Eulerian geometric integration of fluids for computer graphics

Mathieu Desbrun

Applied Geometry Lab
Caltech

Discrete Differential Modeling

Computing thru finite-dim. version of continuous theory

- we leverage *differential geometric understanding* ... **for computational purposes**
  - geometry as a *guiding principle* to discretization
    - coordinate free and intrinsic representations
    - dynamics through discrete variational principles
  - of both academic and practical interests
    - conserved quantities, symmetries, structural identities

- Today: Eulerian simulation of fluids
  - circulation-preserving method of characteristics
  - integration via discrete volumorphisms

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p$$

$$\nabla \cdot u = 0$$

Spatial discretization

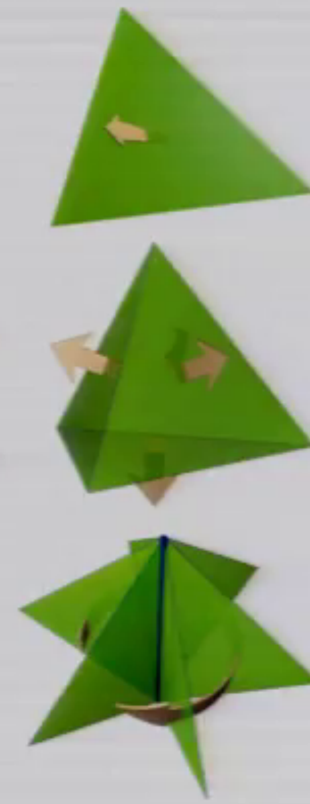
Eulerian grid

- domain discretization = *simplicial complex*
  - or any type of grid...

Spatial discretization via discrete forms

- fluxes through faces for velocity
  - intrinsic (coordinate-free) and Eulerian
- net flux for divergence
  - what comes in...must come out
- flux-based spin for vorticity
  - torque created on a "paddle wheel"

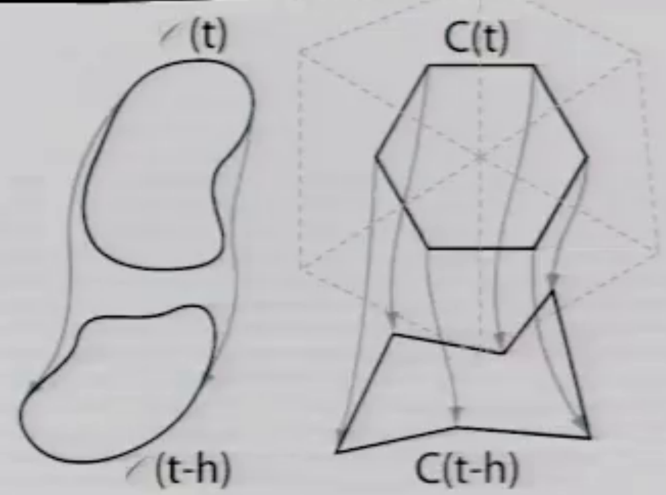
Discrete
Exterior
Calculus



Converting velocity to/from vorticity

- Laplace operator; discrete de Rham complex

Time Integration?

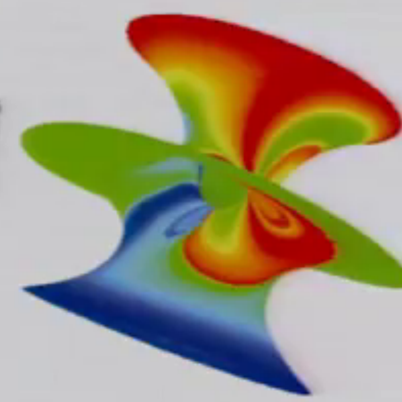


By preserving important structures!

- circulation preservation is key
  - crucial for visual impact
    - volutes in smoke
    - vortices in liquids

