Multiscale Simulation of Porous Media Flow: Obstacles, Opportunities and Open-source

SIAG/GS Early Career Prize Lecture

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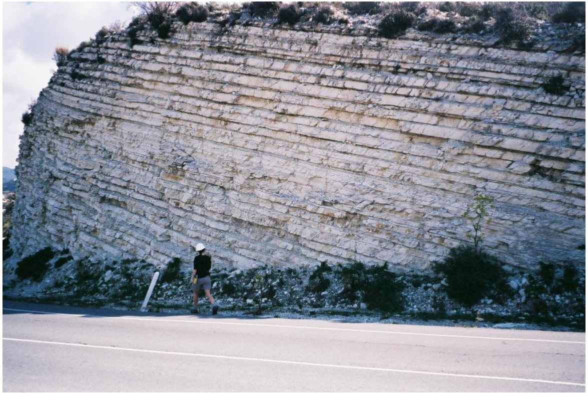
SIAM Geosciences, March 2019

Porous media flow – a multiscale problem



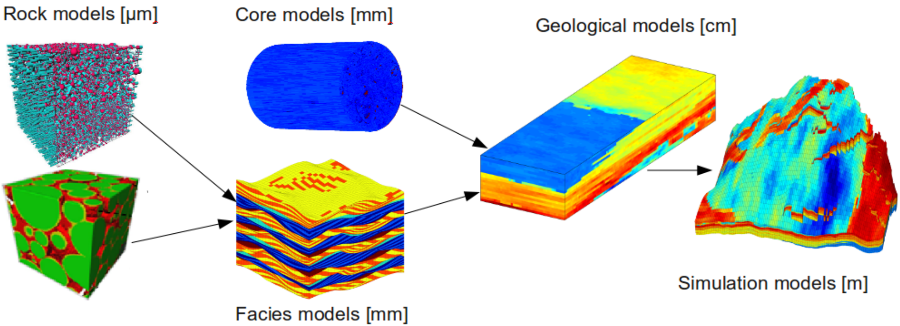
Layered geological structures typically occur on both large and small scales

Porous media flow – a multiscale problem

The scales that impact fluid flow in subsurface rocks range from

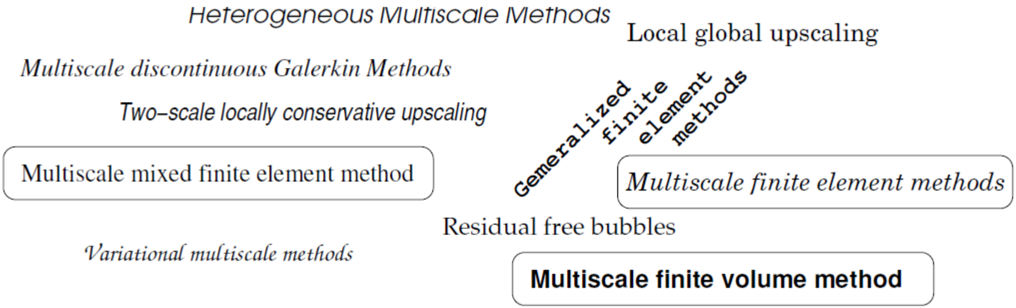
- the micrometer scale of pores and pore channels
- via dm-m scale of well bores and laminae sediments
- to sedimentary structures that stretch across entire reservoirs

Porous rocks are heterogeneous at all length scales (no scale separation)



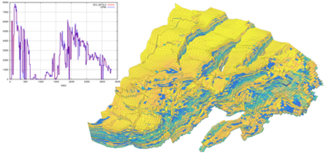
Multiscale methods

Numerical methods that attempt to model physical phenomena on coarse grids while honoring small-scale features in an appropriate way consistent with the local property of the differential operator



From Poisson's equation to reservoir simulation

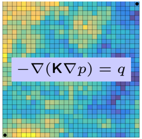
Flow physics



$$
\begin{aligned}
0 =& \partial_t(\phi b_o S_o) + \nabla \cdot (b_o \vec{v_o}) - b_o q_o \\
0 =& \partial_t(\phi b_w S_w) + \nabla \cdot (b_w \vec{v_w}) - b_w q_w \\
0 =& \partial_t \left[\phi(b_g S_b + b_o r_{so} S_o)\right] + \nabla \cdot (b_g \vec{v}_g) \\
& + \nabla \cdot (b_o r_{so} \vec{v_o}) - b_g q_g - b_o r_{so} q_o
\end{aligned}
$$

???

MsFV, MsMFE, 2003



$$-\nabla(\mathbf{K}\nabla p) = q$$

Geology

Multiscale finite-volume methods – status in 2013

Extensive research over the past 15 years – more than 60 papers by Jenny, Lee, Tchelepi, Lunati, Hajibeygi, and others:

- correction functions to handle non-elliptic features
- extension to compressible flow
- adaptivity in updating of basis functions
- iterative formulation with smoothers (Jacobi, GMRES, ...)
- algebraic formulation
- fracture models (embedded/hierarchical, etc)

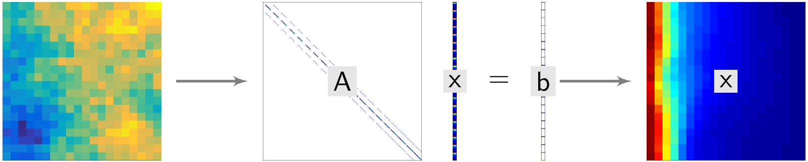
⋮

Strong focus on the ability to converge to a fine-scale solution has gradually made MsFV similar to multigrid methods

Multiscale finite-volume methods – the key concept

Fine-scale system

$$-\nabla \cdot \mathbf{K}\nabla p = q$$
$$Ax = b$$



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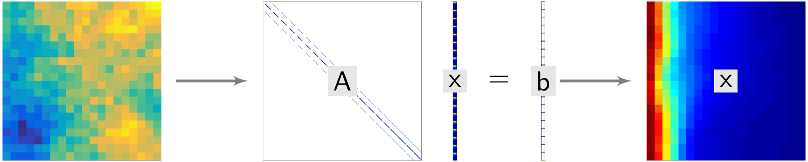
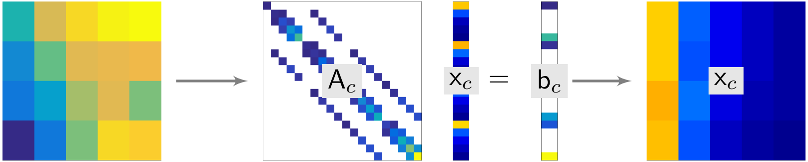


Upscaling

$$-\nabla \cdot \mathbf{K}_c\nabla p_c = q_c$$
$$A_c x_c = b_c$$



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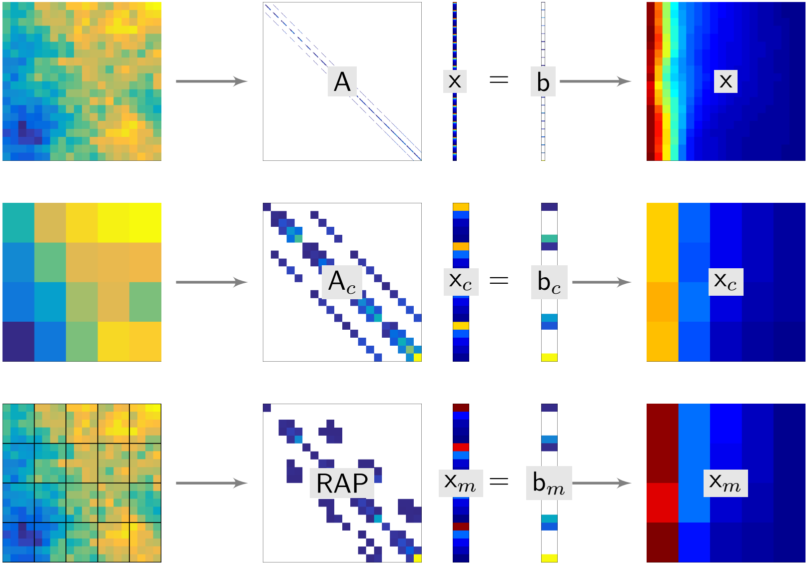
Multiscale method

$$P = \text{basis}(A)$$
$$x \approx P x_m$$

Restriction: R



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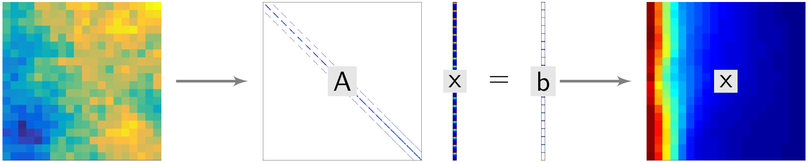

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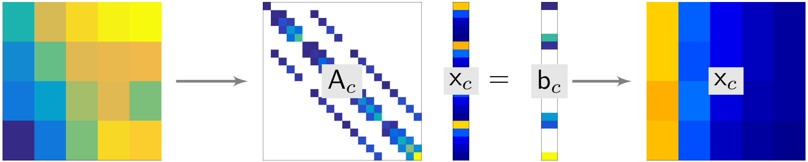

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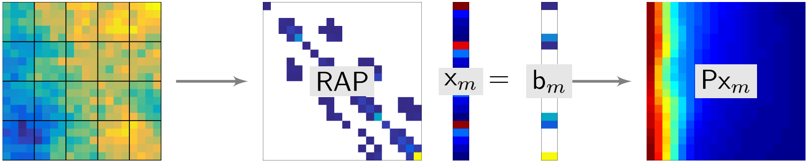
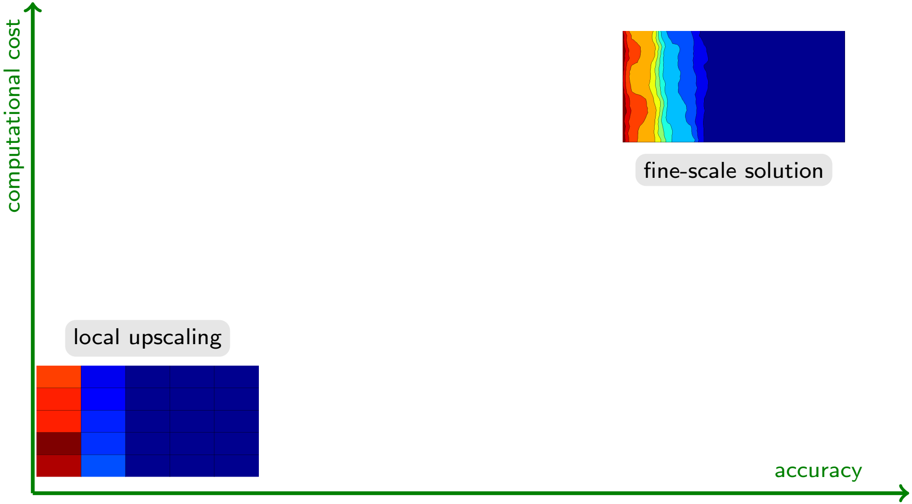


