Periodic Orbits to Gross Pitaevskii with Vortices following Point Vortex Flow

Raghav Venkatraman

Indiana University

The Gross-Pitaevskii Equations

Seek non-constant time periodic solutions to the Gross-Pitaevskii (GP) equations

$$iu_t(x,t) = \Delta u(x,t) + \frac{u(x,t)(1-|u(x,t)|^2)}{\varepsilon^2}, \qquad (x,t) \in \mathbb{D} \times \mathbb{R},$$

posed on the unit disc $\mathbb{D}$, subject to the Dirichlet boundary conditions (BC)

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$$\boxed{\deg(g_n, \partial\mathbb{D}, 0) = n.}$$

Hamiltonian Structure

The flow (GP-BC) conserves the *Ginzburg-Landau Energy*,

$$E_\varepsilon(u) := \frac{1}{2}\int_{\mathbb{D}} |\nabla u|^2 + \frac{1}{2\varepsilon^2}(1-|u|^2)^2\,dx. \tag{0.1}$$

Here $0 < \varepsilon \ll 1$. Energetically, minimizers u_ε of GL prefer $|u_\varepsilon| \approx 1$. In the limit $\varepsilon \to 0$, topological restrictions from the boundary condition force *vortices*.

Renormalized Energy *à la* Bethuel-Brezis-Helein

- The sequence of minimizers u_ε *converge* as $\varepsilon \to 0$ to a *nice* function u_*, *away from exactly* n distinct points– vortices.

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- More generally, for any positive number $N \geq n$, integers $d_i, i = 1, \cdots, N$ satisfying $\sum d_i = n$, and distinct points $a_i, i = 1, \cdots, N$, and a boundary condition g taking values in $\mathbb{S}^1$ with $\deg(g, \partial\mathbb{D}, 0) = n$

$$W(a_1, \cdots, a_N; d_1, d_2, \cdots, d_N; g) := -\pi \sum_{i \neq j} d_i d_j \log |a_i - a_j| + \text{ boundary terms.}$$

Point Vortex Flow

The Hamiltonian system on $\mathbb{C}^N$ associated to W :

$$d_j \left(\frac{da_j}{dt} \right)^{\perp} = -\frac{1}{\pi} \nabla_{a_j} W, \qquad j = 1, \cdots, N. \qquad (PVF)$$

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Crucial to us: (PVF) captures *effective dynamics* of vortices to GP as $\varepsilon \to 0^+$, *up to first collision time*.

Made rigorous by Colliander-Jerrard/Lin-Xin/Jerrard-Spirn.

Rigorous results on the hydrodynamic/mean field limit of GP: Jerrard-Spirn/Serfaty.

Main Question

Given a time periodic solution to (PVF), can we construct time-periodic solution to (GP), whose vortices follow the given periodic solution?

- Large time behavior for GP for $\varepsilon > 0$: given solutions to (PVF) with vortices that never collide, can we construct solutions to (GP) that *follow* these point vortices for all time, as $\varepsilon \to 0^+$? Especially interesting when vortices of opposite degrees persist.

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Our (modest) contribution: in the very special case of (GP-BC), using variational and symmetry arguments, we show that for a very large class of time-periodic solutions, called relative equilibria to (PVF), there exist time-periodic solutions to (GP) following them.

Relative Equilibria

• Definition: Uniformly rotating periodic solutions to the system (PVF).

• Obtained by pursuing the ansatz $a_j(t) = a_j e^{i(-\tilde{\omega} t)},$ where $\tilde{\omega} \in \mathbb{R}$.

• Results in nested rings of vortices, each with equal numbers of rings, and all vortices of a ring having the same degree.

• Different rings may be *aligned* or *staggered*.

Relative Equilibria

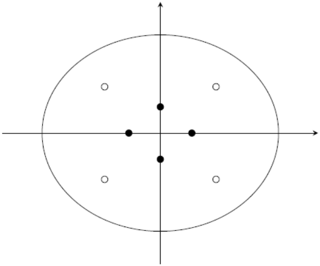

Figure 2: An staggered configuration. The solid and hollow bullets indicate possibly different degrees. $k = 4$. Not to scale.

Figure: A staggered configuration

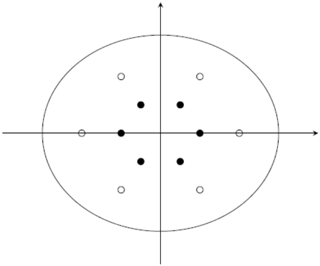

Figure 1: An aligned configuration. The solid and hollow bullets indicate possibly different degrees. $k = 6$. Not to scale.

Figure: An aligned configuration

Rotational Ansatz

Starting Point: Make a rotational ansatz:

$$u(x,t) = R(-k\omega t)v\big(R(\tfrac{\omega}{m}t)x\big),$$

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$$u(x,t) = R(-k\omega t)v\big(R(\tfrac{\omega}{m}t)x\big),$$

Here: $R(\beta)$ is the counterclockwise rotation matrix by an angle β; k, m are integers and $\omega \in \mathbb{R}$.

Thanks to Bob Jerrard for suggesting this ansatz in the case $n = 1$ of a single vortex.