For each (n-2)-simplex

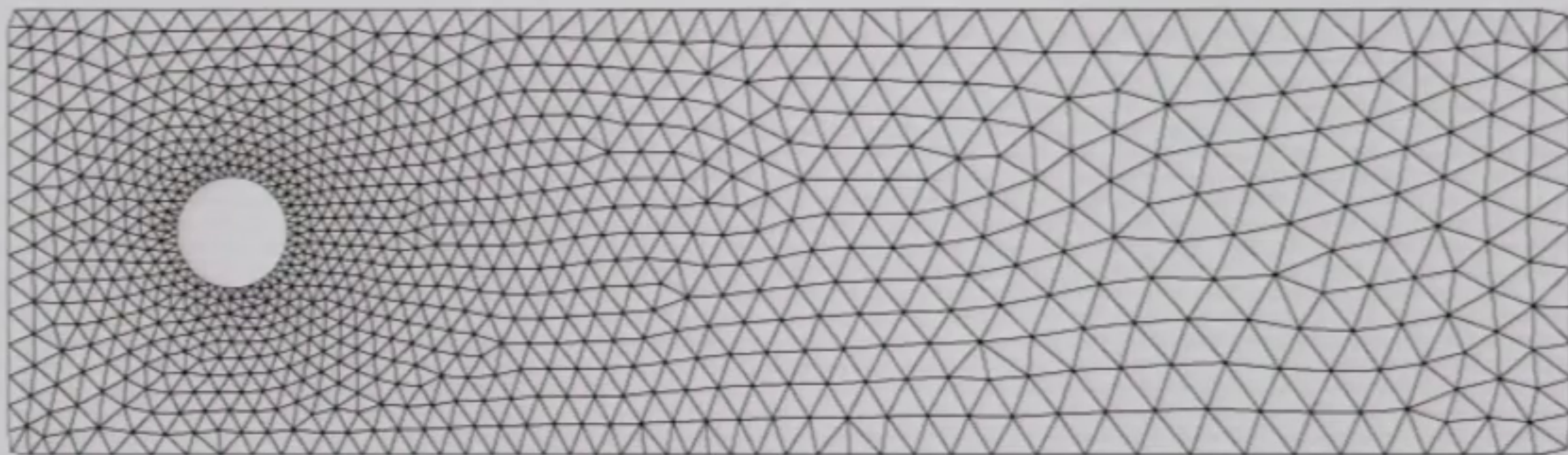
- backtrack loop in current velocity
  - deduce new circulation
    - i.e., new discrete vorticity
- find new velocity field
  - simple Poisson equation



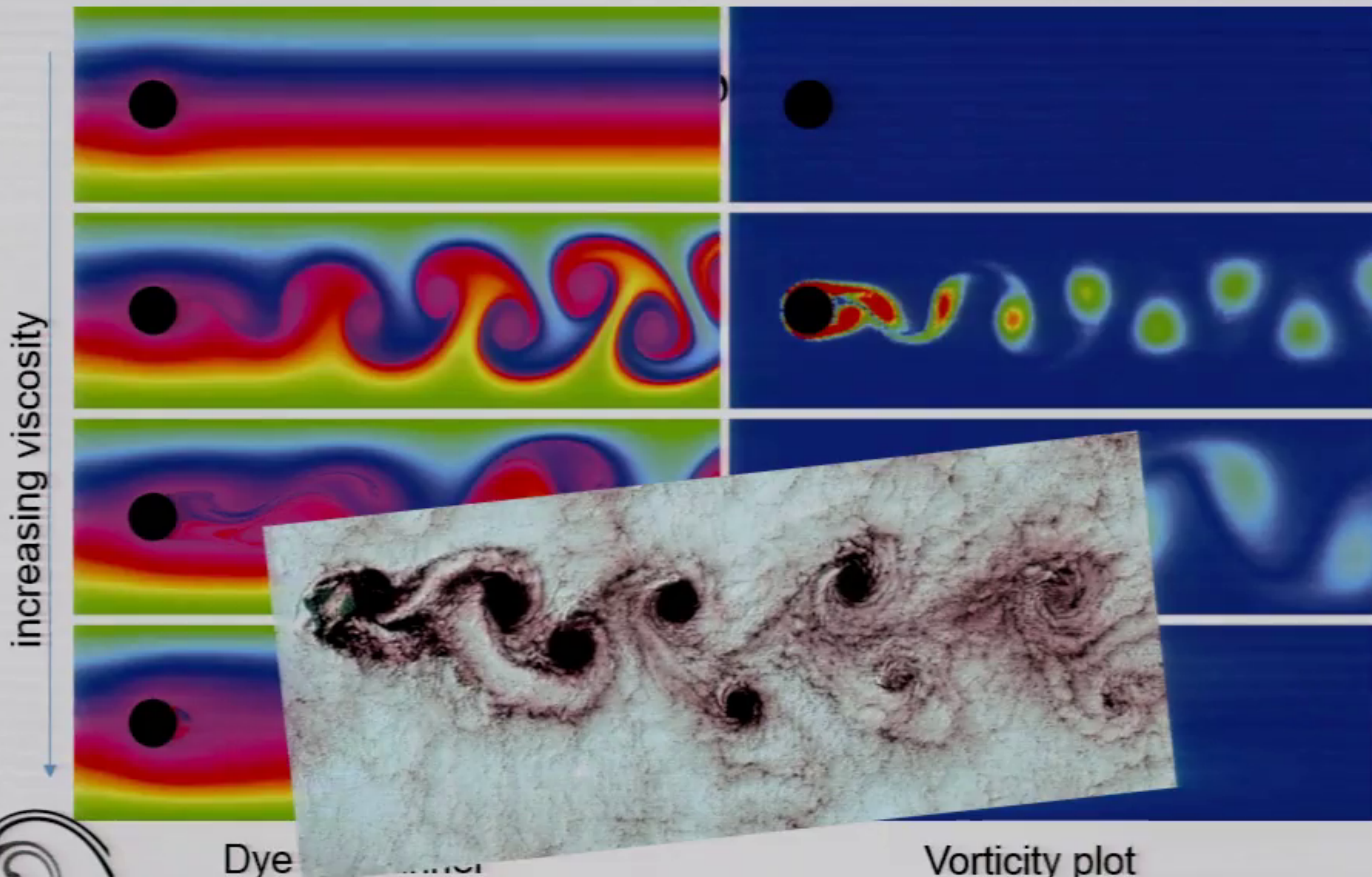
Circulation preserved for any discrete loop, even on curved spaces