Qualitatively correct → small residual



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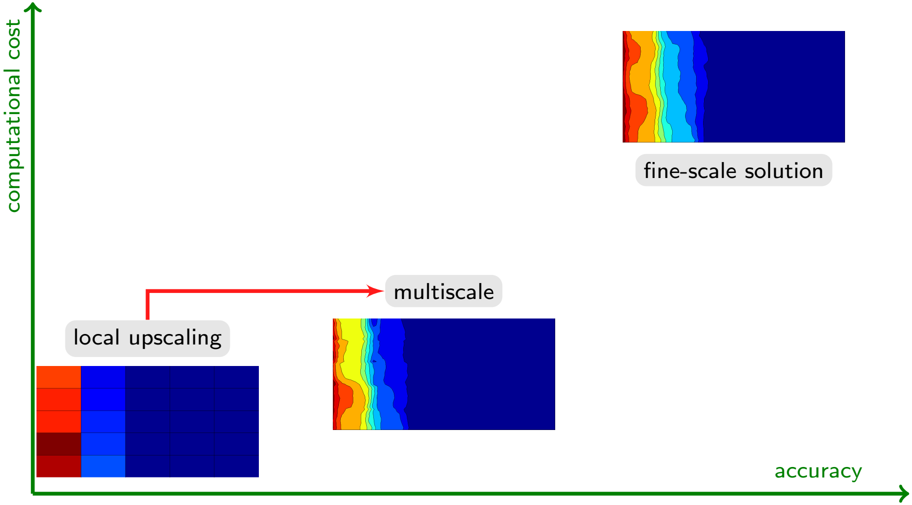
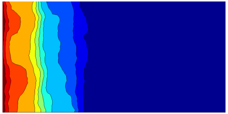


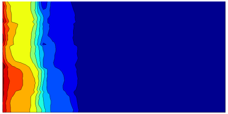
Qualitatively correct → small residual

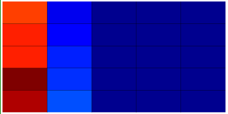
Residual iteration:

$$p^* = p^\nu + S(q - Ap^\nu)$$
$$p^{\nu+1} = p^* + A_{ms}^{-1}(q - Ap^*)$$

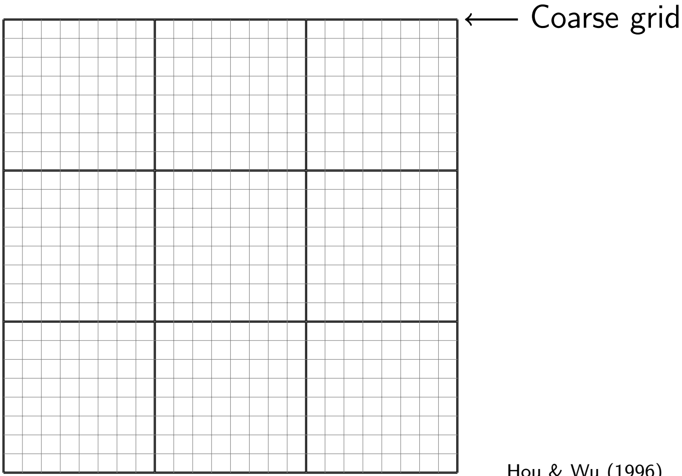
S is some inexpensive smoother.
At any point, solve local problems → locally conservative fine-scale velocity.
Hajibeygi, Jenny, Tchelepi, Wang, ... (2008–2015)







Multiscale basis functions by localization



Hou & Wu (1996)
Jenny, Lee, Tchelepi (2003)

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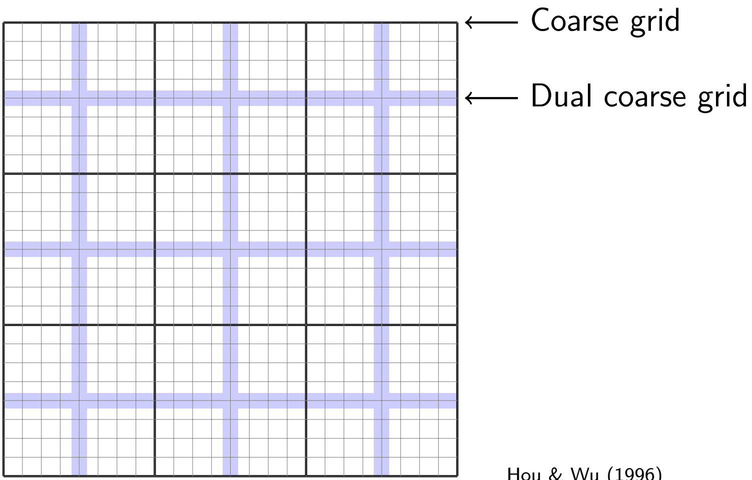

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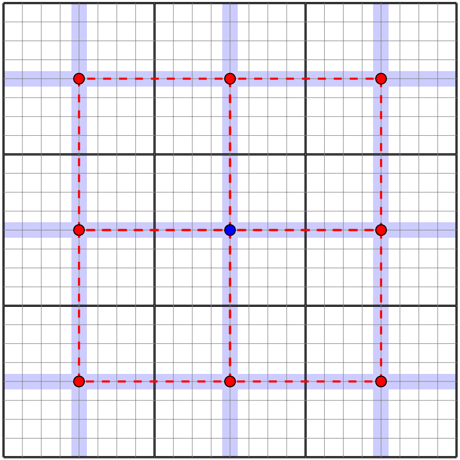


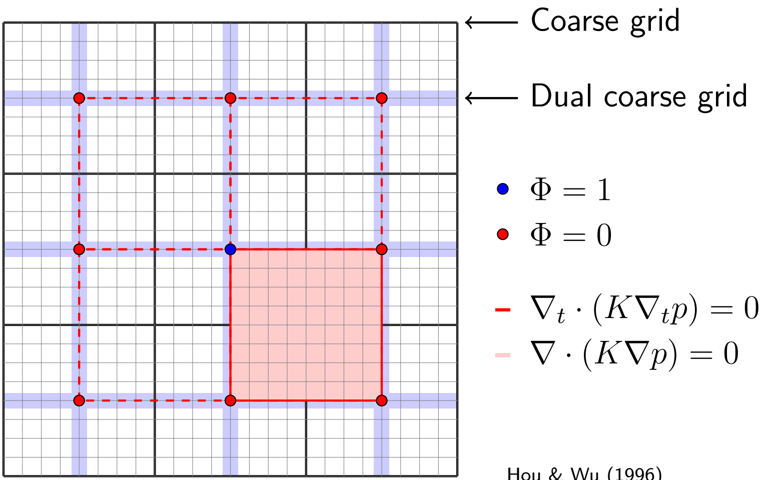
← Coarse grid

← Dual coarse grid

- $\Phi = 1$ (blue)
- $\Phi = 0$ (red)

Hou & Wu (1996)
Jenny, Lee, Tchelepi (2003)

Multiscale basis functions by localization



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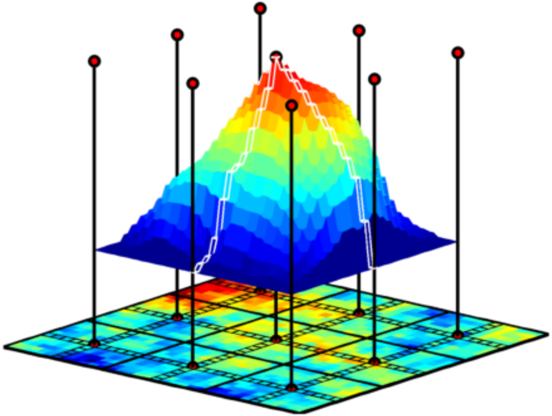
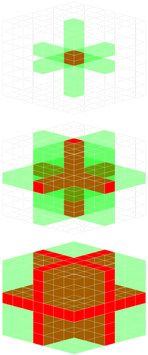
The multiscale finite-volume (MsFV) method



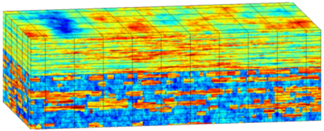


Operator formulation (Lunati & Lee, 2009)

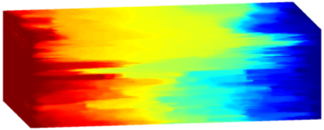
- *Localization assumption:* lower-dimensional local solution as approximation
- Generally: successive approximation 1D → 2D → 3D

The missing piece: industry-standard simulation models

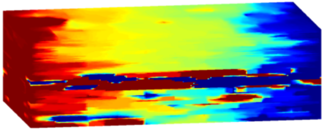
- MsFV only applied to Cartesian grids and idealized versions of real flow physics
- Localization $\rightarrow$ unstable multipoint coarse-scale stencil gives oscillatory solutions
- Requirement of consistent dual-primal partition makes coarsening difficult
- For industry use: needs automated and robust coarsening



