A Half-order numerical scheme for nonlinear SDEs with one-sided Lipschitz drift and Hölder continuous diffusion coefficients

Yucchen Dong
Work with: Prof. Zhongqiang Zhang

Department of Mathematical Sciences
Worcester Polytechnic Institute

ydong@wpi.edu

SIAM Conference on UQ 2018

Apr.19, 2018

Overview

1. Motivation
2. Numerical Simulations
3. Strong Convergence
4. Conclusions and Future Work

Background and Motivation

Consider the following one dimensional SDE

$$dX_t = (1 - X_t^3)dt + \sigma X_t^\gamma dW_t$$

where $X_0 \geq 0$ and $\frac{1}{2} < \gamma < 1$.

- $(1 - x^3)$ is not Lipschitz continuous and leads to blow up of solutions in paths
- X_t^γ, $\frac{1}{2} < \gamma < 1$ is only Hölder continuous.

Issue 1: Euler scheme does not work for non-Lipschitz continuous drift

Let's consider the following SDE:

$$dX_t = -X_t^3\,dt + dW_t. \tag{1}$$

Starting from $X(0) = X_0 = 1/h$, h is time step size. Then

$$X_1 = X_0 - X_0^3 h + \xi_1\sqrt{h} \approx -\frac{1}{h^2}, \quad \xi_1 \sim \mathcal{N}(0,1)$$

$$X_2 = X_1 - X_1^3 h + \xi_2\sqrt{h} \approx \frac{1}{h^5}, \quad \xi_2 \sim \mathcal{N}(0,1)$$

$$X_3 = X_2 - X_2^3 h + \xi_3\sqrt{h} \approx -\frac{1}{h^{14}}, \quad \xi_3 \sim \mathcal{N}(0,1)$$

- The moments of numerical solutions explode when using explicit schemes.

Numerical Example 1

Example (CIR model)

Consider the following SDE

$$dX_t = (1 - X_t^3)dt + X_t^\gamma dW_t, \quad X_0 = 0.5. \tag{4}$$

where $\gamma = 0.5$ and $\gamma = 0.8$.

From the drift coefficients, we can get $\alpha = 3$.

