Topological Advection:
Efficiently Quantifying Chaotic Advection in Sparse Trajectory Data

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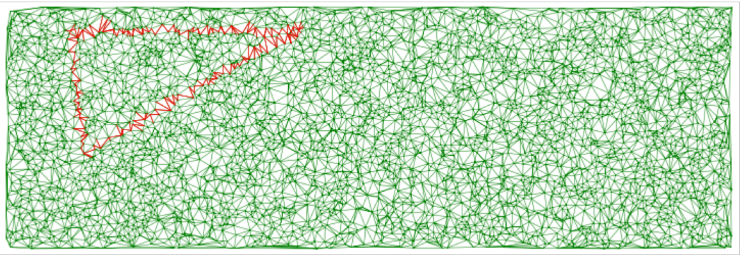
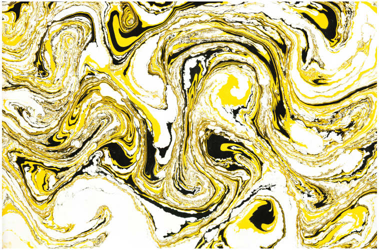
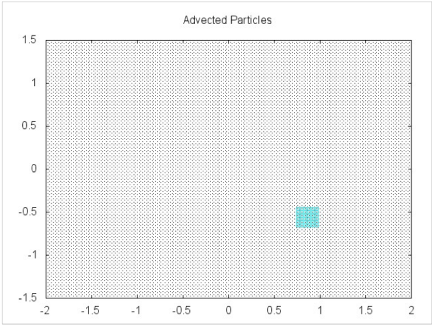
SIAM DS - May 22nd 2019



Motivation: 2 Advection Problems (2D)

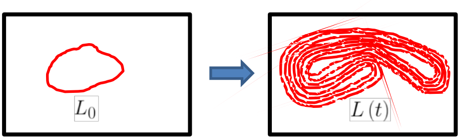




1) Quantifying Chaotic Advection: Topological Entropy

- For a proscribed velocity vector field, we can integrate passively advected particles to obtain a flow map (M)
- Topological entropy can be defined as the exponential growth rate of the number of distinguishable orbits over repeated applications of M.
- Computationally unwieldly. So, an alternative definition in 2D: The maximum exponential growth rate of material curves.

$$L(t) \approx L_0\, e^{ht}$$



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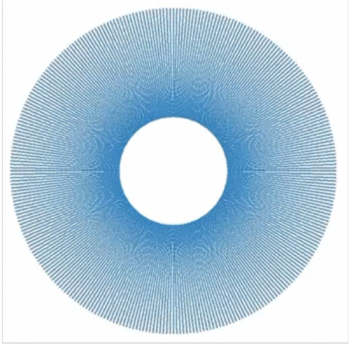
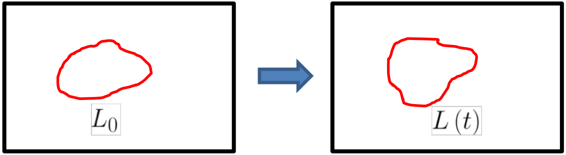


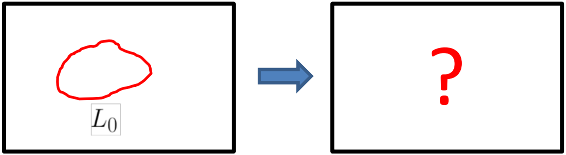
2) Coherent Structure Detection

- Coherent Structure/Set: Region that does not significantly mix with the rest of the fluid
- Boundary acts as a barrier to transport, and experiences little stretching (subexponential)
- Verify that a region constitutes a coherent set: advect material curve boundary and check for stretching

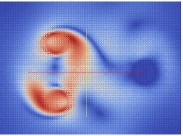


Advection of Material Curves

Common presentation of these two advection problems:



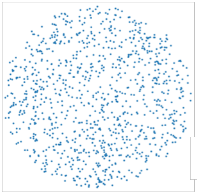
General Problem Statement: Given an initial closed curve or family of non-intersecting curves, find the final curves that these map to under advection