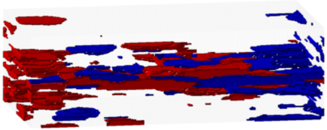
SPE 10: $\log(K)$



Reference solution

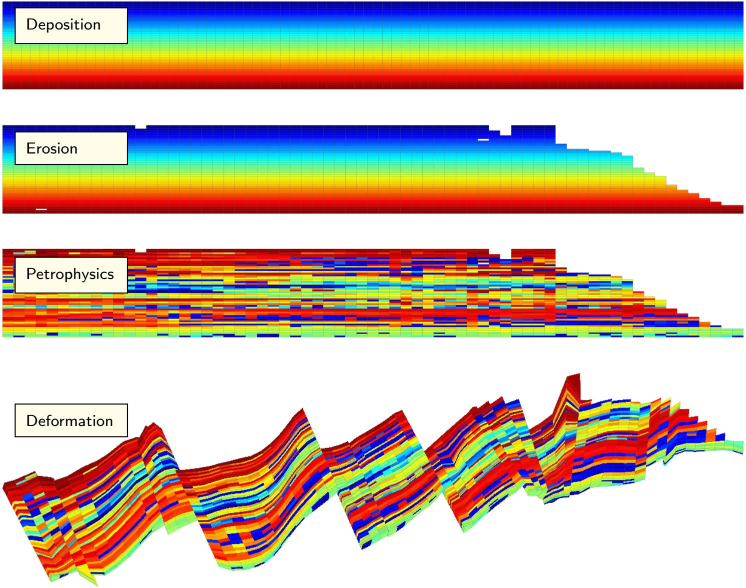


MsFV solution



MsFV $p \notin [0, 1]$

Grids: mimicking geological processes



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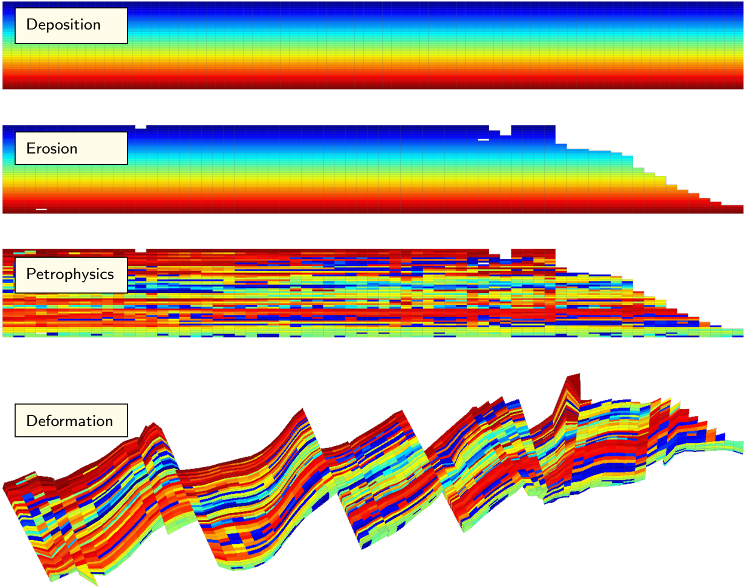

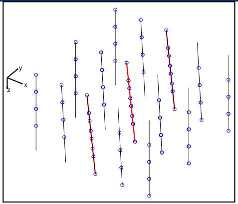

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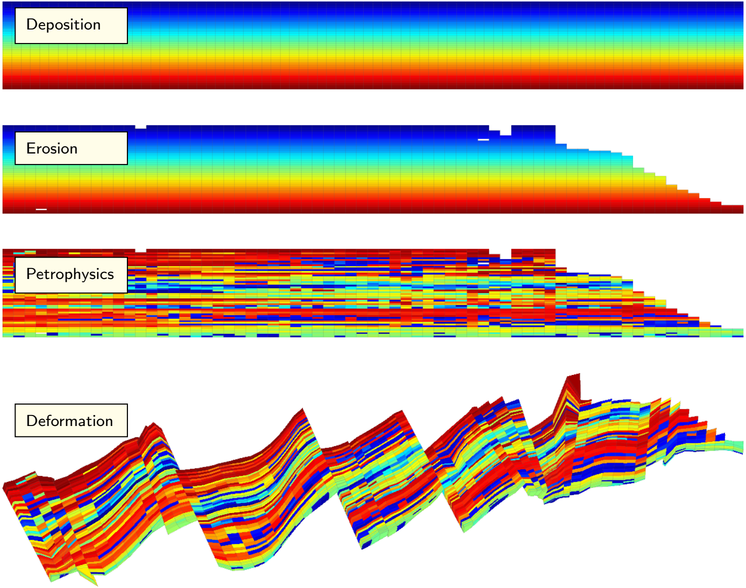

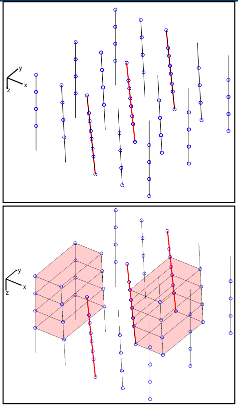

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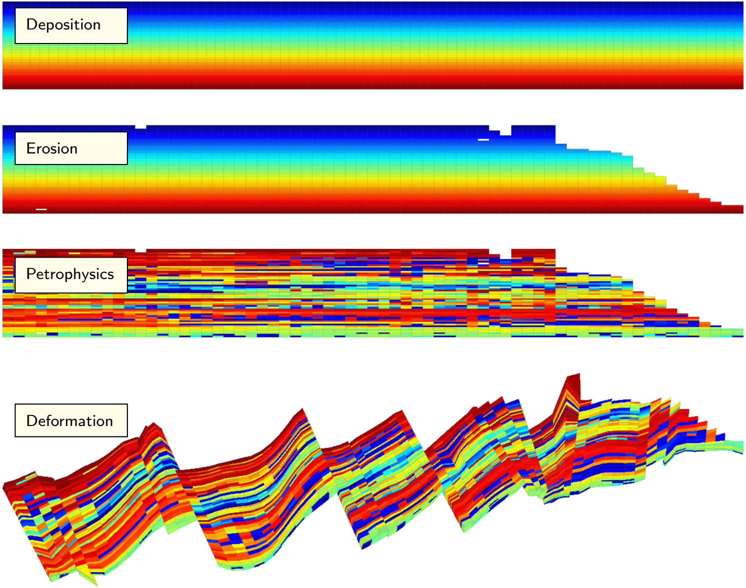
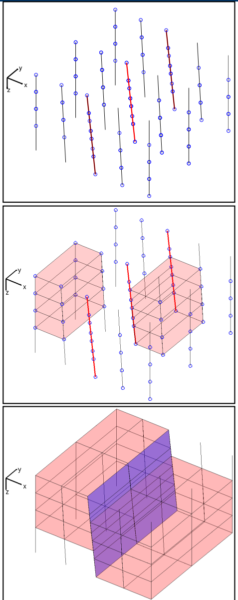
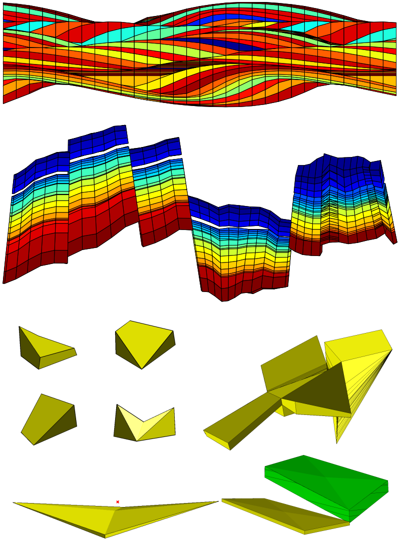




Why is this challenging?



Grid models represent the reservoir geology and are known to have many obscure features

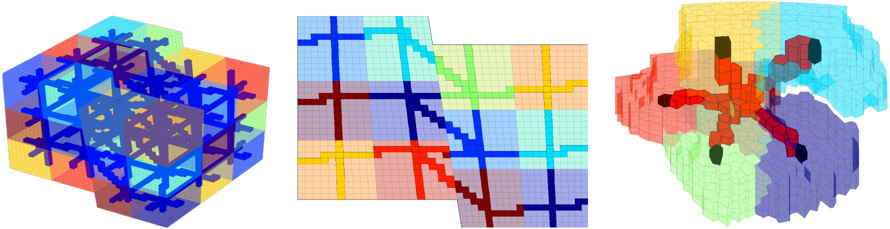
Challenges:

- Industry standard: corner-point / stratigraphic grids
- Grid topology is unstructured because of faults, pinch-outs, erosion, inactive cells, etc
- Geometry: deviates from box shape, high aspect ratios, many faces/neighbors, small faces, ...
- Petrophysics: orders of magnitude variations between neighboring cells

Coarse grids:

- *Will* be unstructured as a general rule
- Will have strange shapes, many special cases to be handled
- Should adapt to features in the reservoir model: petrophysical properties, faults, flow units, flow directions, wells, ...

First contribution: MsFV for unstructured grids

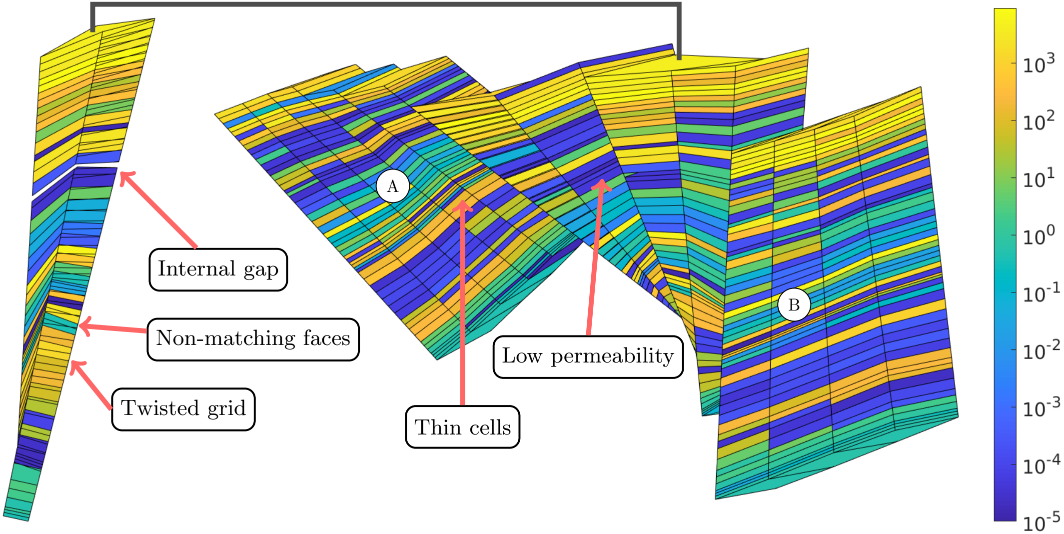


Algorithm for generating admissible primal–dual partitions on general grids

- geometrical partitioning using triangulation
- automated on rectilinear, curvilinear, triangular, and Voronoi grids
- semi-automated on corner-point grids and grids with non-matching faces
- monotonicity issues for heterogeneous permeability

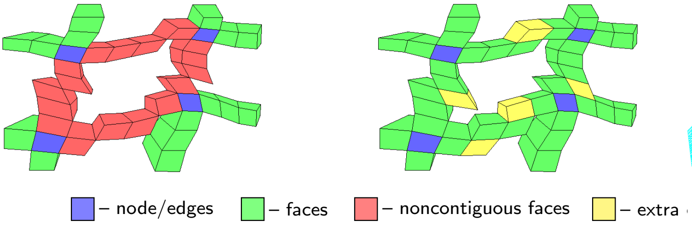
Møyner & Lie: The multiscale finite-volume method on stratigraphic grids, SPE J. (2013)

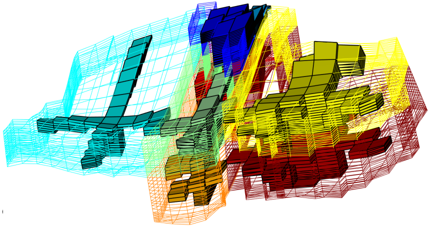
Localization assumption – difficult to impose



Challenge: How to solve *representative* local problem from A to B?

