heat exchange and exit times

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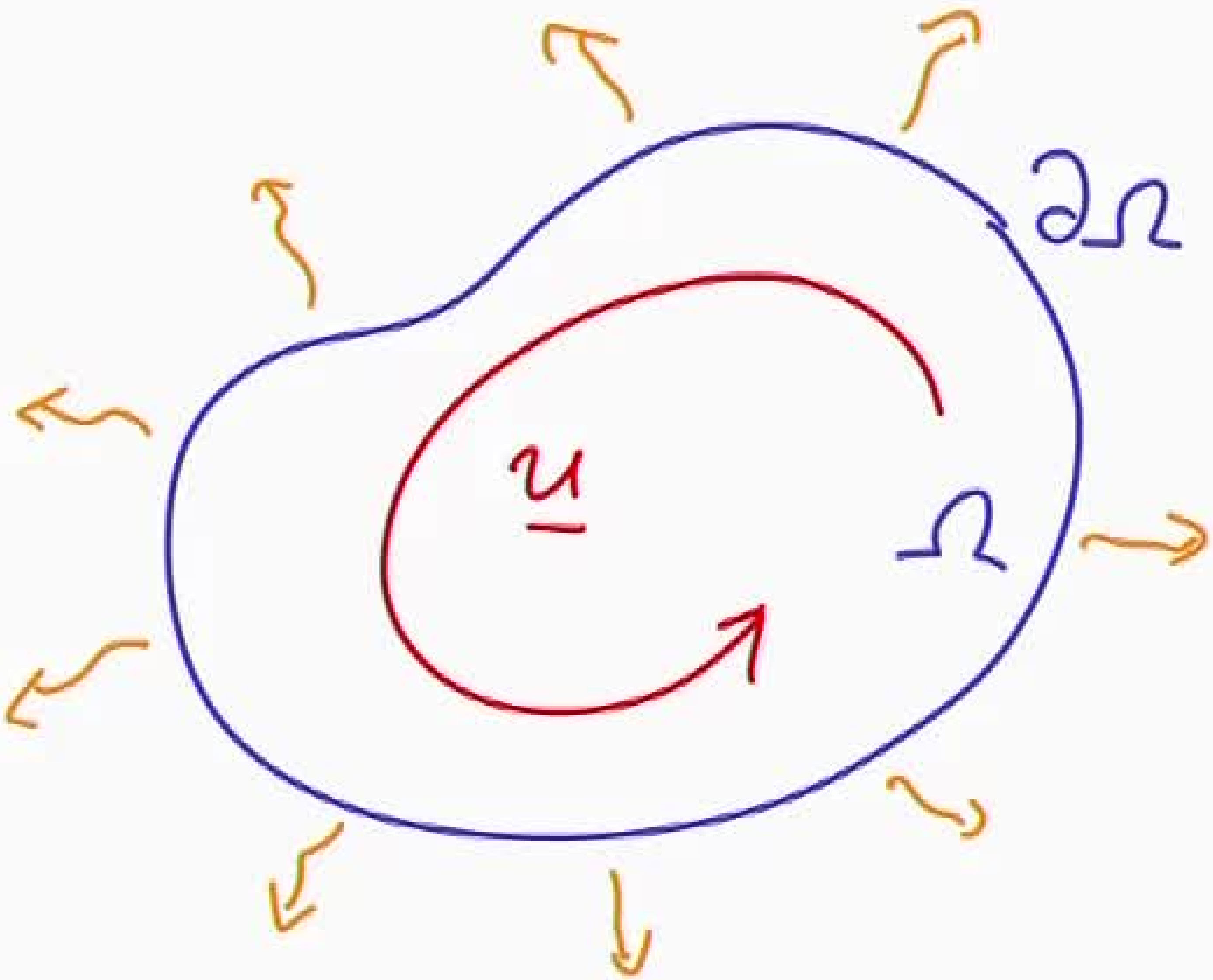
advection–diffusion equation in a bounded region

Advection and diffusion of heat in a bounded region Ω, with Dirichlet boundary conditions:

$$\partial_t \theta + \boldsymbol{u} \cdot \nabla \theta = D \Delta \theta, \qquad \boldsymbol{u} \cdot \hat{\boldsymbol{n}}|_{\partial\Omega} = 0, \qquad \theta|_{\partial\Omega} = 0,$$

with $\nabla \cdot \boldsymbol{u} = 0$ and $\theta(\boldsymbol{x}, t) \geq 0$.