Example: Vortex Shedding

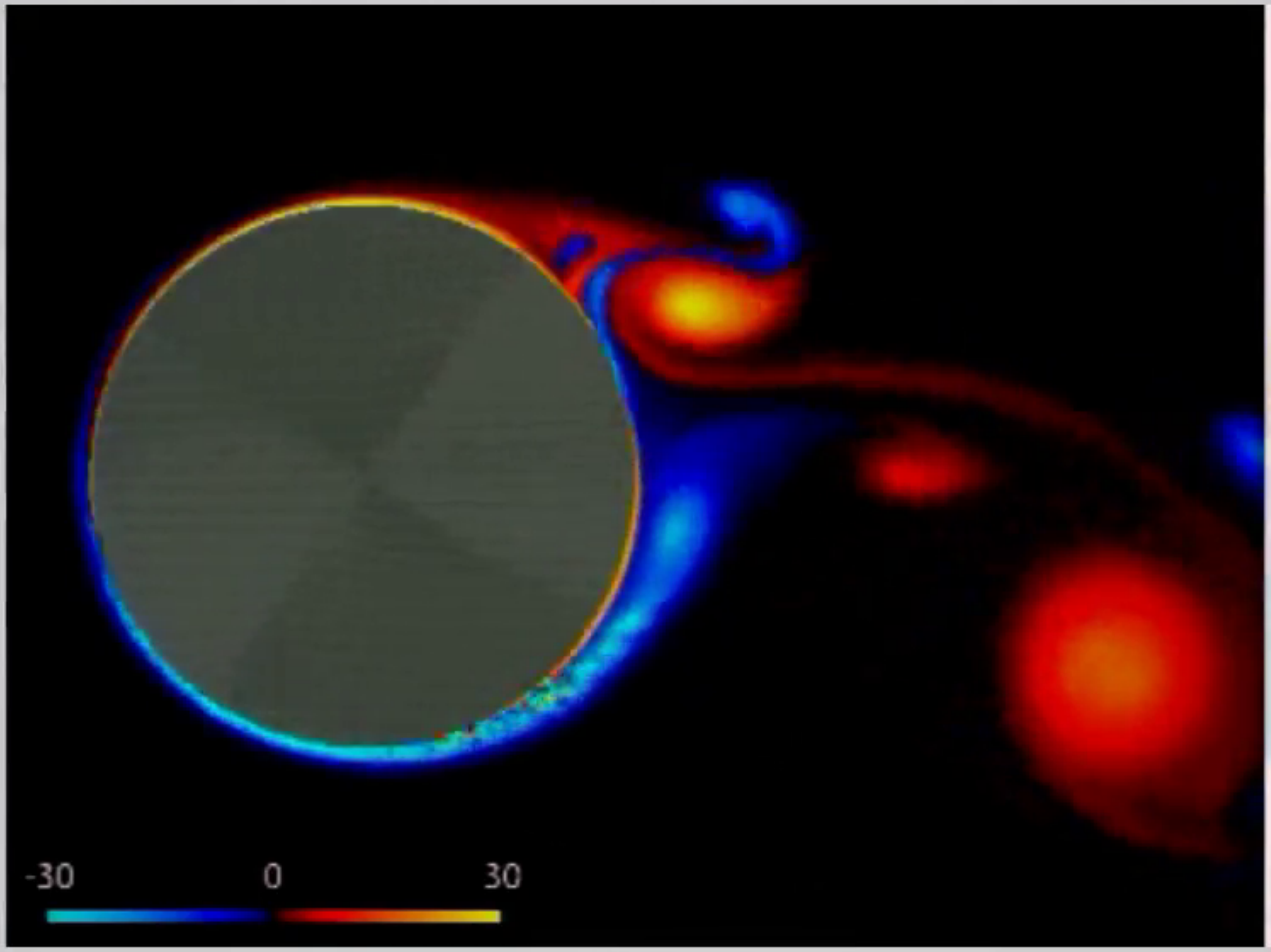
Traditional test: flow past a circle



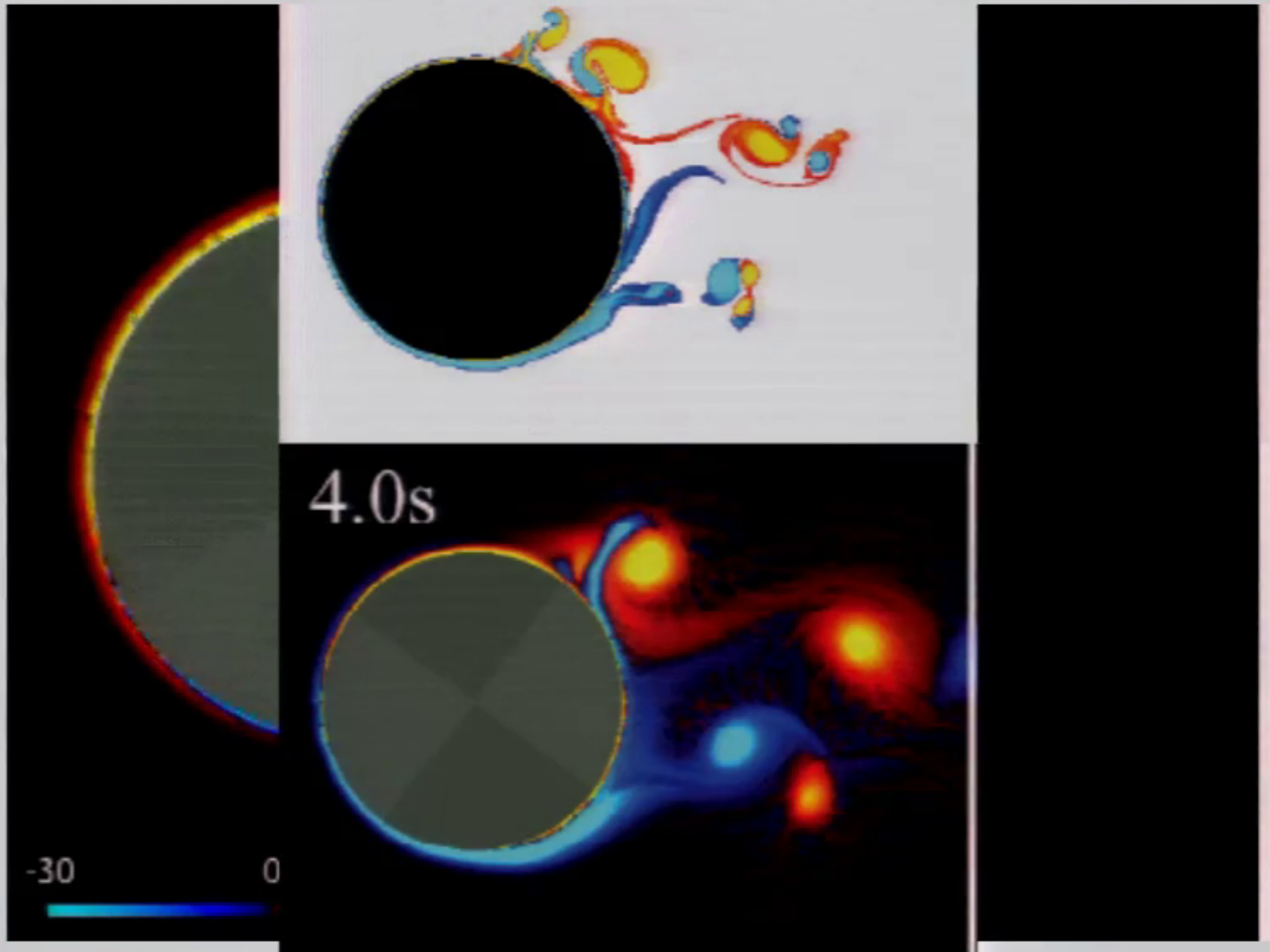
Example: Vortex Shedding



Rotating Obstacle [Re = 15,000]



Rotating Obstacle [Re = 15,000]



Smoking Bunny

Geometry of Fluids

Euler equations represent geodesic flows

- rarely used in CFD, yet geometrically appealing

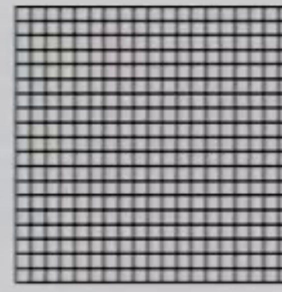
Lie group of volume preserving diffeomorphisms

- motion = geodesic on this group
  - [Lin, Newcomb, Bretherton, Arnold, Marsden et al.]