Difficulties – thick edges

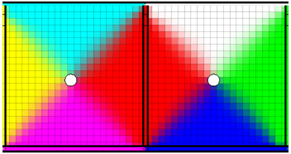




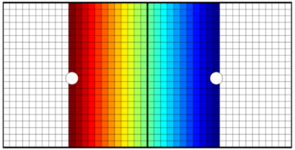
- Edge cells must be connected through faces to solve reduced flow problem between block centers ⟶ thick and interwoven edges, ill suited for numerics
- Difficult to do either graph or geometric algorithms

Difficult to automate partitioning for complex grids

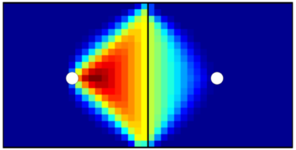
Rethinking the prolongation operator: First attempt



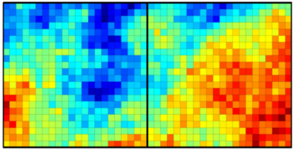
partition of unity function



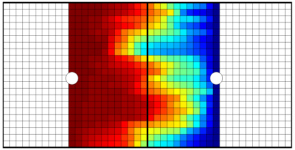
solution



part of basis function



permeability field



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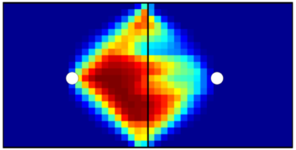

part of basis function

- Simultaneously worked on flow-diagnostics for rapid evaluation of models
- Combine partition-of-unity tracer solutions with local upscaling for basis
- Trades some accuracy for robustness - and works on general grids
- In the end, too complicated! Back to the drawing board...

Møyner & Lie: A multiscale two-point flux-approximation method, J. Comput. Phys (2014)
Møyner, Krogstad & Lie: The application of flow diagnostics for reservoir management, SPE J. (2014)

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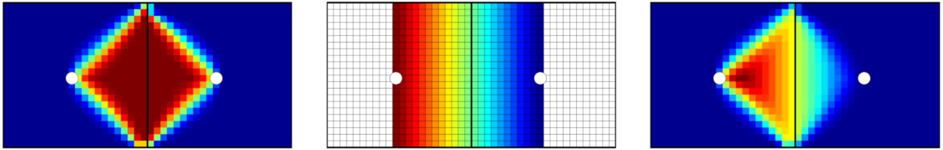
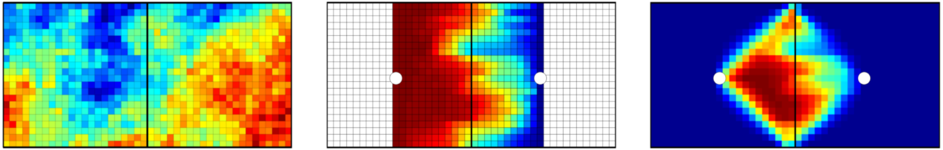


partition of unity function | solution | part of basis function



permeability field | solution | part of basis function

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Rethinking the prolongation operator: Another attempt

What are our requirements on the prolongation operator P?

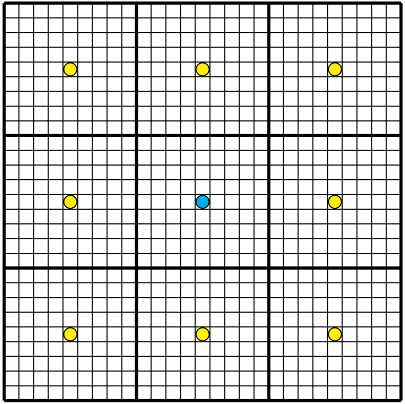
- Partition of unity: represent constant fields

  $$\sum_{j=1}^{N} P_{ij} = 1 \longrightarrow \text{Exact interpolation of constant modes}$$

- Algebraically smooth: minimize $\|AP\|_1$ implies that $APp_c \approx Ap$ locally
- Localization: coarse system $A_c = RAP$ becomes denser as the support of basis functions grows

Approach inspired by smoothed aggregation multigrid (Vaněk et al, 1996)

MsRSB: multiscale restriction-smoothed basis



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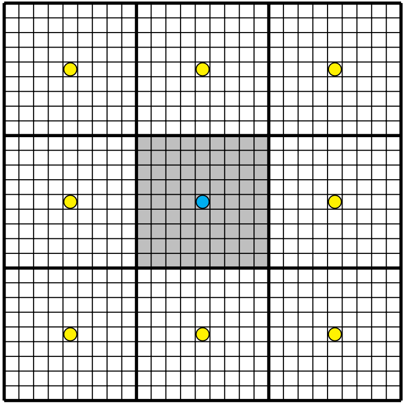
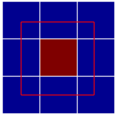


Set P_j to one inside block j



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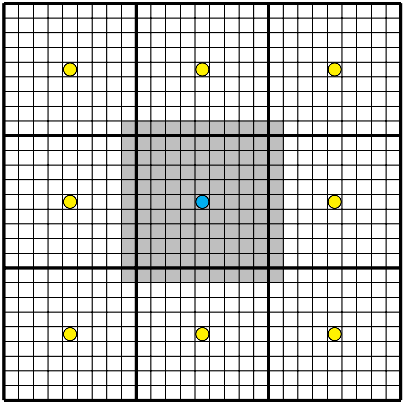
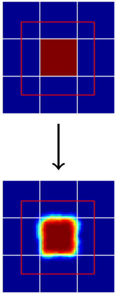


Set P_j to one inside block j

Jacobi increment: $d_j = -\omega D^{-1} A P_j^n$



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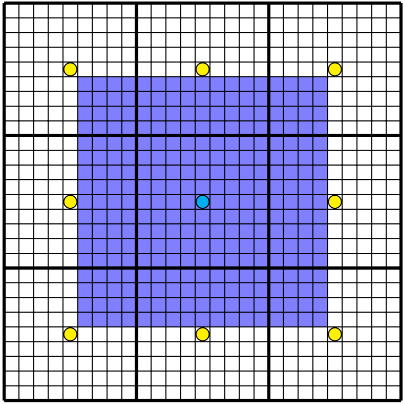


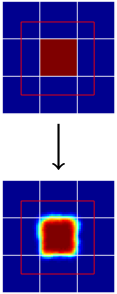
Set P_j to one inside block j

Jacobi increment: $d_j = -\omega D^{-1} A P_j^n$

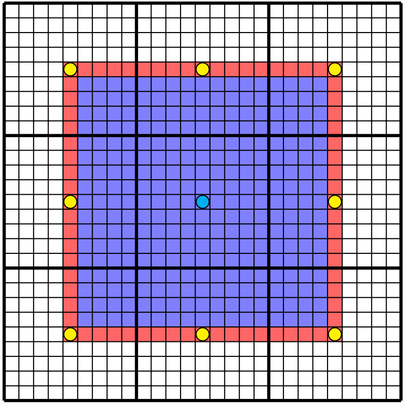
Localize update:

$$\hat{d}_{ij} = \begin{cases} d_{ij} \\ \\ \end{cases}$$

Indices: i = cell, j = ●, k = ●



MsRSB: multiscale restriction-smoothed basis



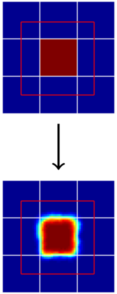
Set P_j to one inside block j

Jacobi increment: $d_j = -\omega D^{-1} A P_j^n$

Localize update:

$$\hat{d}_{ij} = \begin{cases} d_{ij} \\ 0 \end{cases}$$

Indices: i = cell, j = ●, k = ●



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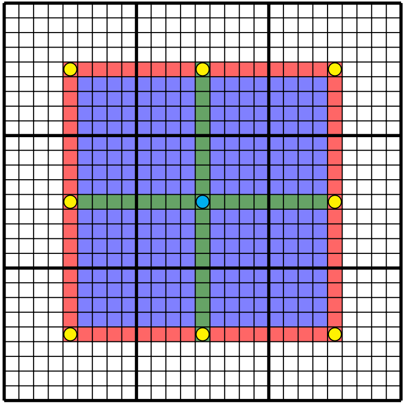


Support region: computed upfront by triangulating coarse centroids

Set P_j to one inside block j

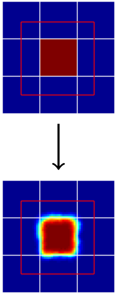
Jacobi increment: $d_j = -\omega D^{-1} A P_j^n$

Localize update:

$$\hat{d}_{ij} = \begin{cases} d_{ij} \\ 0 \\ \frac{d_{ij} - P_{ij}^n \sum_k d_{ik}}{1 + \sum_k d_{ik}} \end{cases}$$

Indices: $i =$ cell, $j =$ ●, $k =$ ●

Apply increment: $P_{ij}^{n+1} = P_{ij}^n + \hat{d}_{ij}$



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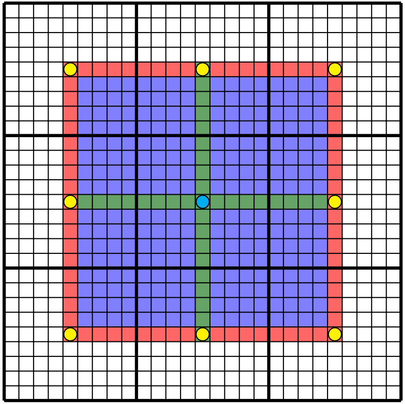

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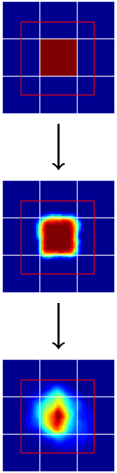
Indices: $i =$ cell, $j =$ ●, $k =$ ●