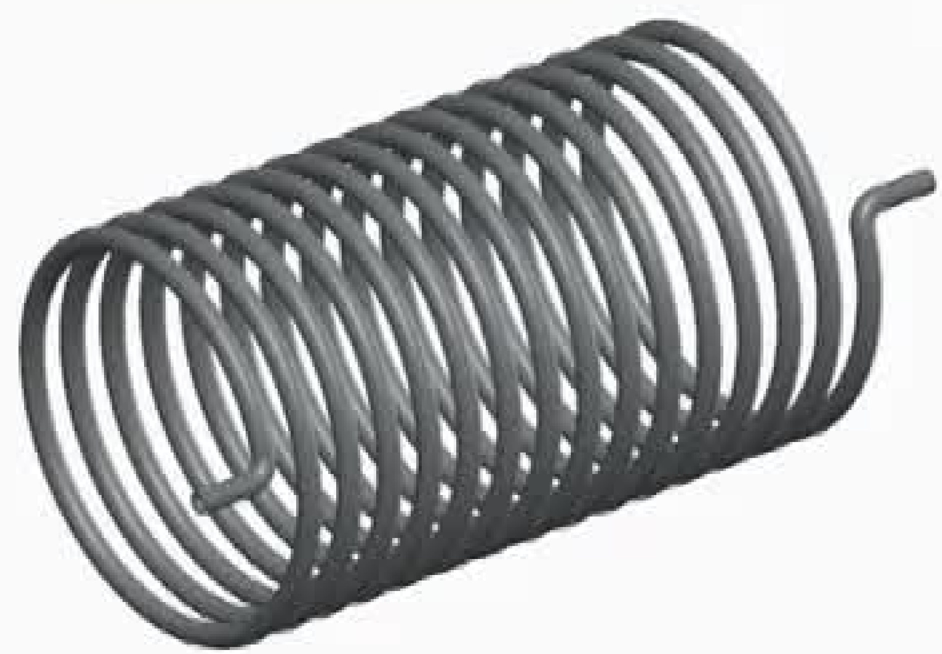


This is the heat exchanger configuration: given an initial distribution of heat, it is fluxed away through the cooled boundaries.

This happens through diffusion (conduction) alone, but is greatly aided by stirring.

heat exchangers

Our domain will be a 2D cross-section of a traditional coil.



heat flux

Write $\langle \cdot \rangle$ for an integral over Ω.

$$\langle \cdot \rangle := \int_\Omega \cdot \, \mathrm{d}V$$

The rate of heat loss is equal to the flux through the boundary $\partial\Omega$:

$$\partial_t \langle \theta \rangle = D \int_{\partial\Omega} \nabla\theta \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S =: -F[\theta] \leq 0. \qquad *$$

Goal: find velocity fields $\boldsymbol{u}$ that maximize the heat flux.

Note that * is not so good for this, since velocity does not appear.

The role of $\boldsymbol{u}$ is to increase gradients near the boundary. What it does internally is not directly relevant. This is in contrast to the traditional Neumann IVP (chaotic mixing, etc).

related problem: mean exit time

Take steady velocity $\boldsymbol{u}(\boldsymbol{x})$. The mean exit time $\tau(\boldsymbol{x})$ of a Brownian particle initially at $\boldsymbol{x}$ satisfies

$$-\boldsymbol{u} \cdot \nabla \tau = D\Delta\tau + 1, \qquad \tau|_{\partial\Omega} = 0,$$

This is a steady advection–diffusion equation with velocity $-\boldsymbol{u}$ and source 1.

Intuitively, a small integrated mean exit time $\langle \tau \rangle = \|\tau\|_1$ implies that the velocity is efficient at taking heat out of the system.

The mean exit time equation is much nicer than the equation for the concentration: it is steady, and it applies for any initial concentration $\theta_0(\boldsymbol{x})$.

relationship between exit time and mean temperature

Recall that $\langle \cdot \rangle$ is an integral over space, and take $\langle \theta_0 \rangle = 1$. The quantity

$$\int_0^\infty \langle \theta \rangle \, \mathrm{d}t$$

is a cooling time. Smaller is better for good heat exchange.

We have the rigorous bounds

$$\int_0^\infty \langle \theta \rangle \, \mathrm{d}t \leq \|\tau\|_\infty \qquad \int_0^\infty \langle \theta \rangle \, \mathrm{d}t \leq \|\tau\|_1 \, \|\theta_0\|_\infty .$$

Thus, decreasing a norm like $\|\tau\|_1$ or $\|\tau\|_\infty$ will typically decrease the cooling time, as expected.

does stirring always help?

[Iyer, G., Novikov, A., Ryzhik, L., & Zlatoš, A. (2010). *SIAM J. Math. Anal.* **42** (6), 2484–2498]

Theorem (Iyer *et al.* 2010)

$\Omega \in \mathbb{R}^n$ bounded, $\partial\Omega \in C^1$. Then

$$\|\tau\|_{L^p(\Omega)} \leq \|\tau_0\|_{L^p(\mathcal{B})}, \qquad 1 \leq p \leq \infty,$$

where $\mathcal{B} \in \mathbb{R}^n$ is a ball of the same volume as Ω, and τ_0 is the 'purely diffusive' solution, $0 = D\Delta\tau_0 + 1$ on $\mathcal{B}$.