Many dynamical systems based on this idea

- extends nicely w/ semidirect product & advected params
  - Euler-Poincaré systems with advection [Holm,Marsden,Ratiu...]
  - magnetohydrodynamics & plasma
  - complex fluids (e.g., liquid crystal)
  - stratified rotating fluids, various Boussinesq approximations

Discrete Setup in a Nutshell [Pavlov]



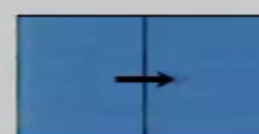
Discretizing Koopman's unitary operator

- matrices pushing forward scalar functions
  - function f stored as one value per cell F_i
  - matrix q encodes volumorphism through $qF \approx \varphi^* f$
- preserve constant functions & L_2 inner product of functions
  - reflecting mass preservation and volume preservation
- i.e., orthogonal, signed doubly stochastic matrices

$$\mathcal{D}(\mathbb{M}) = \{q \in \mathbb{R}^{N\times N} \mid q^T q = \mathrm{Id}, \sum_j q_{ij} = 1 \ \forall i\}$$

$$\mathfrak{D}(\mathbb{M}) = \{A \in \mathbb{R}^{N\times N} \mid A^T + A = 0, \sum_j A_{ij} = 0 \ \forall i\}$$

Use Lagrange-d'Alembert, Lin constraints

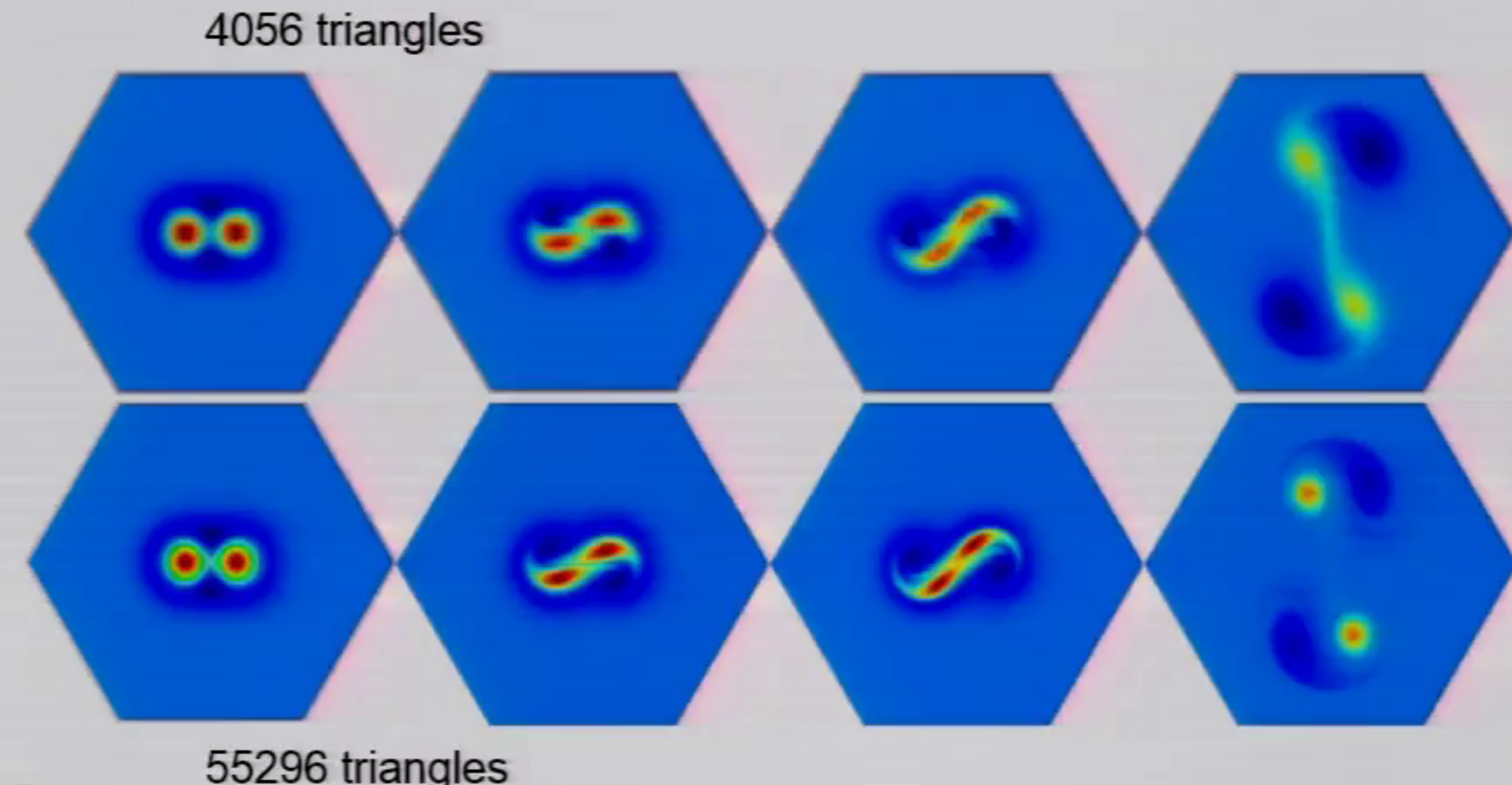


  - and non-holonomic constraints to keep velocity local
  - you get: $2h^2 \dot{A}_{ij} + [A, A^\flat]_{ij} + p_j - p_i = 0$

$$\dot{v}^\flat + \mathbf{L}_v v^\flat + \mathbf{d}p = 0$$

2D Obstacle Course

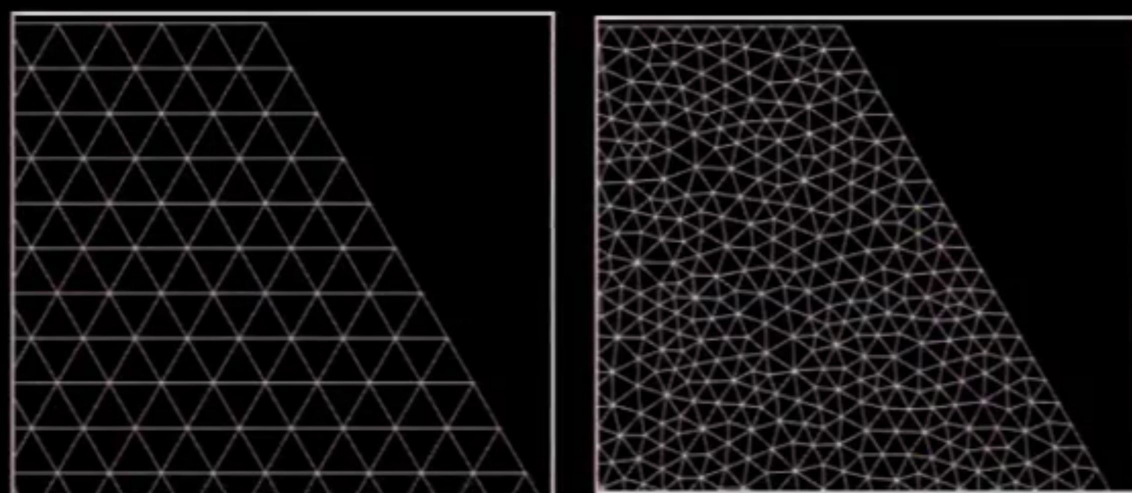
Two Taylor vortices at a distance near bifurcation