Apply increment: $P_{ij}^{n+1} = P_{ij}^n + \hat{d}_{ij}$

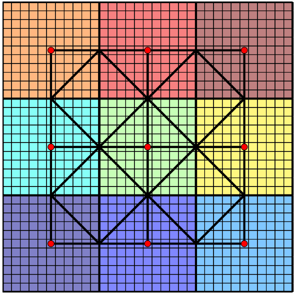
Repeat until convergence



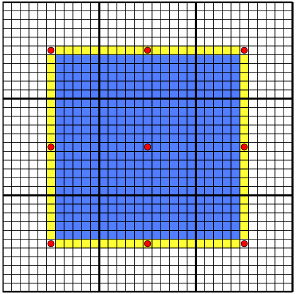
MsRSB: Support regions

Basis functions require **a coarse grid** and **a support region**

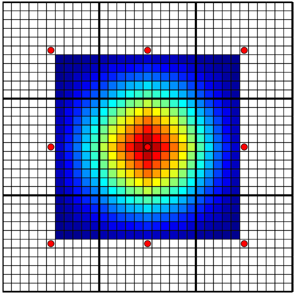
- Support region: logical indices, topological search, distance measures,..
- Region constructed using triangulation of nodal coarse neighbors, resulting in an MPFA stencil on the coarse scale
- Main point: Easy to implement in 3D for fully unstructured meshes



Coarse grid and local triangulation



Interaction region and boundary



Resulting basis function

MsRSB: Support regions

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- Support region: logical indices, topological search, distance measures,..
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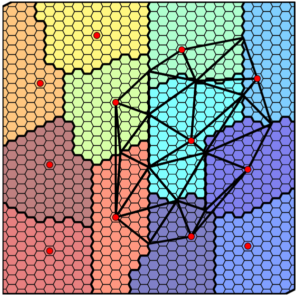
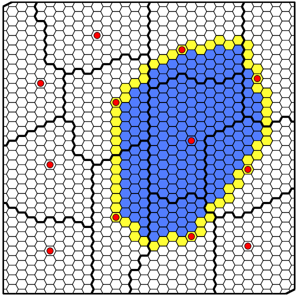
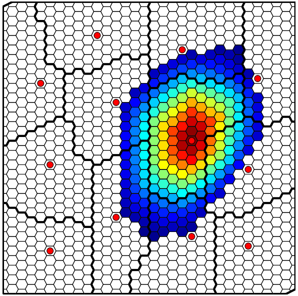


Coarse grid and local triangulation

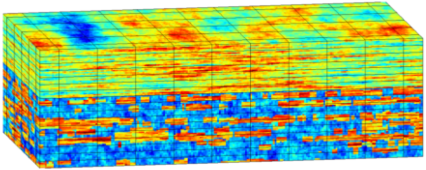


Interaction region and boundary

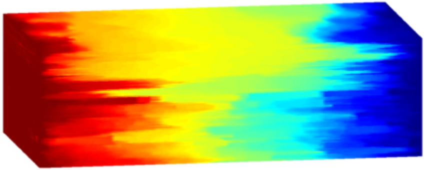


Resulting basis function

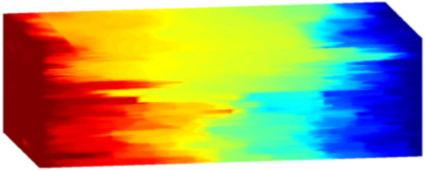
SPE10 – Full model 2



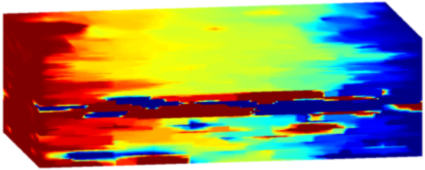
Horizontal permeability



Reference solution



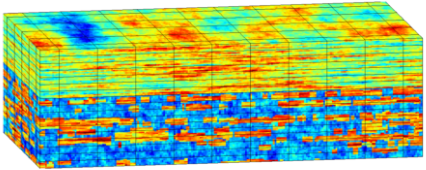
MsRSB



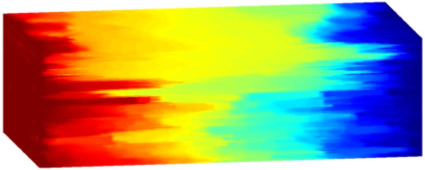
MsFV,

Error	Grid	p (L_2)	p (L_∞)	v (L_2)	v (L_∞)
MsFV	$6 \times 11 \times 17$	3.580	128.461	2.288	11.957
MsRSB	$6 \times 11 \times 17$	0.039	0.309	0.397	0.487

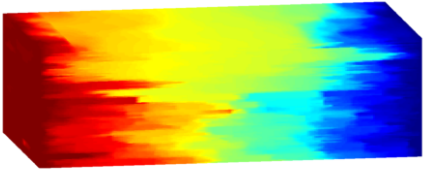
SPE10 – Full model 2



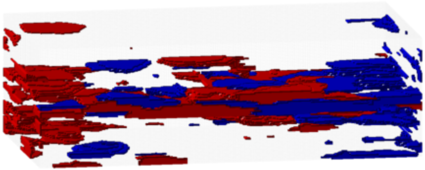
Horizontal permeability



Reference solution



MsRSB



MsFV, $p \notin [0, 1]$

Error	Grid	p (L_2)	p (L_∞)	v (L_2)	v (L_∞)
MsFV	$6 \times 11 \times 17$	3.580	128.461	2.288	11.957
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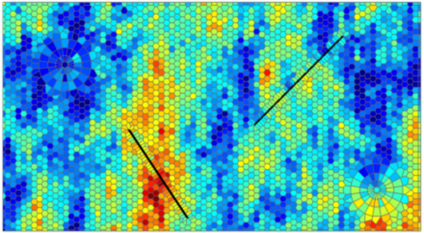
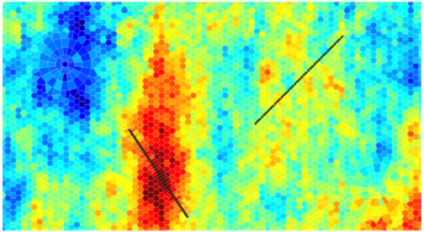
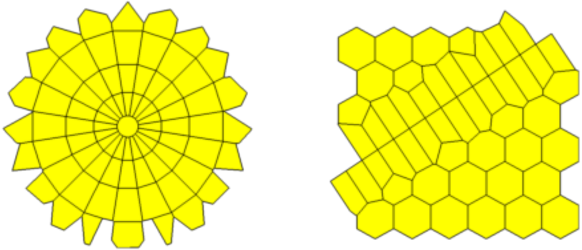
Example: unstructured PEBI grid



Porosity and grid



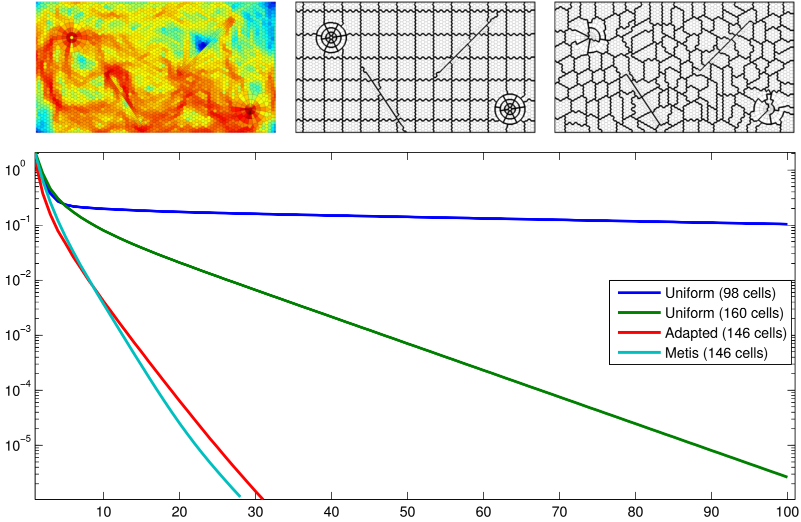
Permability from SPE 10, Layer 35



Detailed view of refinement

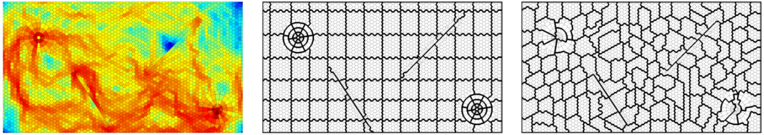
- Unstructured grid designed to minimize grid orientation effects
- Two embedded radial grids near wells
- Fine grid adapts to faults
- The faults are sealed, i.e. allow no fluid flow through

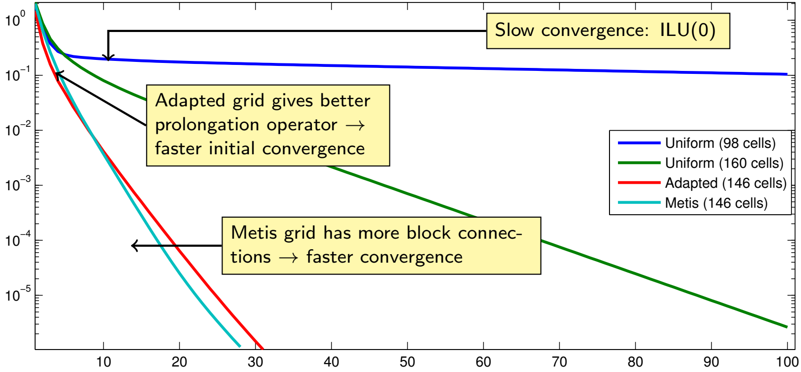
Example: unstructured PEBI grid



Two-step preconditioner, ILU(0) as 2nd stage, Richardson iterations

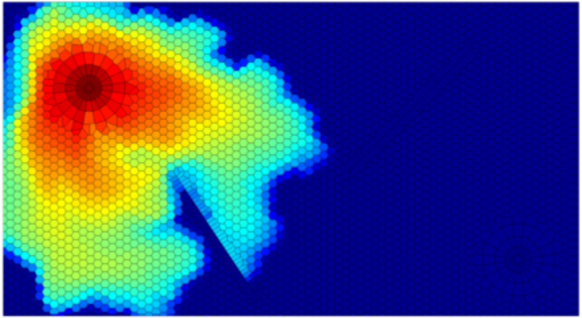
Example: unstructured PEBI grid



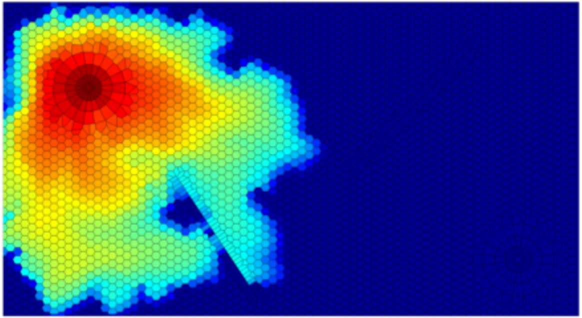


Two-step preconditioner, ILU(0) as 2nd stage, Richardson iterations

Two-phase flow, PEBI grid

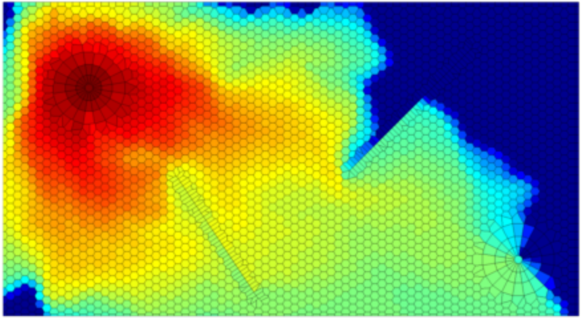


Fine-scale (around 6000 dof)