That is, measured in any norm, the exit time is maximized for a disk with no stirring. So for a disk stirring always helps, or at least isn't harmful.

They also prove that, surprisingly, if Ω is not a disk, then it's always possible to make $\|\tau\|_{L^\infty(\Omega)}$ increase by stirring. (Related to unmixing flows? [IMA 2010 gang; see review Thiffeault (2012)])

optimization problem

Let's formulate an optimization problem to find the best incompressible $\boldsymbol{u}$.

Advection–diffusion operator and its adjoint:

$$\mathcal{L} := \boldsymbol{u} \cdot \nabla - D\Delta, \qquad \mathcal{L}^\dagger = -\boldsymbol{u} \cdot \nabla - D\Delta\,.$$

Minimize $\langle \tau \rangle$ over steady $\boldsymbol{u}(\boldsymbol{x})$ with fixed total kinetic energy $E = \frac{1}{2}\|\boldsymbol{u}\|_2^2$.

The functional to optimize:

$$\mathcal{F}[\tau, \boldsymbol{u}, \vartheta, \mu, p] = \langle \tau \rangle - \langle \vartheta \left(\mathcal{L}^\dagger \tau - 1\right)\rangle + \tfrac{1}{2}\mu\left(\|\boldsymbol{u}\|_2^2 - 2E\right) - \langle p \nabla \cdot \boldsymbol{u} \rangle$$

Here ϑ, μ, p are Lagrange multipliers.

Euler–Lagrange equations

Introduce streamfunction ψ to satisfy $\nabla \cdot \boldsymbol{u} = 0$:

$$u_x = -\partial_y \psi \,, \qquad u_y = \partial_x \psi .$$

The variational problem gives the Euler–Lagrange equations

$$\begin{aligned} \mathcal{L}^\dagger \tau &= 1, & \tau|_{\partial\Omega} &= 0; \\ \mathcal{L}\vartheta &= 1, & \vartheta|_{\partial\Omega} &= 0; \\ \mu \Delta \psi &= J(\tau, \vartheta), & \psi|_{\partial\Omega} &= 0; \\ \langle |\nabla \psi|^2 \rangle &= 2E, \end{aligned}$$

with the Jacobian

$$J(\tau, \vartheta) := (\nabla \tau \times \nabla \vartheta) \cdot \hat{\boldsymbol{z}} \,.$$

a judicious transformation

Transform to new functions η, ξ

$$\tau = \tau_0 + \tfrac{1}{2}(\eta + \xi), \qquad \vartheta = \tau_0 + \tfrac{1}{2}(\eta - \xi)$$

where recall that τ_0 is the solution without flow (purely diffusive).

Then by using the Euler–Lagrange equations we can eventually show

$$\langle \tau \rangle = \langle \tau_0 \rangle - \tfrac{1}{4}\langle |\nabla \xi|^2 \rangle - \tfrac{1}{4}\langle |\nabla \eta|^2 \rangle.$$

Hence, solutions to E–L equations cannot make $\langle \tau \rangle$ increase. So stirring is always better than not stirring.

the nonlinear *ansatz*

For a disk the purely diffusive solution is $\tau_0 = \frac{1}{4}(1 - r^2)$. We then make the *ansatz*

$$\xi = \sqrt{2\mu}\, B(r) \cos m\theta, \qquad \eta = B(r) \sin m\theta, \qquad \psi = \xi/\sqrt{2\mu},$$

and look for solutions of that form.

Inserting this into the full system gives solutions provided the radial functions $B(r)$ satisfy the nonlinear eigenvalue problem

$$\boxed{r^2 B'' + rB' + (r^2\lambda - m^2)B = \tfrac{1}{2} m^2 B^3, \quad \lambda = m/\sqrt{2\mu}.}$$

The left-hand side is Bessel's equation.

Note that it is rather unusual for such a linear-type ansatz to give nonlinear solutions. We also have no guarantee that this is the true optimal solution.

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small-E solutions

For small energy E, exact solution in terms of Bessel functions $J_m(\rho_{mn} r)$, where ρ_{mn} are zeros:

$$\langle\tau\rangle/\langle\tau_0\rangle = 1 - (4m^2/\pi\rho_{mn}^4)E + \mathrm{O}(E^2).$$