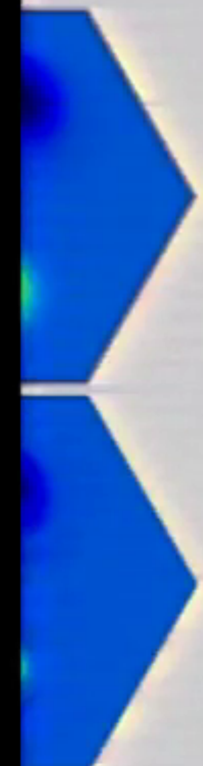
2D Obstacle Course

Two T [illegible] n

Varying Discretization

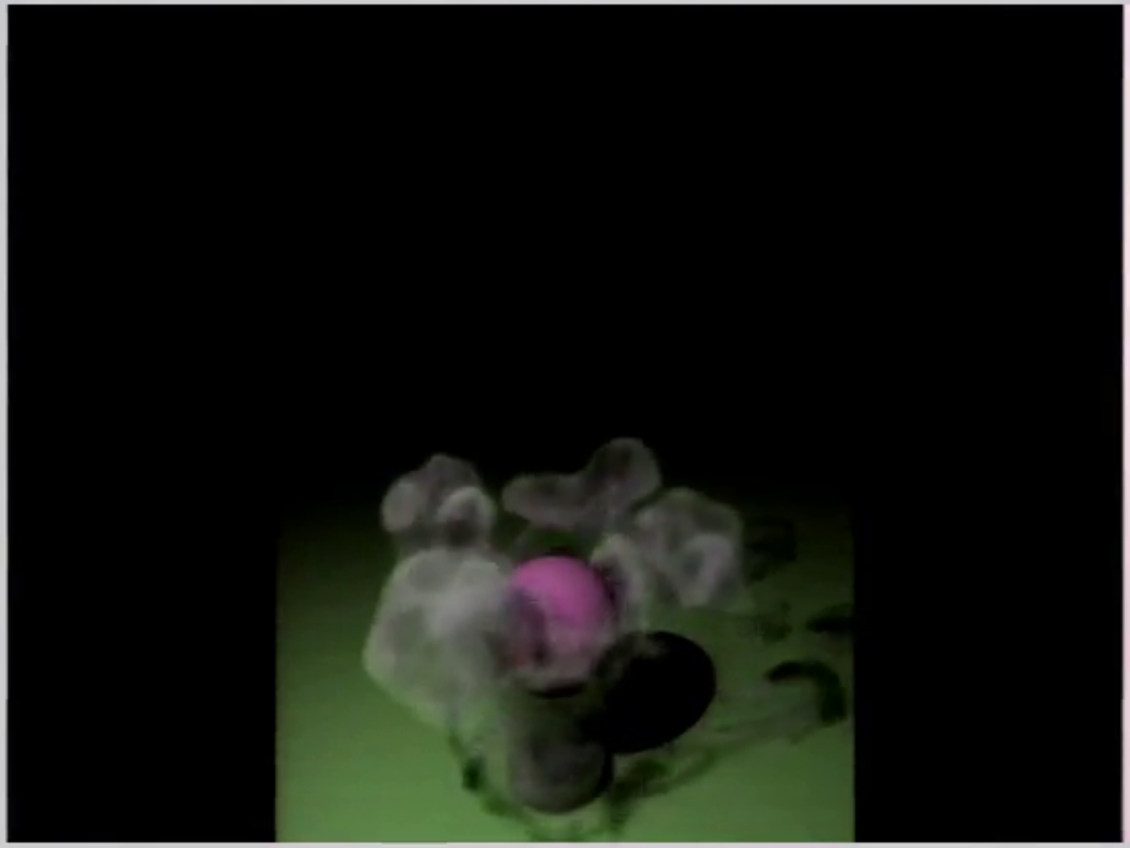


Because our integrator preserves energy exactly, the liveliness of the flow does not depend heavily on the particular choice of temporal or spatial discretization.



On 3D Tet Meshes [Mullen]

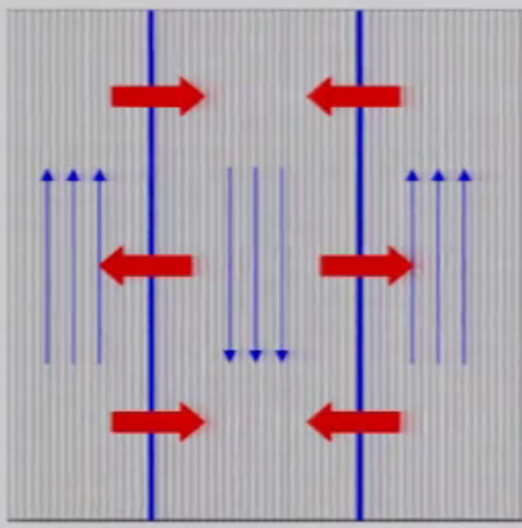
Heated smoke rises around round obstacle