An Elliptic PDE: Variational Formulations

Plugging in the ansatz into (GP) yields an elliptic PDE.

$$\Delta v(y) + \frac{v}{\varepsilon^2}(1 - |v|^2)(y) = \omega \left(kv + \frac{1}{m} y^\perp \cdot \nabla v^\perp \right)(y), \qquad y \in \mathbb{D},$$

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The boundary condition is *compatible* with the rotating frame ansatz above, iff $k|n$ and $m = \frac{n}{k}$. Inspired by relative equilibria, look for v with $k-$fold symmetry.

In case $n = 0$, use the ansatz with $k = 0$ and m an arbitrary integer, reflecting $m-$fold symmetry.

Conserved Quantities

- Hamiltonians:

  Gross-Pitaevskii: Ginzburg-Landau Energy:

$$E_\varepsilon(u) := \int_{\mathbb{D}} \frac{|\nabla u|^2}{2} + \frac{(1-|u|^2)^2}{4\varepsilon^2}\, dx.$$

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- Momenta:

  Gross-Pitaevskii:

$$J(v) := -\frac{1}{2}\int_{\mathbb{D}} k|v|^2 + \frac{1}{m} v \cdot (x^\perp \cdot \nabla) v^\perp \, dx.$$

  Point Vortex Flow:

$$J_0(\mathbf{b}, d) = -\frac{1}{2}\sum_{j=1}^{N} d_j |b_j|^2$$

Constrained Minimization Approach

The elliptic PDE above has a variational formulation based on momentum-constrained minimization. Since this is a minimization procedure, can only yield $+1$ vortices.

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- A: admissible set reflecting symmetry of the chosen relative equilibrium, and
- $J(u) \approx J_0(a_1, \cdots, a_n)$.

$\omega = \omega_\varepsilon$ arises as a Lagrange multiplier. This approach follows work by Gelantalis and Sternberg.

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• Complete the proof using the vortex balls construction and the Jacobian estimate.

Limitations of the constrained minimization approach

- Unable to treat multiple ring solutions/staggered ring solutions.
- Unable to show $\omega_\varepsilon \to \omega$ where ω is the speed corresponding to the limit.

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Good news: no need to control ω_ε any more.

Bad news: we lose the constrained minimization formulation from above: can't specify constraint value and Lagrange multiplier!

Linking

Definition: Fix a Banach space V, a closed subset $S \subset V$ and a submanifold Q, and denote its relative boundary by ∂Q. The sets S and ∂Q are said to link if

- $S \cap \partial Q = \emptyset$
- For any continuous map $h : V \to V$ such that $h|_{\partial Q} = id$, there holds $h(Q) \cap S \neq \emptyset$.

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In the context of Ginzburg Landau, a linking method was used by F-H. Lin to construct critical points of GL *near* critical points of W.

Main Theorem

Theorem (V., '16)

Let $(\mathbf{a}, d)$ *be a relative equilibrium, with speed* ω_0*. Write* $\mathbf{a}(t) := \mathbf{a}e^{i\omega_0 t}$*. For each* $\varepsilon > 0$ *sufficiently small (depending on* $\mathbf{a}$*), there exists a non-trivial time periodic solution* u_ε *to (GP-BC), with the same period of rotation as the given relative equilibrium, such that the Jacobian*

$$Ju_\varepsilon(\cdot, t) \rightharpoonup \pi \sum_{i=1}^{N} d_i \delta_{a_i(t)}$$

as $\varepsilon \to 0$*, in* $W^{-1,1}(\mathbb{D})$*, for each time* $t \in \mathbb{R}$*.*

Here, one can think of $Ju_\varepsilon(\cdot, t) := \det(\nabla u_\varepsilon(\cdot, t))\, dx$.

Some Details in the Proof of the Main Theorem

Main Goal: Find a critical point of the functional $\mathcal{E}_\varepsilon := E_\varepsilon - \omega_0 J$ *near* a given critical point $(\mathbf{a}, d)$ of $\mathcal{H}^{n,\omega_0}(\mathbf{a}, d) := \frac{1}{\pi} W(\mathbf{a}, d) - \omega_0 J_0(\mathbf{a}, d)$

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- Critical value of $\mathcal{E}_\varepsilon$: Using this family, we can give an $\inf - \sup$ characterization of the critical value, which, upto multiples of $\pi \log \frac{1}{\varepsilon}$ and $O(1)$ terms, is the energy $\mathcal{H}^{n,\omega_0}(\mathbf{a}, d)$.

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- Conclusion of Critical Value Step: $\mathcal{E}_\varepsilon$ is Palais-Smale, so pass to the large time limit, holding ε fixed. Obtain a critical point v_ε satisfying

$$\left| \mathcal{E}_\varepsilon(v_\varepsilon) - N\pi \log \frac{1}{\varepsilon} - \mathcal{H}^{n,\omega_0}(\mathbf{a}) - N\gamma \right| = o_\varepsilon(1)$$

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- Above argument only says energies are close. We need v_ε to have zeroes close to the given critical point of $\mathcal{H}$. Follows from Pohazaev-type identities and letting $\varepsilon \to 0^+$.

Afterthought: some examples

- For each integer k, there exists a solution to Gross-Pitaevskii with boundary condition $g \equiv 1$, and zeroes on the vertices of concentric $k-$gons, one with $+1$ vortices, and the other, staggered, with -1 vortices.

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- Given n, corresponding to the degree of the b.c., fix a divisor k of n. Then there exists a periodic orbit to (GP-BC) containing $\frac{n}{k}$ aligned rings, each with k $+1$ vortices.

- When n is a prime, there's only one relative equilibrium to (GP-BC) with all $+1$ vortices. Stability??