Multiscale (146 dof)

Two-phase flow, PEBI grid



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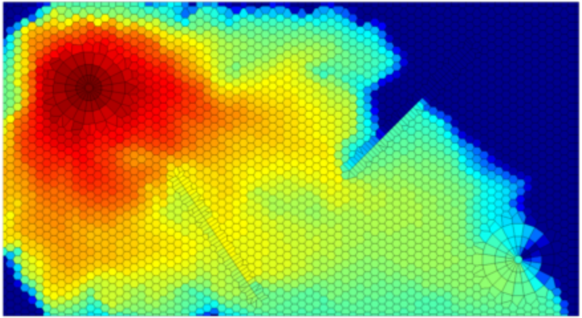
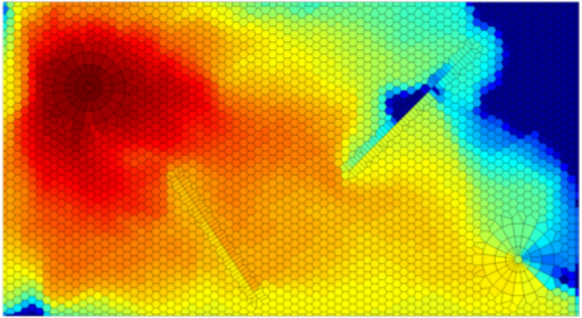
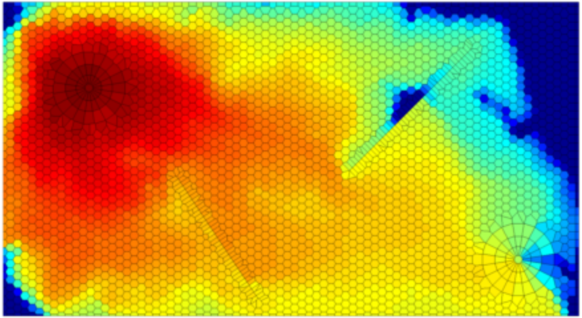


Multiscale (146 dof)

Two-phase flow, PEBI grid

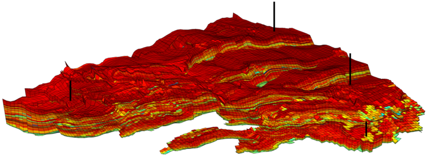


Fine-scale (around 6000 dof)

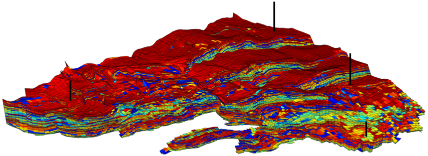


Multiscale (146 dof)

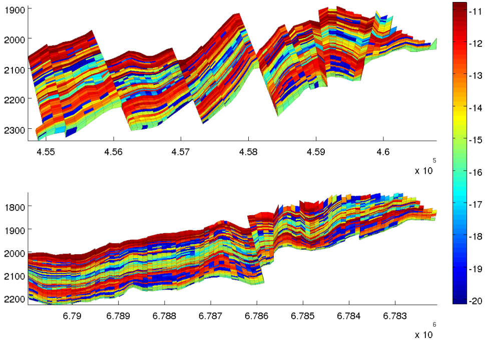
Introduction: The Gullfaks field



Horizontal permeability



Vertical permeability



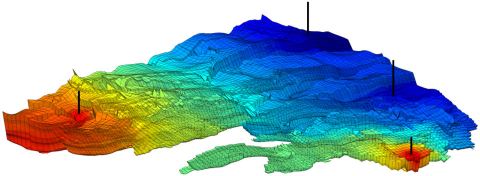
Vertical cross sections

- Field from the North Sea in Norway – challenging anisotropic permeability and grid
- Model includes cells with nearly 40 faces
- 216 000 cells with a large number of faults and eroded layers
- Synthetic well configuration with four vertical wells

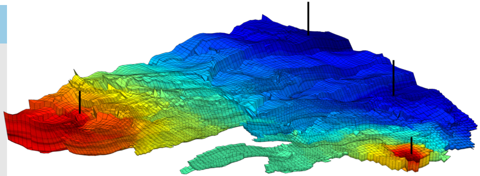
The Gullfaks field – incompressible flow

- First coarsening strategy: Uniform blocks, split over faults
- Second coarsening strategy: Use Metis with same number of DoF

Grid type	DoF	p (L_2)	p (L_∞)
$15 \times 15 \times 20$	416	0.032	0.102
Metis	416	0.032	0.100
$10 \times 10 \times 10$	1028	0.028	0.597
Metis	1028	0.015	0.112

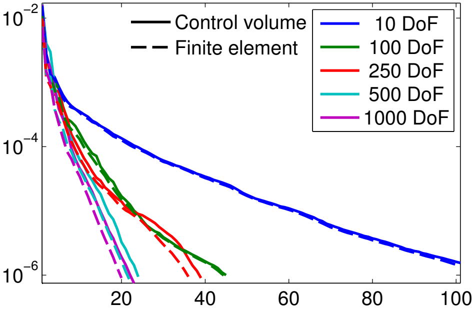


Fine scale (216 000 DoF)

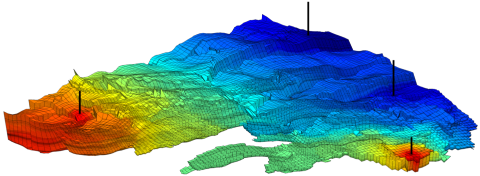


MsRSB (416 DoF)

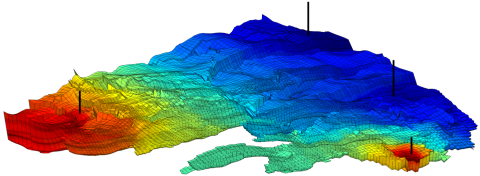
The Gullfaks field – incompressible flow



GMRES + MS-ILU(0)



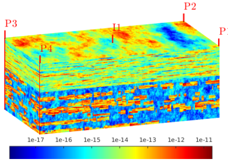
Fine scale (216 000 DoF)

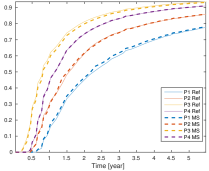


MsRSB (416 DoF)

Extensions of MsRSB

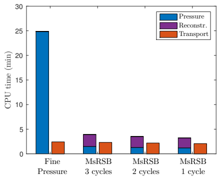
Black-oil & compressible multiphase flow

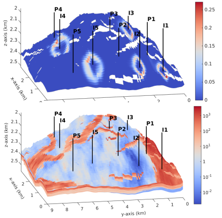




Møyner & Lie, SPE J. (2017)

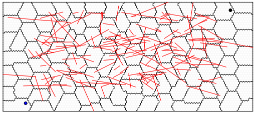
Polymer EOR with non-linear velocity

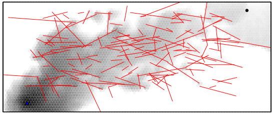




Hilden et al, TiPM. (2017)

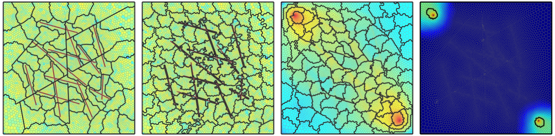
Embedded fracture models





Shah et al, JCP (2016)

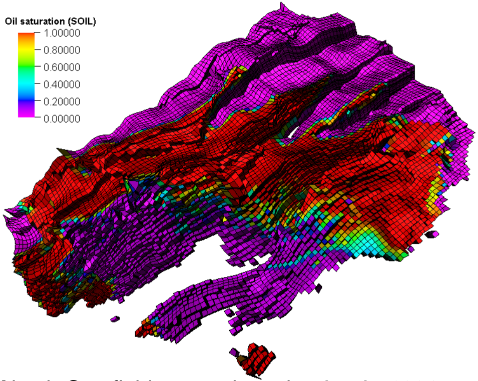
Multiple feature-specific operators

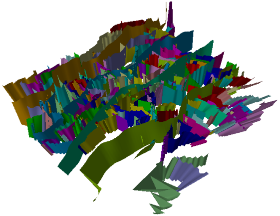


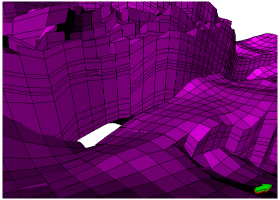
Lie et al, SPE J (2016)

Klemetsdal et al, COMG (2019)

