1 = 2.5$, $v_2 = 3$, $r_1 = r_2 = 1$, $w = 4$). The result of simulation of the system after the convolution regularization and starting from the same initial conditions, $x_{01} = x_{02} = 2$, is given in the following:



Regularization by convolution of the water tank system with a small noise amplitude ($\lambda = 0.01$)

- The result of another simulation with $v_1 = v_2 = 3$, $w = 4$, $r_1 = r_2 = 1$, $x_{01} = 3.5$, $x_{02} = 3$, $\lambda = 0.01$, is given in the following



Regularization by convolution of the water tank system with a small noise amplitude ($\lambda = 0.01$) and with the following numerical values: $v_1 = v_2 = 3$, $w = 4$, $x_{01} = 3.5$, $x_{02} = 3$.

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- The behavior after the Zeno point depends on the noise amplitude ($\dot{x}$ is a function of λ). When $\lambda \to 0$, in the interior of the Zeno region ($x_1 < r_1$ and $x_2 < r_2$) we obtain a solution following the vector field

$$\dot{x} = \frac{1}{2}(f_{q_1}(x) + f_{q_2}(x)) = \begin{pmatrix} \frac{w}{2} - v_1 \\ \frac{w}{2} - v_2 \end{pmatrix}$$

- We show that our approach is equivalent to a kind of averaging in the Zeno region, the averaging of vector fields is

$$\begin{aligned} \dot{x}(t) &= \frac{1}{T}\sum_{i=1}^{2}\int_{t_i}^{t_{i+1}} f_{q_i}(x)dt \\ &= \frac{1}{\Delta t_1 + \Delta t_2}\left(\int_{\Delta t_1} f_{q_1}(x)dt + \int_{\Delta t_2} f_{q_2}(x)dt\right) \\ &= \frac{1}{\Delta t_1 + \Delta t_2}\left(\begin{pmatrix} w - v_1 \\ v_2 \end{pmatrix}\Delta t_1 + \begin{pmatrix} v_1 \\ w - v_2 \end{pmatrix}\Delta t_2\right). \end{aligned}$$

where, $T = \sum_{i=0}^{1}(t_{i+1} - t_i) = \sum_1^2 \Delta t_i$ is the period of the cycle.

- In the interior of the Zeno region the length of stay in each of the discrete states in the Zeno cycle is equal (corresponding to physical symmetry of the switching). Therefore, in the interior of the Zeno region we have $\Delta t_1 = \Delta t_2 = \Delta t$, and the averaging of vector fields becomes

$$f_{q_z}(x) = \begin{pmatrix} \frac{w}{2} - v_1 \\ \frac{w}{2} - v_2 \end{pmatrix} = \frac{1}{2}\left(f_{q_1}(x) + f_{q_2}(x)\right).$$

- This averaging after the Zeno point is valid only in the interior of the Zeno region ($x_{\mathbf{1}} < r_{\mathbf{1}}$ and $x_{\mathbf{2}} < r_{\mathbf{2}}$) or when the length of stay in each of the discrete states in the Zeno cycle is equal.
- When the system enters the Zeno region, it will be possible for the regularized system to return to the domain of a discrete state in the cycle.

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Conclusion

- Extension of the behavior of a Zeno hybrid system is possible by adding noise to the model.
- There are different ways of including noise. Here, we propose adding a noise term through a convolution, but a choice among different probability distributions and noise amplitudes still needs to be made.
- However, a useful idealization is obtained by taking the limit as the noise amplitude goes to zero.
- We have shown how to include the noise into HDSs to determine the probabilities of remaining in each domain of evolution or switching to another discrete state.
- Reasonable results can be expected using the probabilities of the system being in the evolution domain or the guard domain, without convolving with the vector fields. The noise is still incorporated into the probabilities, but the method is easier to use.
- We have also shown (elsewhere) how we can transform piecewise affine systems to stochastic piecewise affine systems.

Thank you for your attention

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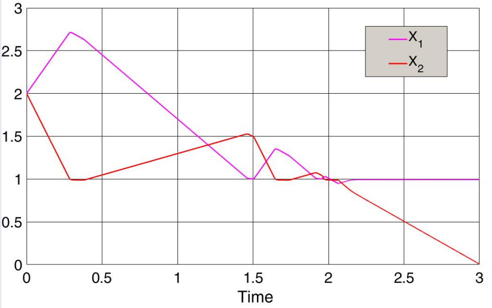
- The result of another simulation with $v_1 = 1.5$, $r_1 = r_2 = 1$, $v_2 = 3.5$, $w = 4$, $x_{01} = 2$, $x_{02} = 2$ is given in the following:



Regularization by convolution of the water tank system with a small noise amplitude ($\lambda = 0.01$), with the parameter values $v_1 = 1.5$, $r_1 = r_2 = 1$, $v_2 = 3.5$, $w = 4$, $x_{01} = 2$, $x_{02} = 2$.

- Because of the control, $\dot{x}_1$ cannot go below 0, so it must be kept at 0, and

$$\begin{aligned} \dot{x}_1 &= 0 \\ \dot{x}_2 &= (w - v_1) - v_2 . \end{aligned}$$

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