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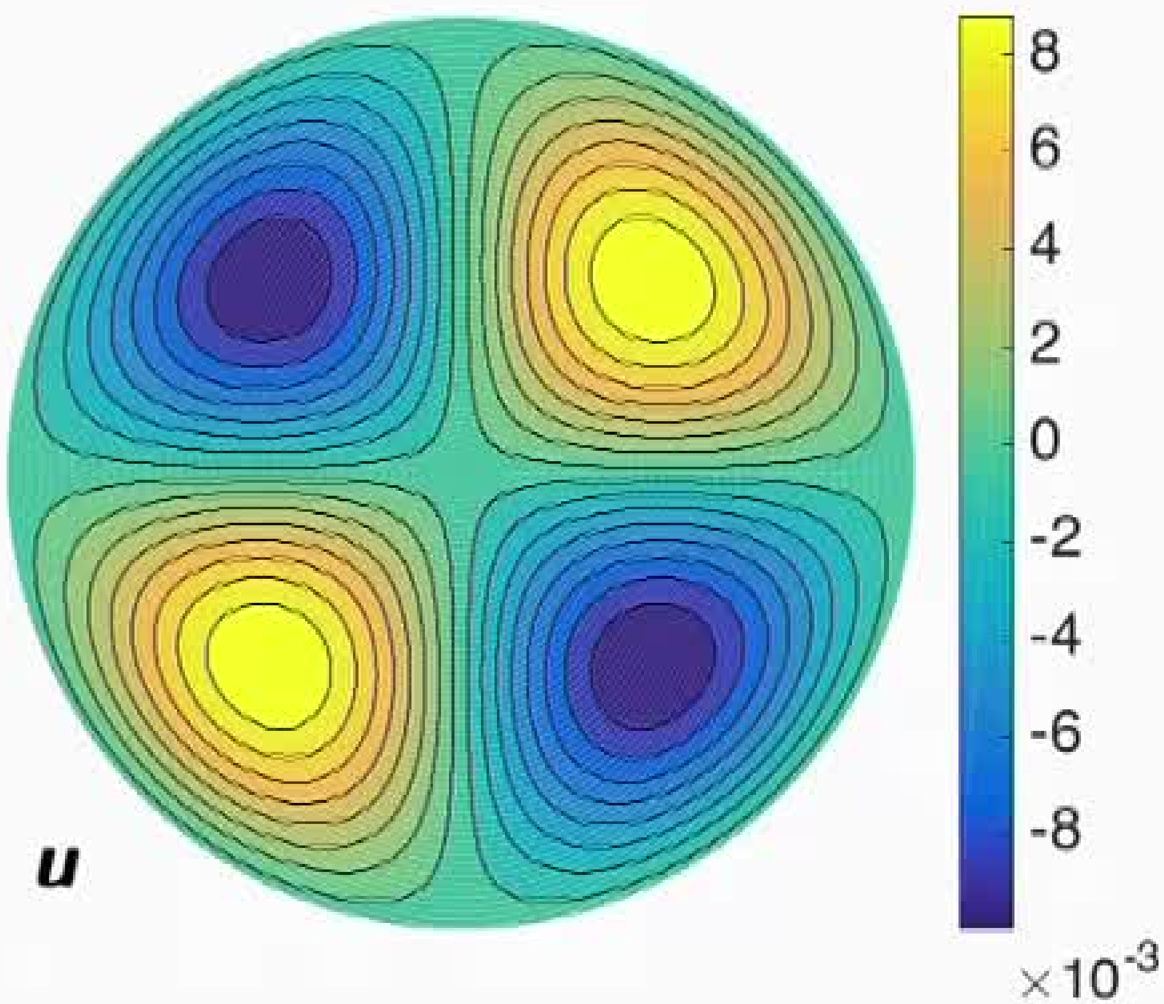