Gullfaks field – simulated in Intersect Multiscale simulator



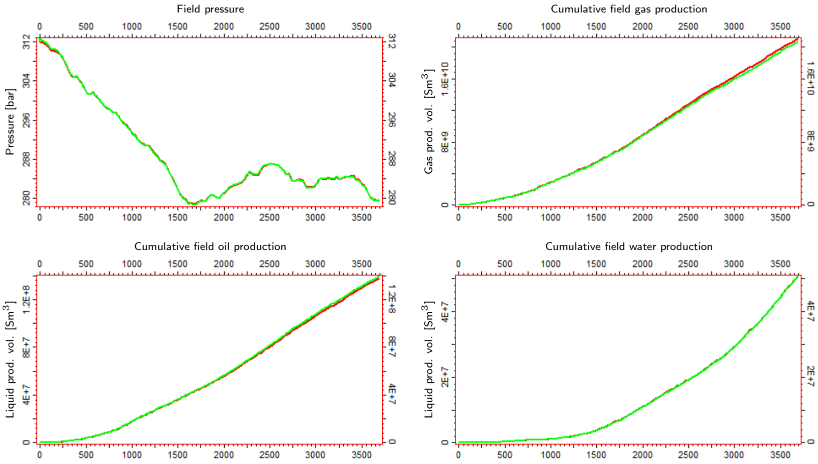




- Giant North Sea field, started production in 1986
- Sedimentology similar to SPE 10, but heavily faulted
- Mainly water injection, but also gas and water-alternating-gas in some areas
- Coarse $80 \times 100 \times 19$ black-oil simulation model with real history

Lie, Møyner, Natvig, Kozlova, Bratvedt, and Watanabe - Successful Application of Multiscale Methods in a Real Reservoir Simulator Environment
Comput. Geosci. (2017)

Gullfaks field – simulated in Intersect Multiscale simulator



Lie, Møyner, Natvig, Kozlova, Bratvedt, and Watanabe - Successful Application of Multiscale Methods in a Real Reservoir Simulator Environment Comput. Geosci. (2017)

Sequential schemes for multicomponent flow

- EoS-coupled compositional flow with N components $i \in \{1, ..., N\}$,

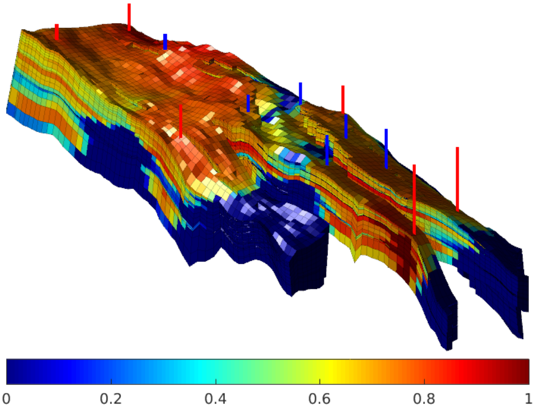
$$\frac{\partial}{\partial t}(\phi \left[\rho_L S_L X_i + \rho_V S_V Y_i\right]) + \nabla \cdot (\rho_L X_i \vec{v}_L + \rho_V Y_i \vec{v}_V) = q_i,$$

$$\vec{v}_\alpha = -\mathbf{K}\lambda_\alpha(\nabla p_\alpha - \rho_\alpha g \Delta z)$$

$$f_i^L(p, T, x_1, ..., x_N) = f_i^V(p, T, y_1, ..., y_N) \sum_{i=1}^{N} x_i = 1, \sum_{i=1}^{N} y_i = 1, S_V + S_L = 1.$$

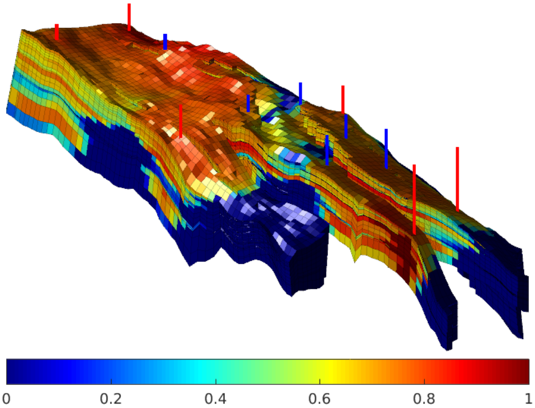
- Multiscale solver is most efficient in a sequential setting
- Objective: Sequential-implicit scheme for EoS-compositional flow
- Many schemes in literature - often simplified compositional descriptions or insufficient details given

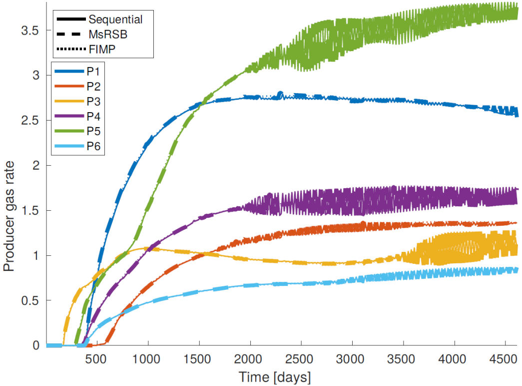
Sequential schemes for multicomponent flow



- First attempt: Total mass-based scheme (Møyner & Tchelepi, RSC17)

Sequential schemes for multicomponent flow

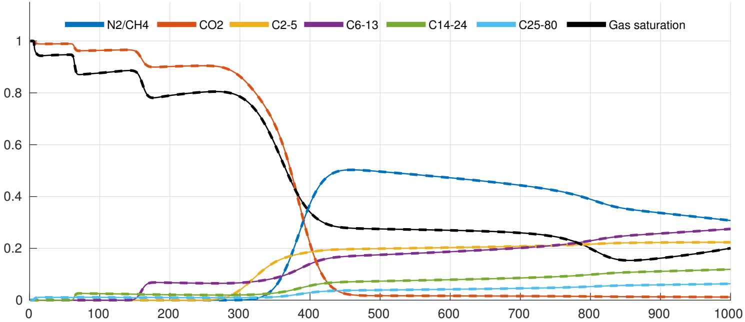




- First attempt: Total mass-based scheme (Møyner & Tchelepi, RSC17) → **failure**.
- Stability of previously suggested scheme fails for general fluid description

Sequential schemes for multicomponent flow

- Again, it was back to the drawing board: Literature study, discussions with Arthur Moncorgé at Total, a few failed prototypes...
- Transport with relaxed saturation closure, volumetric pressure equation
- Exact mass-conservation without outer loop
- Compatible with natural or molar variables



Møyner & Tchelepi - A Mass-Conservative Sequential Implicit Multiscale Method for General Compositional Problems, SPE J. (2018)

Historical works: Acs et al (1985), Watts (1986), Trangenstein & Bell (1989), Coats (1995), ...

Sequential schemes for multicomponent flow

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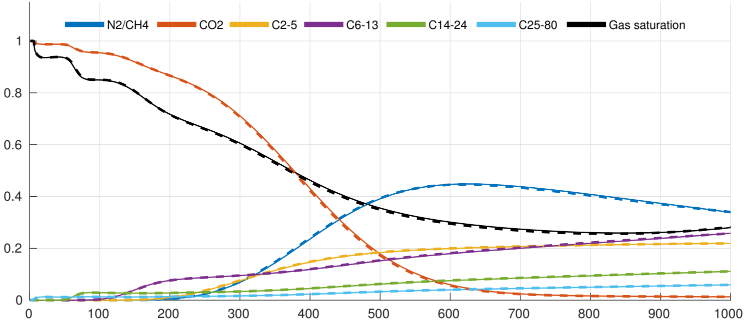
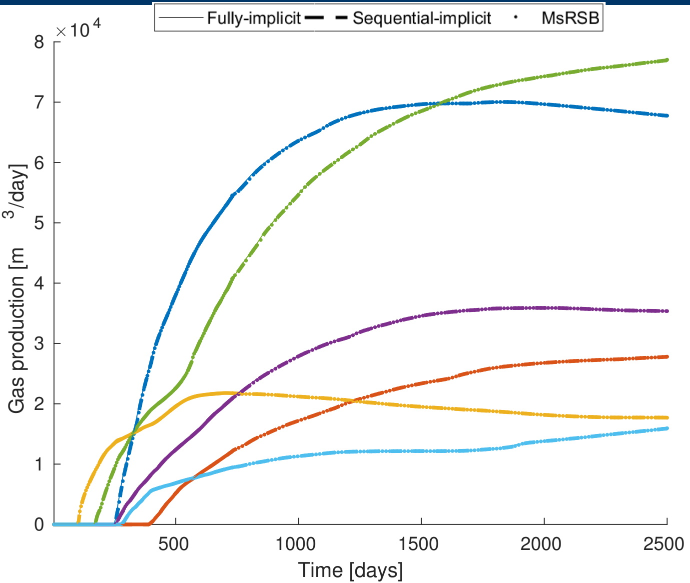
- Exact mass-conservation without outer loop
- Compatible with natural or molar variables



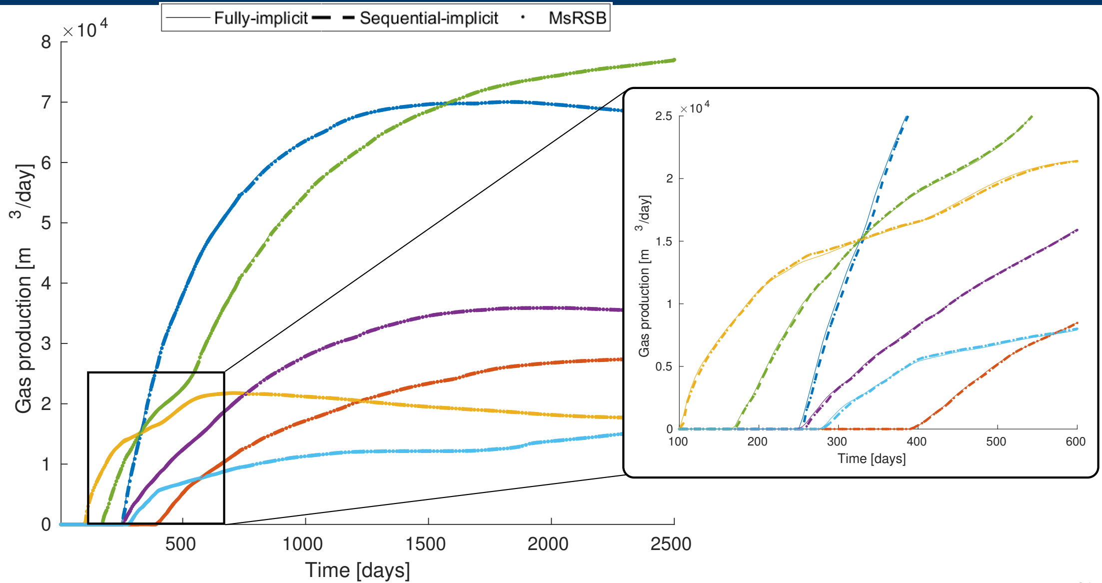
Møyner & Tchelepi - A Mass-Conservative Sequential Implicit Multiscale Method for General Compositional Problems, SPE J. (2018)

Historical works: Acs et al (1985), Watts (1986), Trangenstein & Bell (1989), Coats (1995), ...