Result

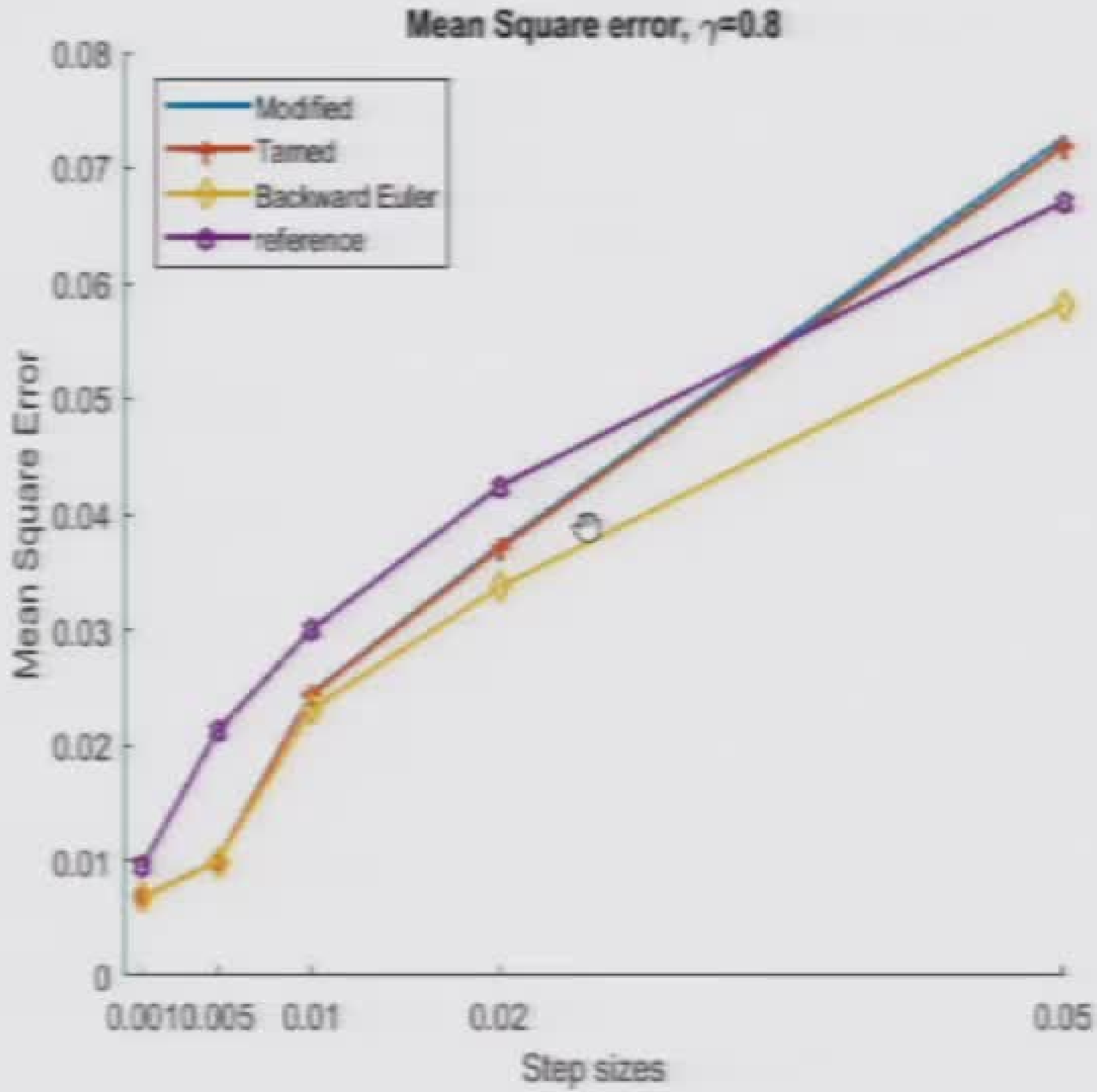

Figure: Example 1 with $\gamma = 0.8$ in different stepsizes

Numerical Example 2

Example (Two factor Heston Model)

Consider the following SDEs

$$
\begin{aligned}
dX_t = &\quad (1 - X_t^3)dt + X_t^\gamma dW_t^1, \\
dS_t = &\quad \mu S_t dt + \sqrt{X_t} S_t (\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2).
\end{aligned} \tag{8}
$$

where W_t^1 and W_t^2 are two independent standard Brownian motions, $\mu = 0.5, \rho = -0.7$ and initial values are $S_0 = 1, X_0 = 0.5$. Also, γ takes value in 0.5 or 0.8.

Assumptions

Consider

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma X_s^\gamma dW_s, \frac{1}{2} < \gamma < 1$$

(i) The initial condition is such that

$$\mathbb{E}[|X_0|^{2p}] \leq K < \infty, \quad \text{for all} \quad p \geq 1. \tag{9}$$

(ii) There is a positive constant β such that

$$(x-y)(b(x)-b(y)) \leq \beta |x-y|^2. \tag{10}$$

(iii) There exist $\mathcal{K}_1 > 0$ and $\alpha \geq 1$ such that for $t \in [0, T]$

$$|b(x)-b(y)|^2 \leq \mathcal{K}_1 (1 + |x|^{2\alpha-2} + |y|^{2\alpha-2})|x-y|^2, \quad x, y \in \mathbb{R}. \tag{11}$$

(iv) The function $b(x)$ is positive when $x = 0$, i.e., $b(0) > 0$.

Half-order Strong Convergence

Define $\theta(t) = \sup_{k\in\{1,2,\dots,N\}}\{t_k : t_k \le t\}$. Let

$$Z_t = Y_{\theta(t)} + \bar{b}(Y_{\theta(t)})(t-\theta(t)) + \sigma(Y_{\theta(t)})(W_t - W_{\theta(t)}). \qquad (12)$$

and the numerical solution can be obtained by $Y_t = |Z_t|$ for $t \in [0, T]$.