"Full" Data (flow map or velocity vector field):

$$M_t(\vec{x})$$

$$\dot{\vec{x}} = \vec{v}(\vec{x}, t)$$



Sparse Data (set of trajectories):

$$[\vec{x}_1(t), \cdots, \vec{x}_N(t)]$$

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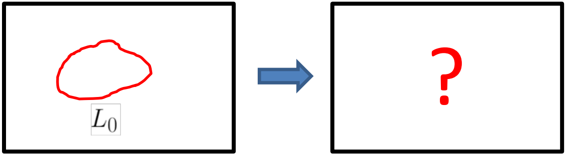

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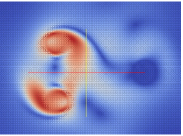

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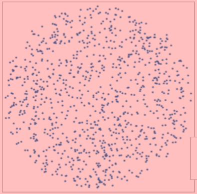

Sparse Data (set of trajectories):

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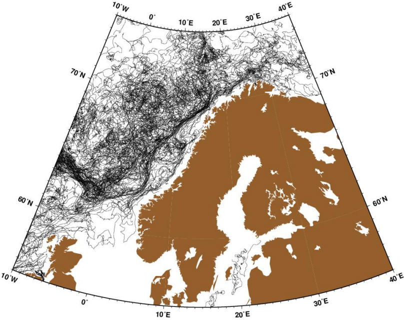
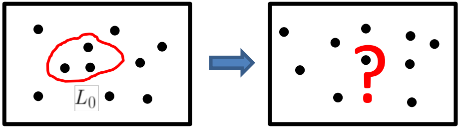
Topological Advection



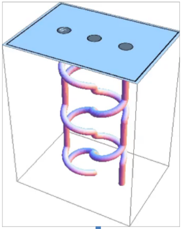
Topological Advection: Given sparse data (advective trajectories), what can we say about the final state of a material curve if we know its initial state? (up to homotopy)

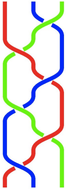


- Main idea: use the topological features of the trajectory set to create efficient algorithms
- Algorithms/Approaches:
  - Bestvina-Handel
  - Braiding/Dynnikov Coordinates
  - E-tec
  - Dual E-tec

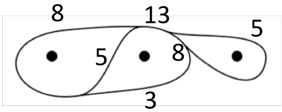
Background: Bestvina-Handel Algorithm

Trajectory Motion as a Braid





$$b = \sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1$$



- Band (material curve) represented as a weighted train-track
- Bestvina-Handel: finds invariant traintrack under the action of the braid, and the transition matrix
- Max eigenvalue of the transition matrix gives the braid dilation, which can then give a lower bound on the topological entropy of the underlying flow

Drawbacks

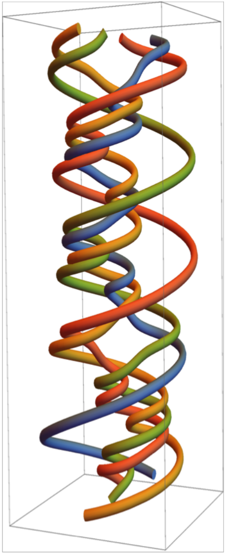
- Only asymptotic behavior is captured
- Requires trajectories to be periodic
- Computationally slow (difficulty with > 20 trajectories)

M. Bestvina, M. Handel. Train tracks for surface homeomorphisms. Topology (1995)

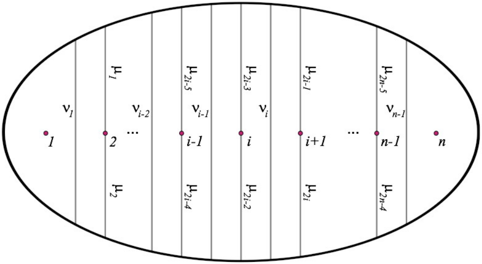
* Based off of a computational proof of the Nielsen-Thurston classification of mapping class groups