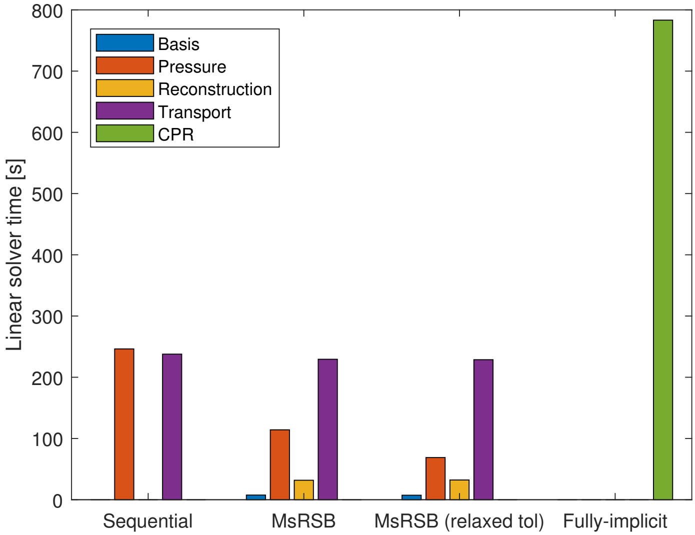
Compositional Norne: Gas production



Compositional Norne: Gas production



Multiscale Norne - Linear solver time



A few lessions learned

There are substantial research challenges and opportunities in applying methods to industrial test cases

- Test your methods on as real problems as possible, as early as possible
- Fail fast – fail often. Solutions never arrive as fully-formed ideas
- Kill your darlings – don't get attached to the first almost working solution

How to do this efficiently? Develop open-source code.

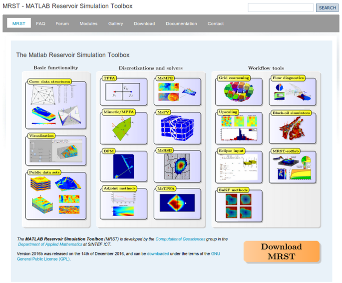
- Build on the work of others, and let others build on your work
- Reproducible science: Version control, systematic storage of test cases
- Releasing code means cleaning up, documenting and making it ready for the next project!

The MATLAB Reservoir Simulation Toolbox

Open-source toolbox for reservoir modelling, developed by SINTEF Digital and used in most of our research

Wide international user base:

- academic institutions, oil and service companies
- USA, Norway, China, Brazil, United Kingdom, Iran, Germany, Netherlands, France, Canada, ...
- 9 000+ unique downloads last two years



http://www.sintef.no/
http://www.bitbucket.org/mrst/

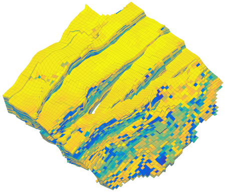
Used in publications:

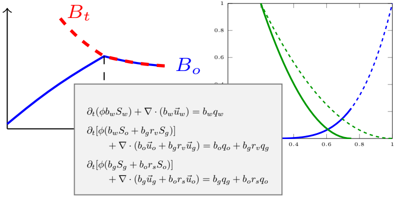
- 130+ master and PhD theses
- 210+ journal/proceedings papers by authors outside our group

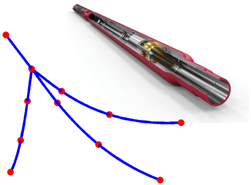
MRST AD-OO – framework for rapid prototyping

General object-oriented framework for porous media flow

- Extended MRST from discretization framework to general multiphysics capabilities
- Modular code - easy to swap components
- Multiphase flow, thermal, tracers, geomechanics, compositional, black-oil, geochemistry, multisegment wells ...

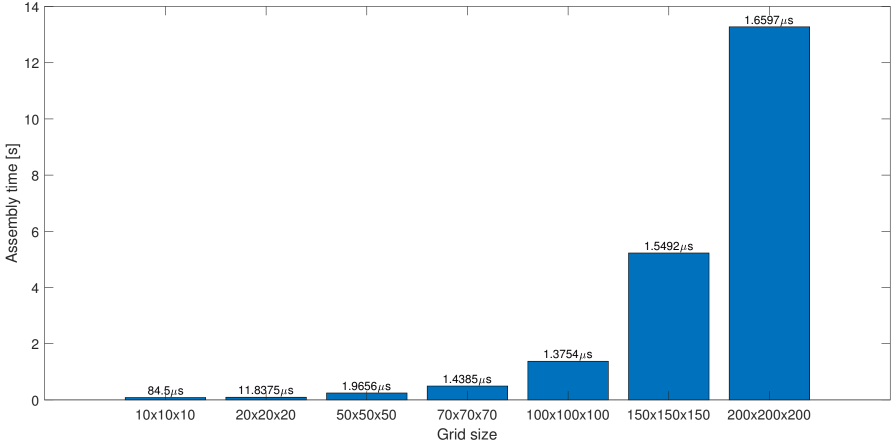


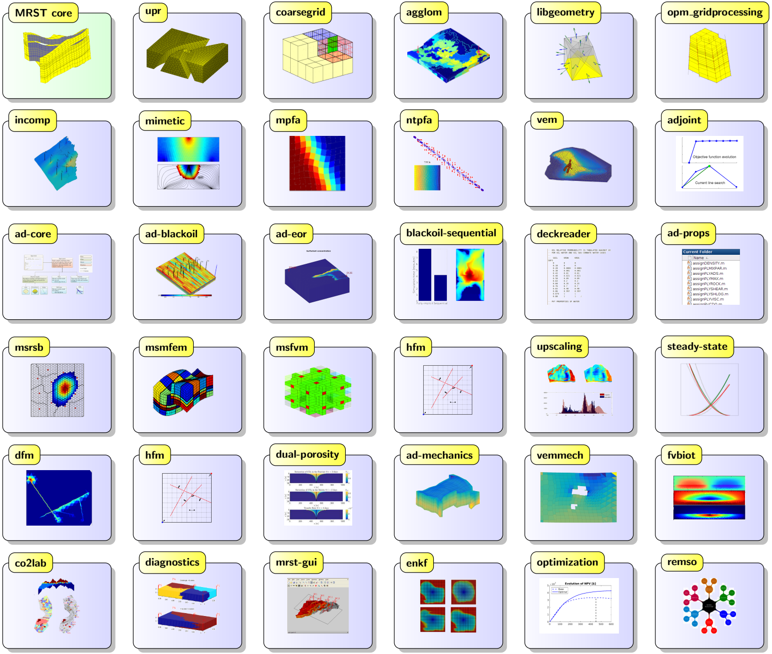




MRST AD-OO – surprisingly efficient

What can you do with MRST-OO? Assembly of three-phase problem with two wells





Grid generation and coarsening

Discretization and solvers for incompressible flow

Discretization and solvers for compressible flow

Upscaling and multiscale methods

Fractured media and geomechanics

Workflow tools

Acknowledgments

Collaborators at SINTEF, NTNU, Stanford, TU Delft, Chevron and Schlumberger

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- VISTA, a basic research programme funded by Equinor and conducted in close collaboration with The Norwegian Academy of Science and Letters

We are hiring – `tinyurl.com/sintef-comg-2019`

Backup

Governing equations for isothermal flow

Compositional model in the Matlab Reservoir Simulation Toolbox (MRST) has typical choices for compositional simulation of hydrocarbons,

- Densities and phase behavior predicted by equation-of-state
- Generalized cubic equation-of-state: Martin's equation
- Lohrenz-Bray-Clark viscosity correlations
- Schur-complement used to obtain N by N system for variable set α,

$$-J\Delta x = \begin{bmatrix} B & C \\ D & E \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} f \\ h \end{bmatrix} \rightarrow A\alpha = (B - CE^{-1}D)\alpha = f - CE^{-1}h = b.$$

Remark: E is easily invertible, as fugacity is local to each cell

Pressure

- Assemble backward Euler accumulation terms and fugacities,

$$C_i = \frac{\Phi}{\Delta t}\left[(\rho_L S_L X_i + \rho_V S_V Y_i)^{n+1} - (\rho_L S_L X_i + \rho_V S_V Y_i)^0\right]$$

$$F_i = f_i^L(p, T, x_1, ..., x_N) - f_i^V(p, T, y_1, ..., y_N).$$

- Reduce this system R to $N \times N$ primary variables p by eliminating secondary variables s,

$$R = \begin{bmatrix} \frac{\partial C}{\partial p} & \frac{\partial C}{\partial s} \\ \frac{\partial F}{\partial p} & \frac{\partial F}{\partial s} \end{bmatrix} \to G = \left(\frac{\partial C}{\partial p} - \frac{\partial C}{\partial s}\frac{\partial F}{\partial s}^{-1}\frac{\partial F}{\partial p}\right)$$

- Find weights that eliminate time-derivatives for non-pressure variables

$$(G^T)^{-1} w = e_i \begin{cases} 1 \text{ if variable i is pressure} \\ 0 \text{ otherwise} \end{cases}$$

- Pressure equation is linear combination of component mass conservation,

$$R_p = \sum_{i=1}^{N} w_i M_i, \text{ which removes any volume error from transport.}$$

Sequential scheme: Transport

- Again, conservation of each component $i \in \{1, ..., N\}$ with fixed pressure,

$$\frac{\partial}{\partial t}(\phi\left[\rho_L S_L X_i + \rho_V S_V Y_i\right]) + \nabla \cdot (\rho_L X_i \vec{v}_L + \rho_V Y_i \vec{v}_V) = q_i,$$

  with fugacity balance in two-phase cells.
- Sum of mole fractions close the system, but saturation closure is relaxed

$$\sum_{i=1}^{N} x_i = 1, \sum_{i=1}^{N} y_i = 1, \quad S_V \geq 0,\ S_L \geq 0.$$

- Removal of one variable and one equation $\rightarrow$ system is still well-posed

Component fluxes are given by fractional flow, accounting for volume error:

$$\vec{v}_{i,L} = \tilde{X}_i F_L(\vec{v}_T + \sum_{\beta \neq L} \lambda_\beta \mathbf{K}(\rho_L - \rho_\beta)),$$

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$\tilde{X}_i = X_i \sum_\alpha S_\alpha$ | $F_L = \frac{\lambda_L(S_L/\sum_\alpha S_\alpha)}{\ldots}$