Theorem

Suppose that Assumption holds and $X_t > 0$ when $t \in [0, T]$. Suppose also that $X_0 = x > 0$. Then there exists a positive constant C depending on σ, p and T but not on Δt such that

$$\sup_{t\in[0,T]} \mathbb{E}[|X_t - Y_t|^{2p}] \le C\Delta t^p, \quad p \ge 1.$$

Proof of Theorem

Lemma

Assume that $\{Z_t\}_{0\leq t\leq T}$ is given by (12) and assumptions 18 hold. If $\frac{1}{2} < \gamma < 1$ and Δt is sufficiently small, then

$$\sup_{t\in[0,T]} \mathbb{P}(Z_t \leq 0) \leq C\exp(-\Delta t^{1-2\gamma}).$$

Lemma

Let $p \geq 0$, the numerical scheme Y_t has bounded positive moments. i.e.

$$\mathbb{E}[|Y_t|^p] < \infty \text{ , for } 0 \leq t \leq T.$$

Especially, for all $p \geq 1$, there exists a positive constant C depending on σ, p and T but not on Δt such that

$$\mathbb{E}[|Y_{\theta(t)} - Y_t|^{2p}] \leq C\Delta t^p.$$

References

A. Alfonsi,
Strong order one convergence of a drift implicit Euler scheme: application to the CIR process,
Statist. Probab. Lett 83(2013) pp.602-607

A. Berkaoui, M.Bossy and A.Diop,
Euler scheme for SDEs with non-Lipschitz diffusion coefficient: strong convergence,
ESAIM Probab. Stat., 12(2008), pp.1-11

G. Ferreyra and P. Sundar
Comparison of solutions of stochastic equations and applications
Stochastic Analysis and Applications 18(2),211-229(2000)

Z.Zhang and H.Ma
Order-preserving strong schemes for SDEs with locally Lipschitz coefficients
Appl. Numer. Math 112(2017),pp.1-16

References

A. Alfonsi,
Strong order one convergence of a drift implicit Euler scheme: application to the CIR process,
Statist. Probab. Lett 83(2013) pp.602-607

A. Berkaoui, M.Bossy and A.Diop,
Euler scheme for SDEs with non-Lipschitz diffusion coefficient: strong convergence,
ESAIM Probab. Stat., 12(2008), pp.1-11

G. Ferreyra and P. Sundar
Comparison of solutions of stochastic equations and applications
Stochastic Analysis and Applications 18(2),211-229(2000)

Z.Zhang and H.Ma
Order-preserving strong schemes for SDEs with locally Lipschitz coefficients
Appl. Numer. Math 112(2017),pp.1-16

References

A. Alfonsi,
Strong order one convergence of a drift implicit Euler scheme: application to the CIR process,
Statist. Probab. Lett 83(2013) pp.602-607

A. Berkaoui, M.Bossy and A.Diop,
Euler scheme for SDEs with non-Lipschitz diffusion coefficient: strong convergence,
ESAIM Probab. Stat., 12(2008), pp.1-11

G. Ferreyra and P. Sundar
Comparison of solutions of stochastic equations and applications
Stochastic Analysis and Applications 18(2),211-229(2000)

Z.Zhang and H.Ma
Order-preserving strong schemes for SDEs with locally Lipschitz coefficients
Appl. Numer. Math 112(2017),pp.1-16

Lemma

Let $p \geq 0$, the numerical scheme Y_t has bounded positive moments. i.e.