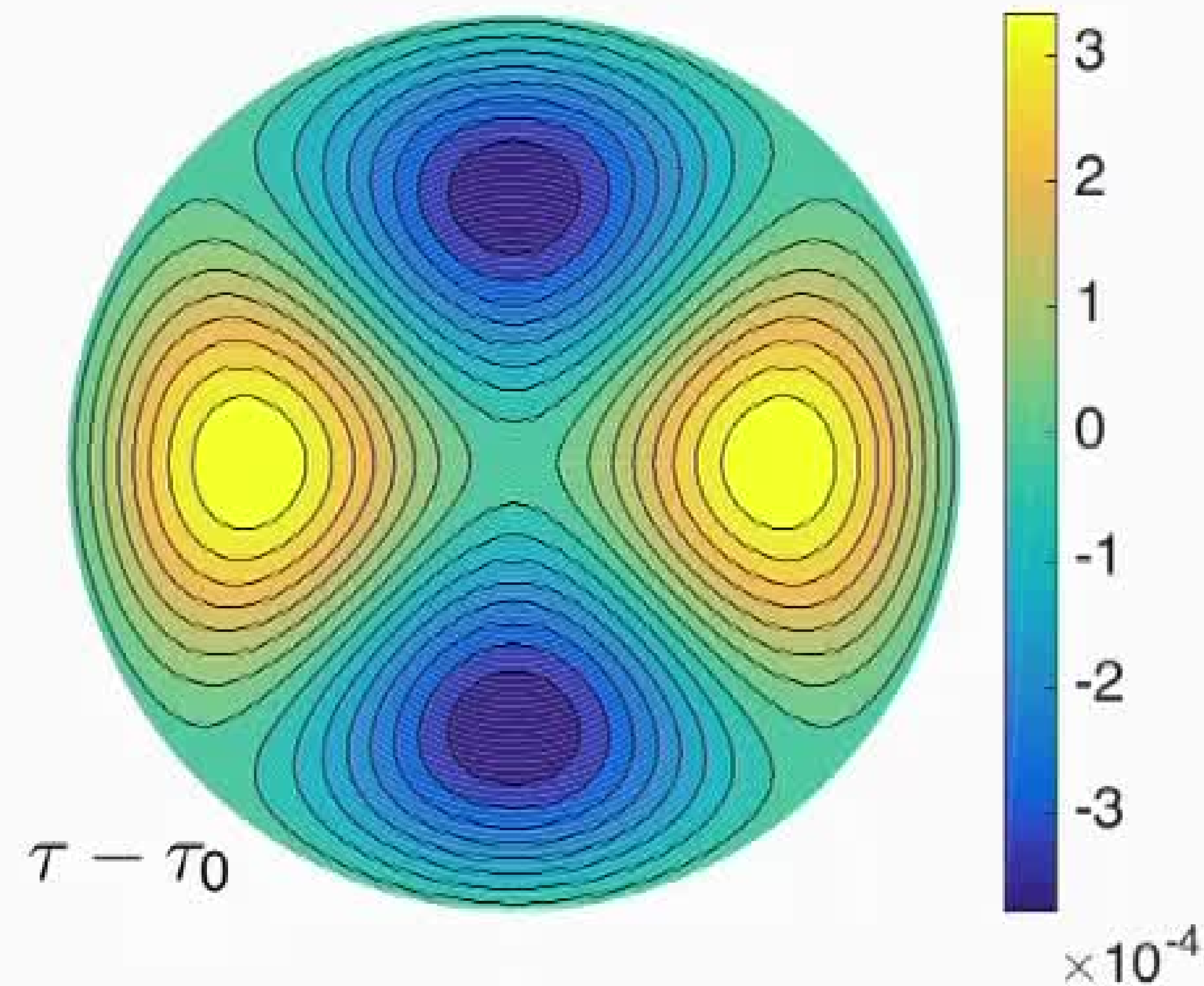
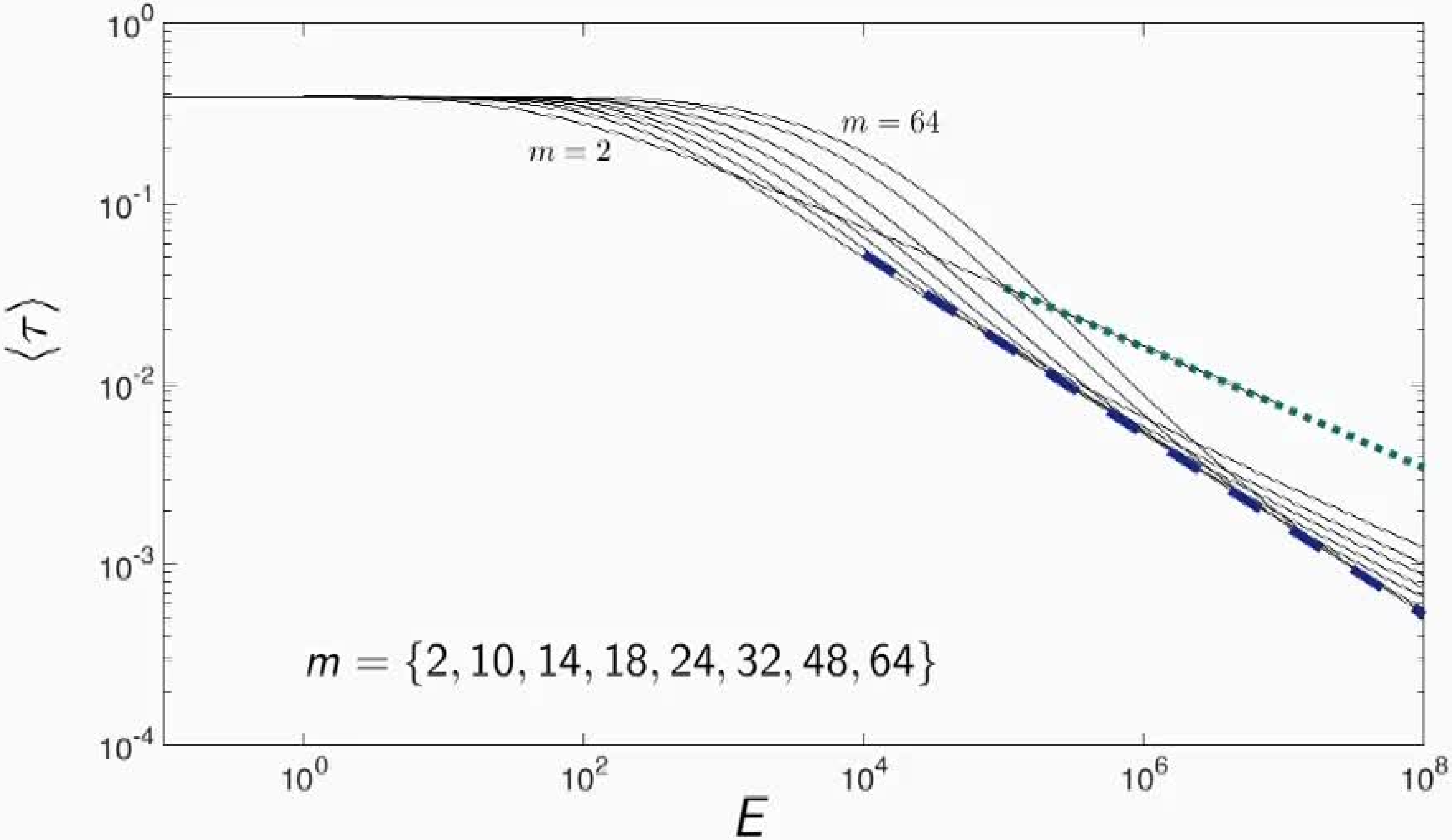