Background: Braid Theory Approach

Trajectory Motion as a Braid



Loop Encoded with
Dynnikov Coordinates



$$a_i = \tfrac{1}{2}\left(\mu_{2i} - \mu_{2i-1}\right), \qquad b_i = \tfrac{1}{2}\left(\nu_i - \nu_{i+1}\right)$$

$$\boldsymbol{u} = (a_1, \ldots, a_{n-2}, b_1, \ldots, b_{n-2})$$

$$L(\boldsymbol{u}) = |a_1| + |a_{n-2}| + \sum_{i=1}^{n-3} |a_{i+1} - a_i| + \sum_{i=0}^{n-1} |b_i| ,$$

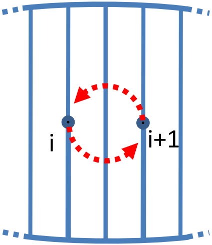
J-O. Moussafir, On computing the entropy of braids, Funct. Anal. Other Math (2006)

Braids of Entangled Particle Trajectories, Jean-Luc Thiffeault, Chaos 2010

Background: Braid Theory Approach

Braid Action on the Loop

$\boxed{\sigma_i}$



Update rules (σ_i generator):

$$
\begin{aligned}
a'_{i-1} &= a_{i-1} - b^+_{i-1} - \left(b^+_i + c_{i-1}\right)^+ \\
b'_{i-1} &= b_i + c^-_{i-1}\,, \\
a'_i &= a_i - b^-_i - \left(b^-_{i-1} - c_{i-1}\right)^- \,, \\
b'_i &= b_{i-1} - c^-_{i-1}\,, \\
c_{i-1} &= a_{i-1} - a_i - b^+_i + b^-_{i-1}\,,
\end{aligned}
$$

Finite-Time Braiding Exponent:

$$\boxed{\mathrm{FTBE}(\mathbf{b}) = \frac{1}{T}\log\frac{|\mathbf{b}\ell_E|}{|\ell_E|}}$$

- FTBE analogous to the topological entropy
- Coherent structure detection

Nice Features

- Much faster than Bestvina-Handel
- Not just asymptotic information
- Aperiodic trajectories can be treated (FTBE)
- Generating braids and their action on loops is split

Drawbacks

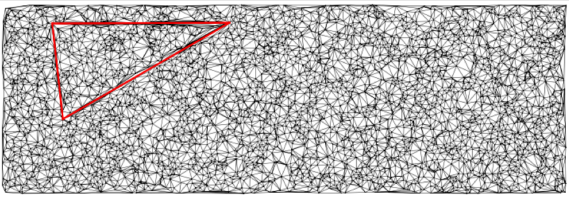
- Computational complexity is $\mathcal{O}(N^2)$ Can do better
- 2D geometric data is lost upon projection

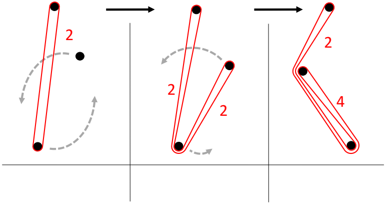
Finite Time Braiding Exponents, Marko Budisic & Jean-Luc Thiffeault, Chaos 2015

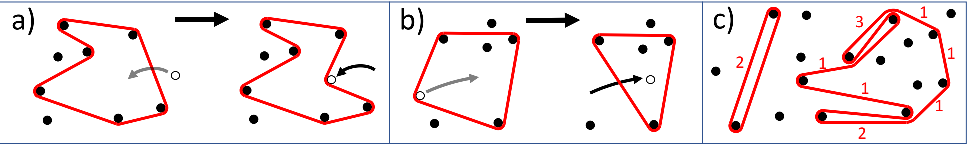
M.R. Allshouse, J-L. Thiffeault, Detecting Coherent Structures Using Braids, Physica D 2012

Big Picture

- Encode a taut (homotopically minimal length) loop as integer edge weights in a triangulation of the points (Initially constrained Delaunay).
- Evolve the triangulation forward according to the point motion, updating edge weights as triangles collapse and reform.
- Get a lower bound on the topological entopy as the exponential increase in total edge weights with time.