$$\mathbb{E}[|Y_t|^p] < \infty \text{ , for } 0 \leq t \leq T.$$

Especially, for all $p \geq 1$, there exists a positive constant C depending on σ, p and T but not on Δt such that

$$\mathbb{E}[|Y_{\theta(t)} - Y_t|^{2p}] \leq C\Delta t^p.$$

Result

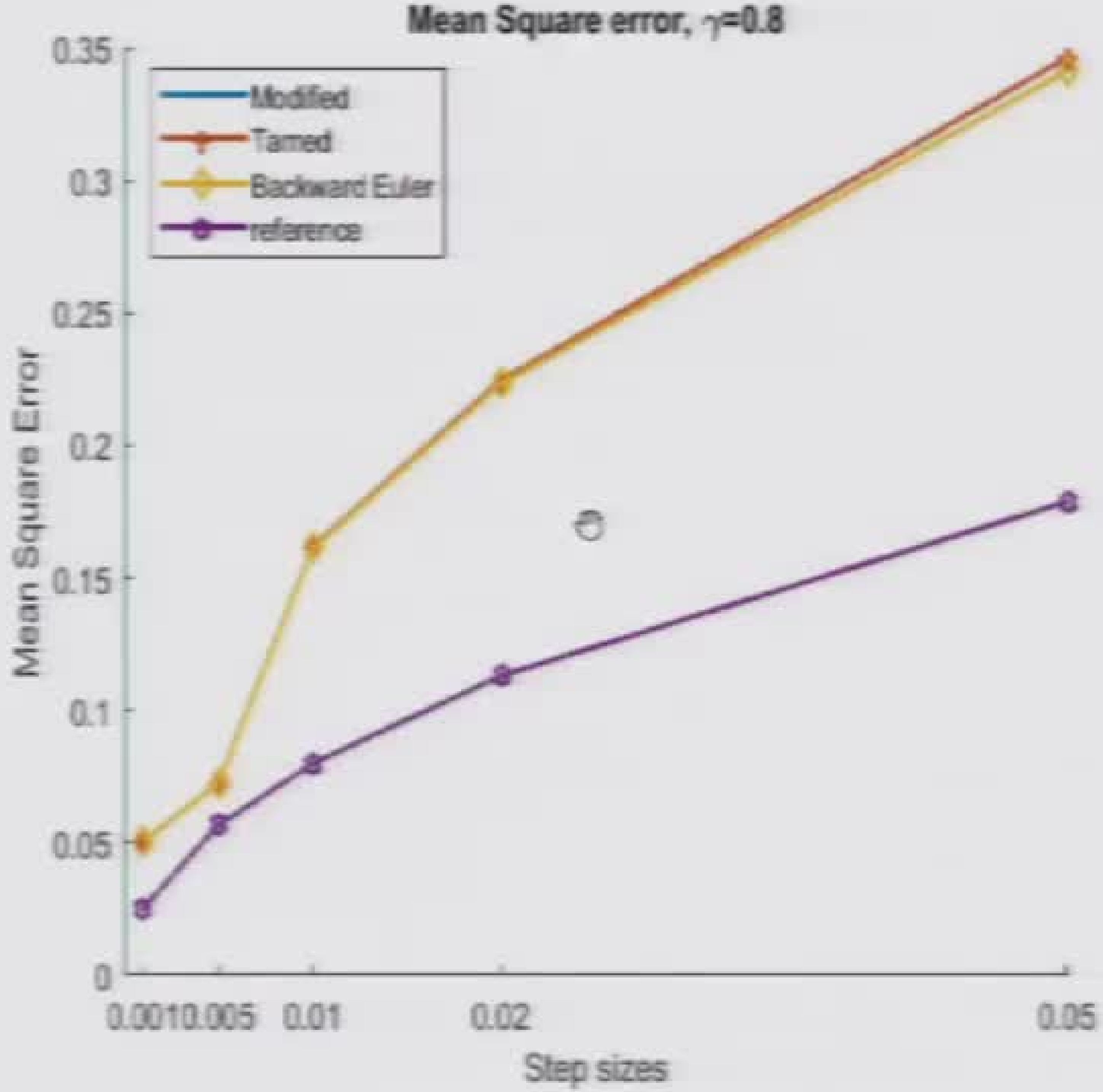

Figure: Example 2 with $\gamma = 0.8$ in different stepsizes

Result

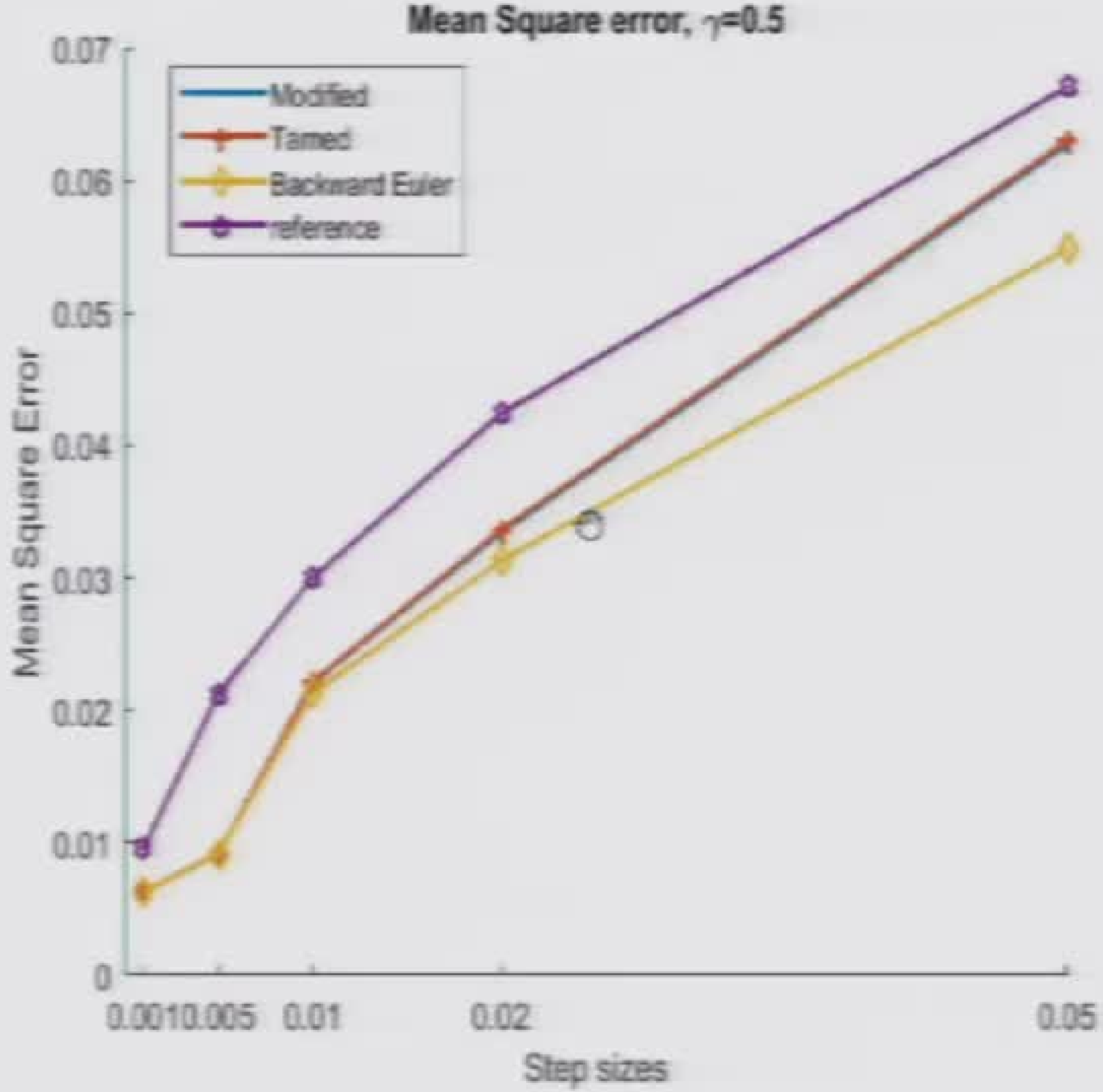

Figure: Example 1 with $\gamma = 0.5$ in different stepsizes

Numerical Schemes (Cont.)