Numerical solution with Matlab's **bvp5c**, using a continuation method:



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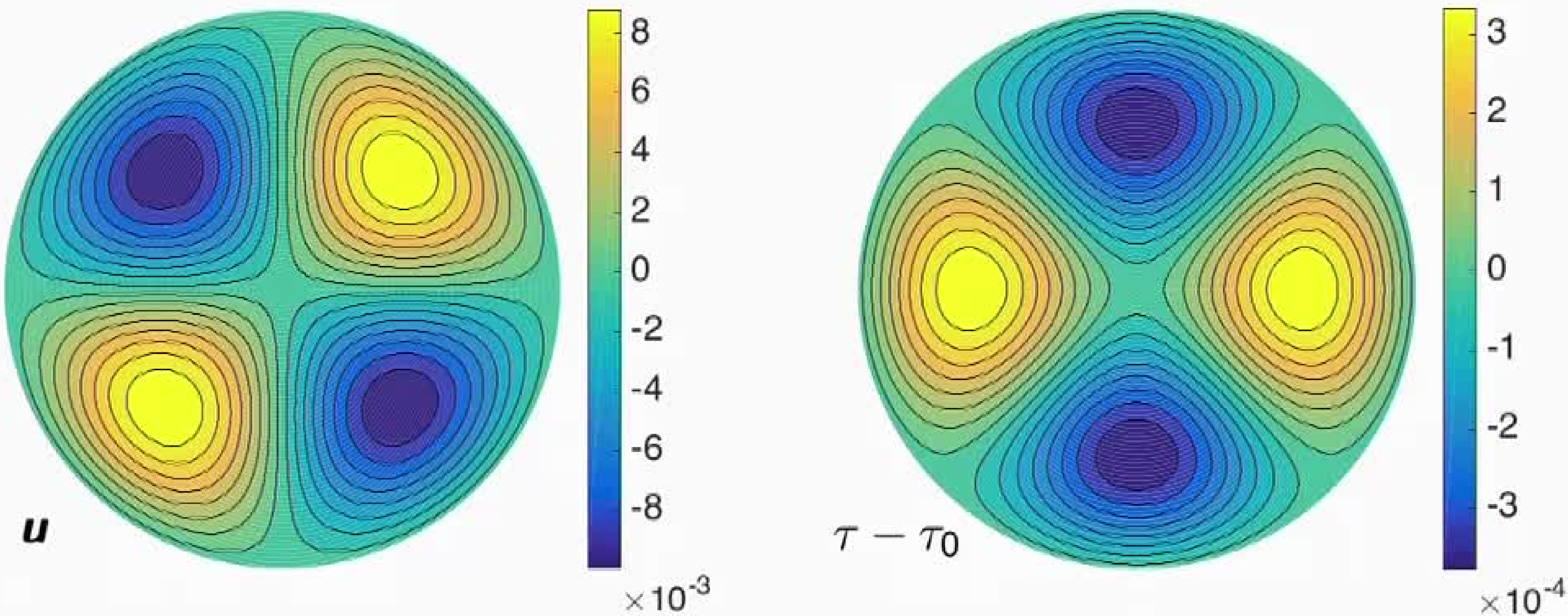