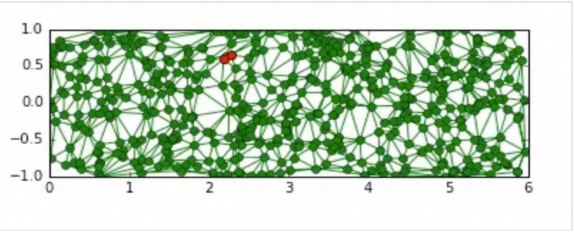




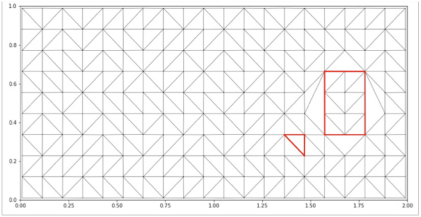
Ensemble-based Topological Entropy Calculation (E-Tec), Eric Roberts, Suzanne Sindi, Spencer Smith, and Kevin Mitchell, Chaos (2019)

Background: E-tec

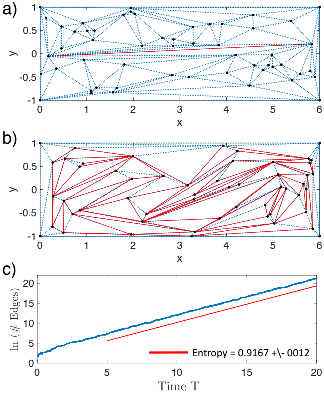
Advection Video



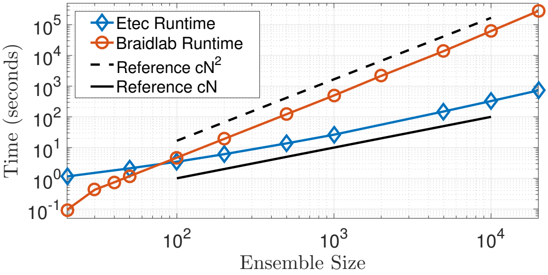
Coherent Structure Detection

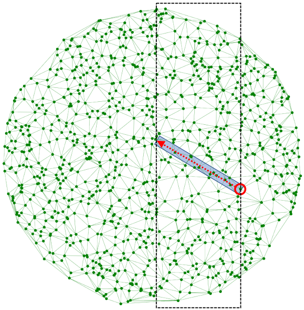


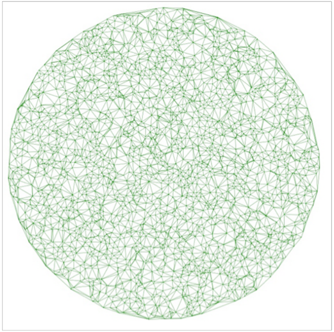
Topological Entropy



Background: E-tec Comparison







Computational Complexity:

$$\mathcal{O}(N + N^k \log N), \text{ with } 1/3 \leq k \leq 3/2,$$

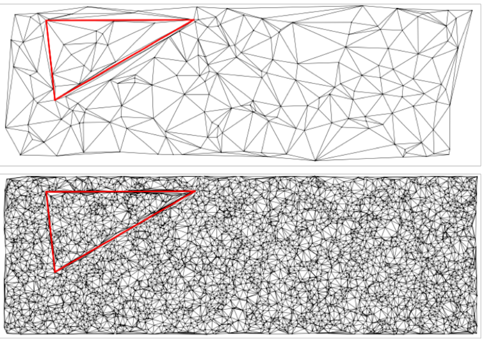
Nice Features	Drawbacks
Much quicker than braiding approach	Conceptually (+ data structures) more complex
Preserves 2D geometry	Triangulation evolution depends on the specific loop
Gives homotopically minimal curve	Requires a constrained Delaunay initial triangulation