Now, we consider several cases of $\bar{b}(x)$:

$$a)\quad \bar{b}(y) = \frac{b(y)}{1 + |y|^{\alpha}\,\Delta t}.$$

$$b)\quad \bar{b}(y) = \frac{\tanh(b(y)\Delta t)}{\Delta t}.$$

Numerical Schemes

Let $\xi_k \sim \mathcal{N}(0,1)$. We test the numerical examples with the following schemes:

$$X_{t_{k+1}} = \left| X_{t_k} + \frac{(1 - X_{t_k}^3)\Delta t}{1 + |X_{t_k}|^3 \Delta t} + X_{t_k}^{\gamma} \sqrt{\Delta t} \xi_k \right|, \quad (5)$$

$$X_{t_{k+1}} = \left| X_{t_k} + \tanh(\Delta t(1 - X_{t_k}^3)) + X_{t_k}^{\gamma} \sqrt{\Delta t} \xi_k \right|, \quad (6)$$

$$X_{t_{k+1}} = X_{t_k} + \Delta t(1 - X_{t_{k+1}}^3) + \sigma X_{t_k}^{\gamma} \sqrt{\Delta t} \xi_k. \quad (7)$$

Numerical Example 2

Example (Two factor Heston Model)

Consider the following SDEs

$$
\begin{aligned}
dX_t = & \quad (1 - X_t^3)dt + X_t^\gamma dW_t^1, \\
dS_t = & \quad \mu S_t dt + \sqrt{X_t} S_t (\rho dW_t^1 + \sqrt{1 - \rho^2} dW_t^2).
\end{aligned} \tag{8}
$$

where W_t^1 and W_t^2 are two independent standard Brownian motions, $\mu = 0.5, \rho = -0.7$ and initial values are $S_0 = 1, X_0 = 0.5$. Also, γ takes value in 0.5 or 0.8.

Assumptions

Consider

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma X_s^\gamma dW_s, \frac{1}{2} < \gamma < 1$$

(i) The initial condition is such that

$$\mathbb{E}[|X_0|^{2p}] \leq K < \infty, \quad \text{for all } p \geq 1. \tag{9}$$

(ii) There is a positive constant β such that

$$(x-y)(b(x)-b(y)) \leq \beta|x-y|^2. \tag{10}$$

(iii) There exist $\mathcal{K}_1 > 0$ and $\alpha \geq 1$ such that for $t \in [0, T]$

$$|b(x)-b(y)|^2 \leq \mathcal{K}_1(1+|x|^{2\alpha-2}+|y|^{2\alpha-2})|x-y|^2, \quad x, y \in \mathbb{R}. \tag{11}$$

(iv) The function $b(x)$ is positive when $x = 0$, i.e., $b(0) > 0$.

Define $\theta(t) = \sup_{k\in\{1,2,\dots,N\}}\{t_k : t_k \le t\}$. Let

$$Z_t = Y_{\theta(t)} + \bar{b}(Y_{\theta(t)})(t-\theta(t)) + \sigma(Y_{\theta(t)})(W_t - W_{\theta(t)}). \qquad (12)$$

and the numerical solution can be obtained by $Y_t = |Z_t|$ for $t \in [0, T]$.

Theorem

Suppose that Assumption holds and $X_t > 0$ when $t \in [0, T]$. Suppose also that $X_0 = x > 0$. Then there exists a positive constant C depending on σ, p and T but not on Δt such that

$$\sup_{t\in[0,T]} \mathbb{E}[|X_t - Y_t|^{2p}] \le C\Delta t^p, \quad p \ge 1.$$

Proof of Theorem

Lemma

Assume that $\{Z_t\}_{0\leq t\leq T}$ is given by (12) and assumptions 18 hold. If $\frac{1}{2} < \gamma < 1$ and Δt is sufficiently small, then

$$\sup_{t\in[0,T]} \mathbb{P}(Z_t \leq 0) \leq C\exp(-\Delta t^{1-2\gamma}).$$