large E case: numerics

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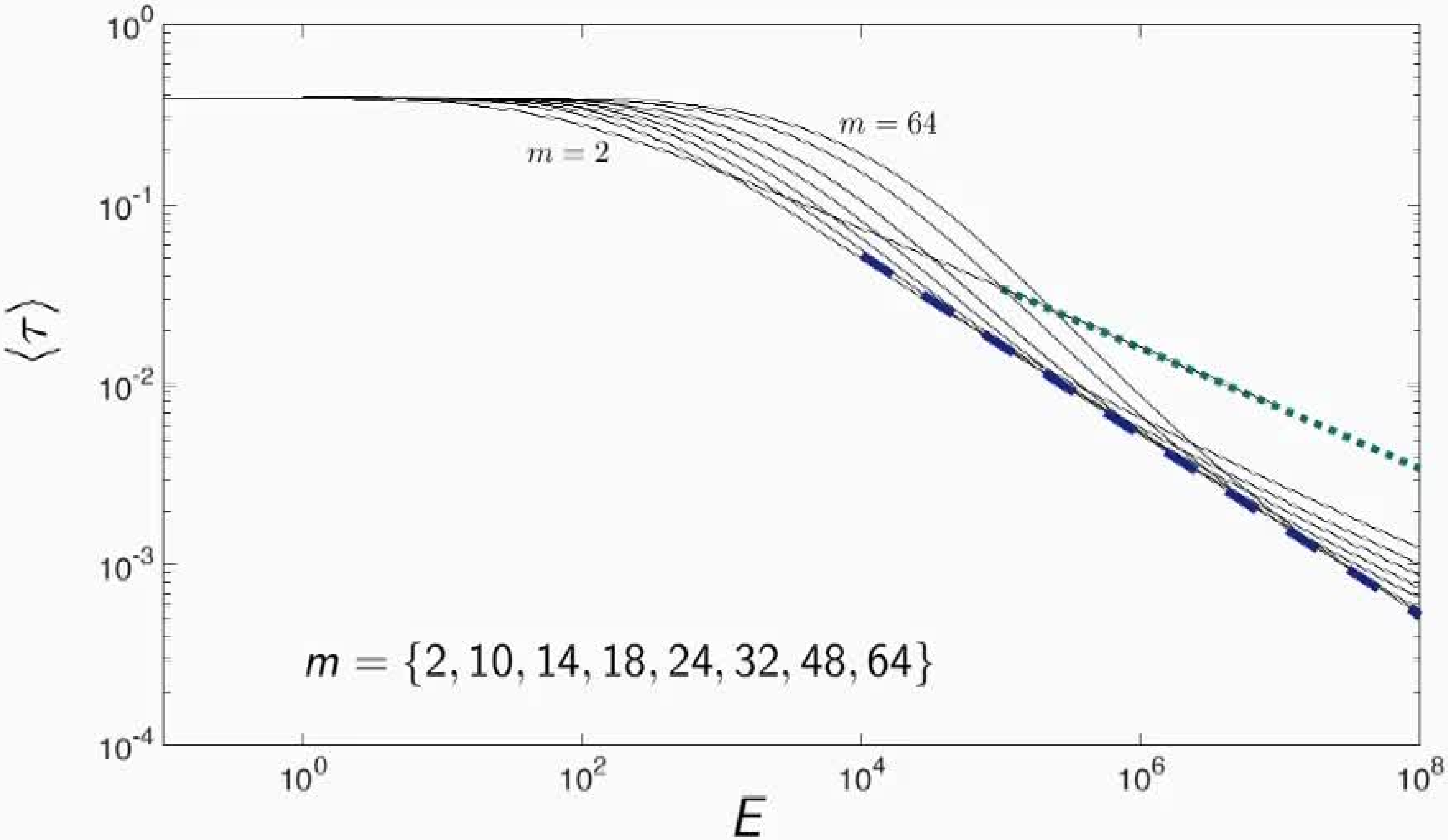
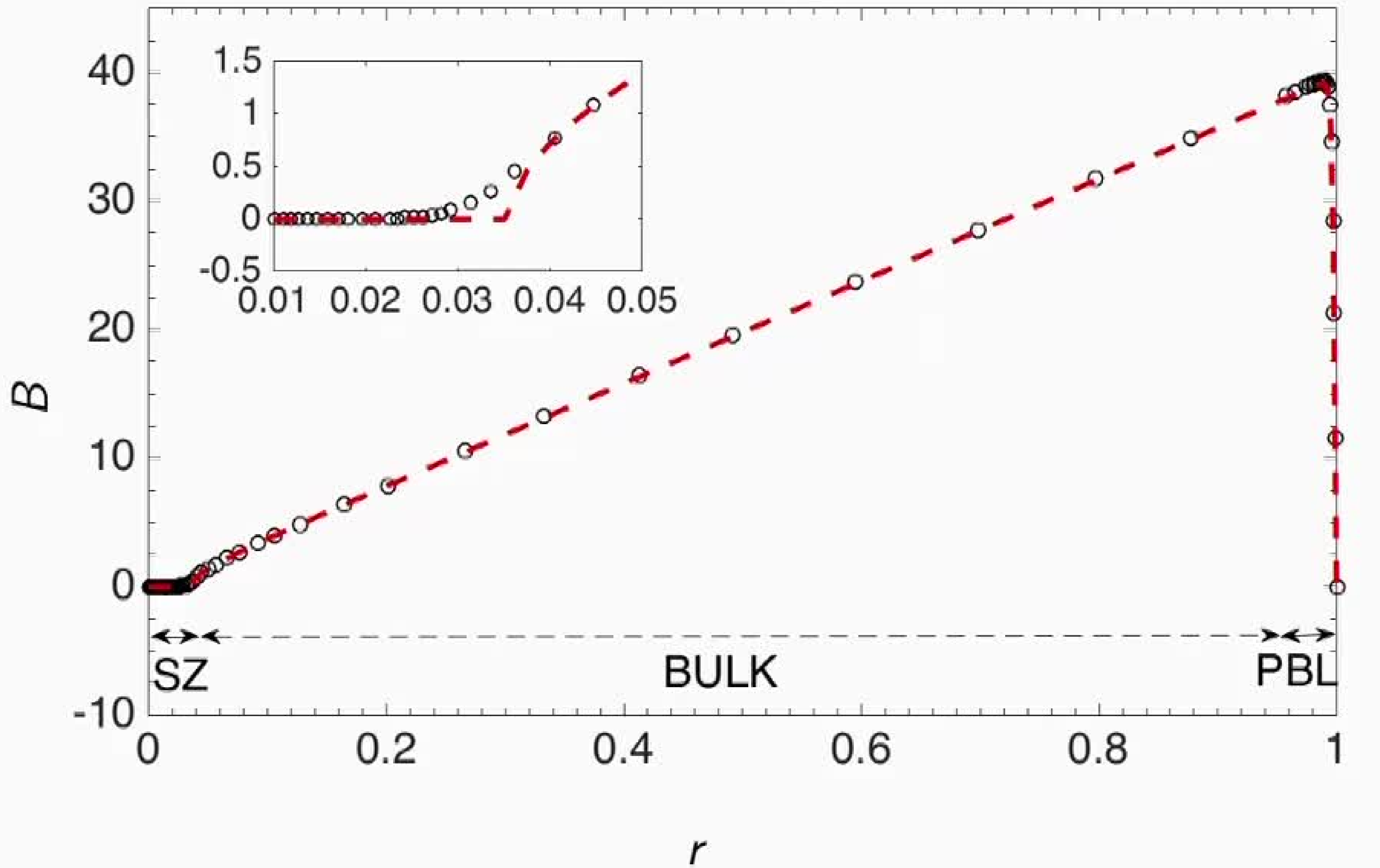