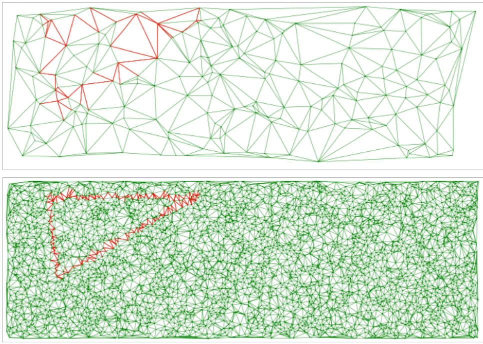
Dual E-tec Ingredients

- Triangulation (Delaunay or other) encodes any band/train-track via edge weights that count transverse intersections.
- Triangulation evolves through a series of events (triangle collapse event or Delaunay event).
- Each event triggers a Whitehead move, and the edge weight update rule is very simple.

E-tec

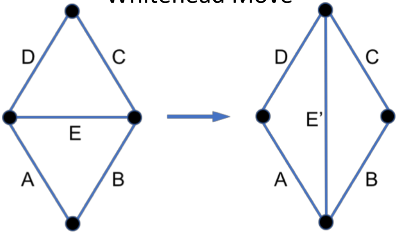
Dual E-tec





Dual E-tec: Triangulation Updating

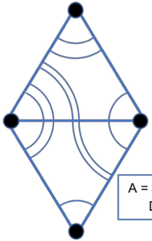
Whitehead Move

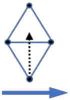


Update Rule

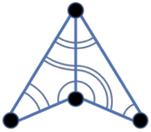
$$\boxed{E^{'} = \max\left(A + C, B + D\right) - E}$$

Example



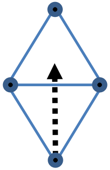


A = 3, B = 4, C = 3,
D = 6, E = 5

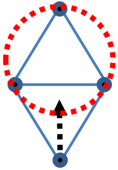


A = 3, B = 4, C = 3, D = 6,
E' = 5 = max(3+3,4+6)-5

Triangulation Evolution



Collapse Event (Whitehead move when triangle goes through zero area)

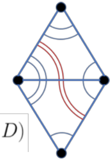


Delaunay Event (Whitehead move when point enters circumcircle of other triangle)

- Very flexible: can choose any procedure for kenetically updating the triangulation via Whitehead moves
- Can accumulate Whitehead move operators and have these act on any initial band (splitting like braid approach)

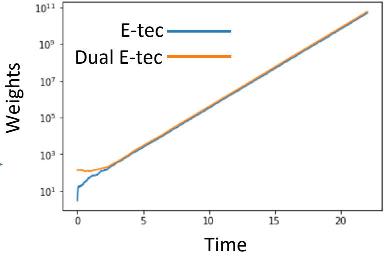
Dual E-tec: features

- Weight accounting: "diagonal" weights



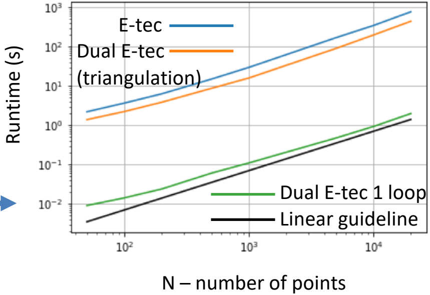
$$D = \frac{1}{2}abs\,(A - B + C - D)$$

- Same estimate of topological entropy as E-tec



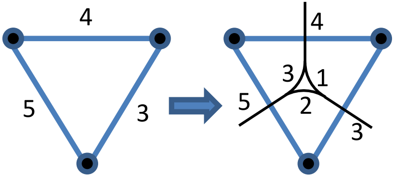
- Operator approach: Each collapse event stored as an operator. Accumulated operators act on any loop.