Larger m worse at small E, then better, then maybe worse again?

structure of the radial solution $B(r)$ for large E



large-E asymptotics: outer solution

Rescaled variables $B = E^{\alpha}\,\tilde{B}$ and $\lambda = E^{\beta}\,\tilde{\lambda}$:

$$r^2\tilde{B}''E^{\alpha} + r\tilde{B}'E^{\alpha} + r^2\tilde{\lambda}\tilde{B}E^{\alpha+\beta} - m^2\tilde{B}E^{\alpha} = \tfrac{1}{2}m^2\,\tilde{B}^3E^{3\alpha}.$$

Outside the boundary layer, the large-E balance must occur between the terms $r^2\tilde{\lambda}\tilde{B}E^{\alpha+\beta}$ and $\tfrac{1}{2}m^2\,\tilde{B}^3E^{3\alpha}$, so $\beta = 2\alpha$.

This gives the outer solution

$$B_{\text{outer}} = E^{\alpha}\,\tilde{B} = \sqrt{2/m^3}\,\tilde{\lambda}\,E^{\alpha}\,r\,.$$

(This does not include the stagnation zone in the center. Neglect for now.)

Cannot satisfy $B_{\text{outer}}(1) = 0$: need boundary layer.

large-E asymptotics: inner solution

Inner variable $r = 1 - \epsilon\rho$:

$$\frac{(1-\epsilon\rho)^2}{\epsilon^2}\,\bar{B}''E^{\alpha} + \frac{(1-\epsilon\rho)}{\epsilon}\,\bar{B}'E^{\alpha} + (1-\epsilon\rho)^2\,\tilde{\lambda}\,\bar{B}\,E^{3\alpha} - m^2\,\bar{B}\,E^{\alpha} = \tfrac{1}{2}m^2\,\bar{B}^3\,E^{3\alpha}.$$

Dominant balance: highest derivative with $E^{\alpha} = \epsilon^{-1}$:

$$\bar{B}'' + \tilde{\lambda}\bar{B} = \tfrac{1}{2}m^2\,\bar{B}^3.$$

This has an exact tanh solution, which after matching with the outer solution as $\rho \to \infty$ gives

$$B_{\text{inner}} = \sqrt{2\tilde{\lambda}/m^2}\,E^{\alpha}\tanh\left(\sqrt{\lambda/2}\,\rho\right)$$