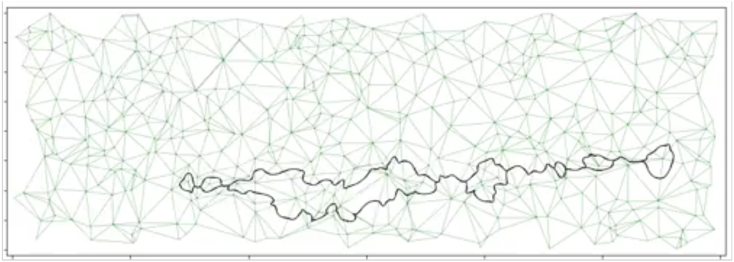
- Same computational complexity as E-tec, but overall faster.



Dual E-tec: features

The edge weights contain enough information to reconstruct a train-track representation of the loop





Nice Features

- Faster than Braiding approach and E-tec
- Preserves 2D geometry
- Operator approach
- Conceptually simple

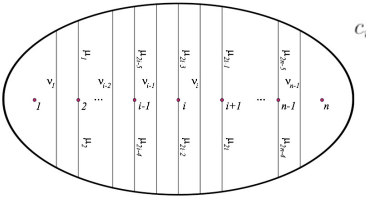
Drawbacks

Does not give the unique minimal length band

Some ambiguity in choice of triangulation

Dual E-tec: Braid approach as special case

- Dynnikov Coordinates – triangulation (disk boundary identified as single point)
- Braid generator “twists” decomposed into series of collapse events
- Simple Whitehead move update rule applied to each event gives the Dynnikov coordinate update rules



Update rules:

$$c_{i-1} = a_{i-1} - a_i - b_i^{+} + b_{i-1}^{-} \,,$$

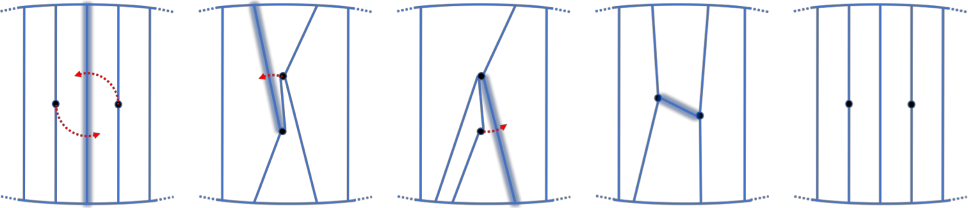
$$a'_{i-1} = a_{i-1} - b_{i-1}^{+} - \left(b_i^{+} + c_{i-1}\right)^{+}$$

$$b'_{i-1} = b_i + c_{i-1}^{-} \,,$$

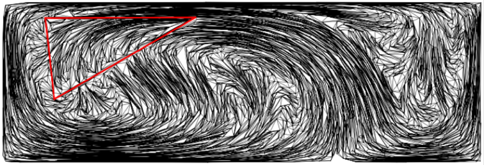
$$a'_i = a_i - b_i^{-} - \left(b_{i-1}^{-} - c_{i-1}\right)^{-} \,,$$

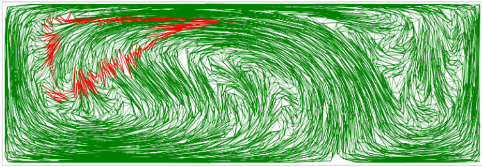
$$b'_i = b_{i-1} - c_{i-1}^{-} \,,$$

Can completely recover the more complicated Dynnikov update rules from the simpler Whitehead move update rule!



Conclusions and Future Work





Conclusions

- Posed general question: Given sparse trajectory data, what can we say about the evolution of a material curve?
- Viewed this as Topological Advection
- Reviewed Bestvina-Handel, Braiding (Dynnikov), and E-tec algorithms

- Introduced new approach "Dual" to E-tec
- Fastest current algorithm for this problem
- Enables operator approach
- Can derive Dynnikov coordinate update rules

Future Work

- Path forward to extending Dual E-tec to 3D and higher dimensions
- Natural way to add and delete points in time
- Separate application to coherent structure detection
- Applications to experimental systems (active nematic paper in review)