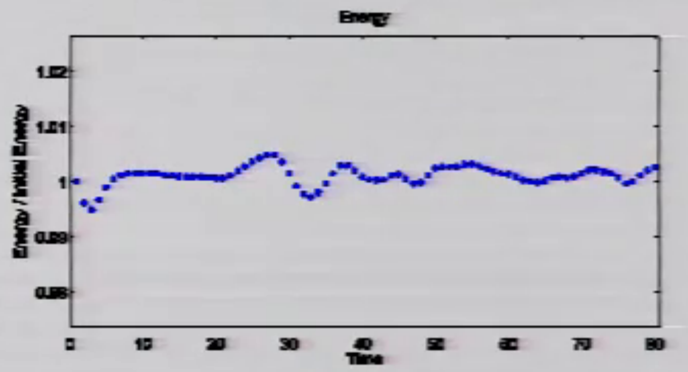


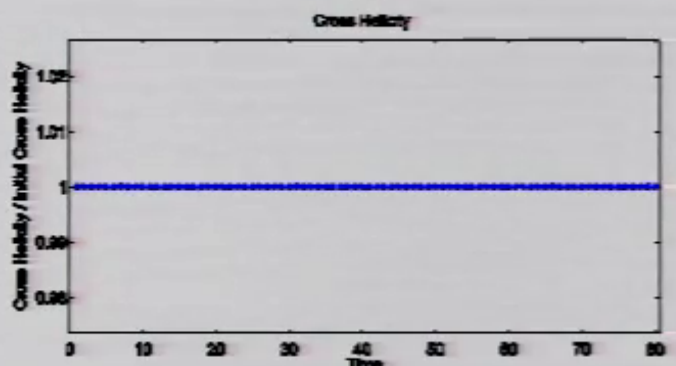
Discrete MagnetoHydroDynamics

As expected, key invariants preserved [Gawlik]

- energy conserved over long runs
- cross-helicity preserved exactly
- no spurious magnetic reconnection (same topology of field lines)
  - when magnetic resistivity is null



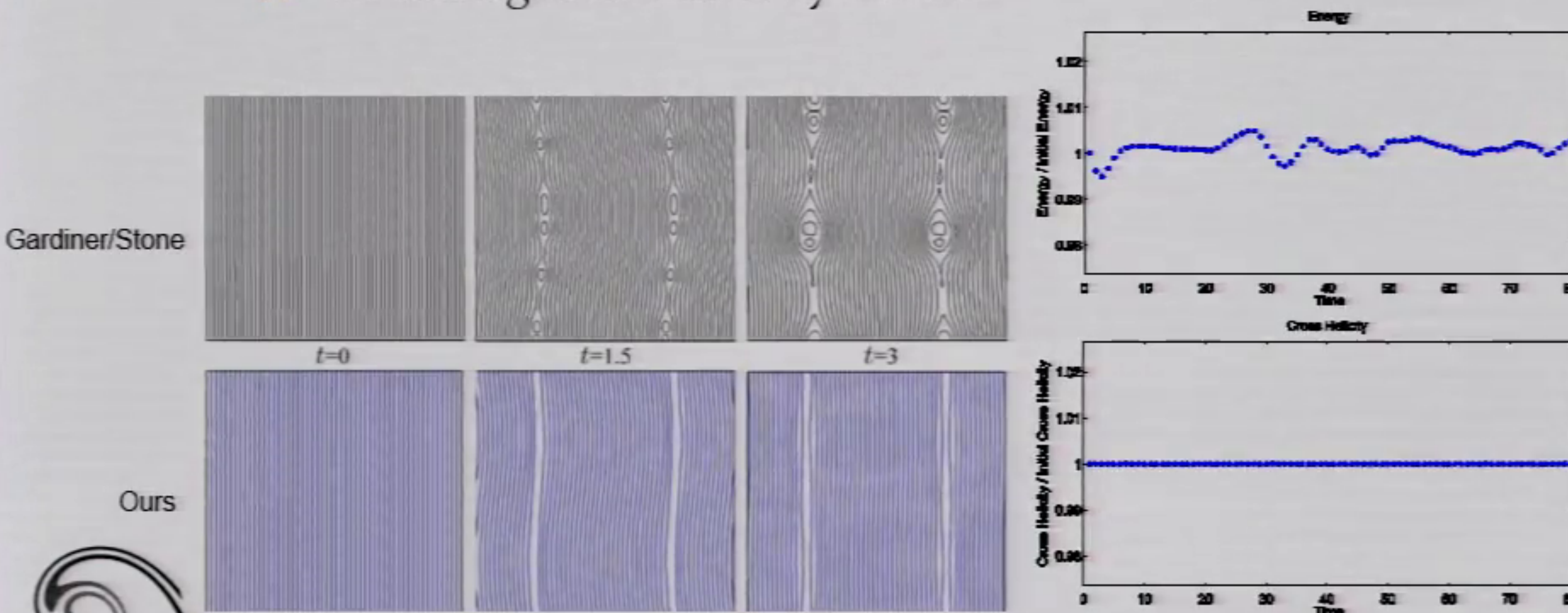




Discrete MagnetoHydroDynamics

As expected, key invariants preserved [Gawlik]

- energy conserved over long runs
- cross-helicity preserved exactly
- no spurious magnetic reconnection (same topology of field lines)
    - when magnetic resistivity is null

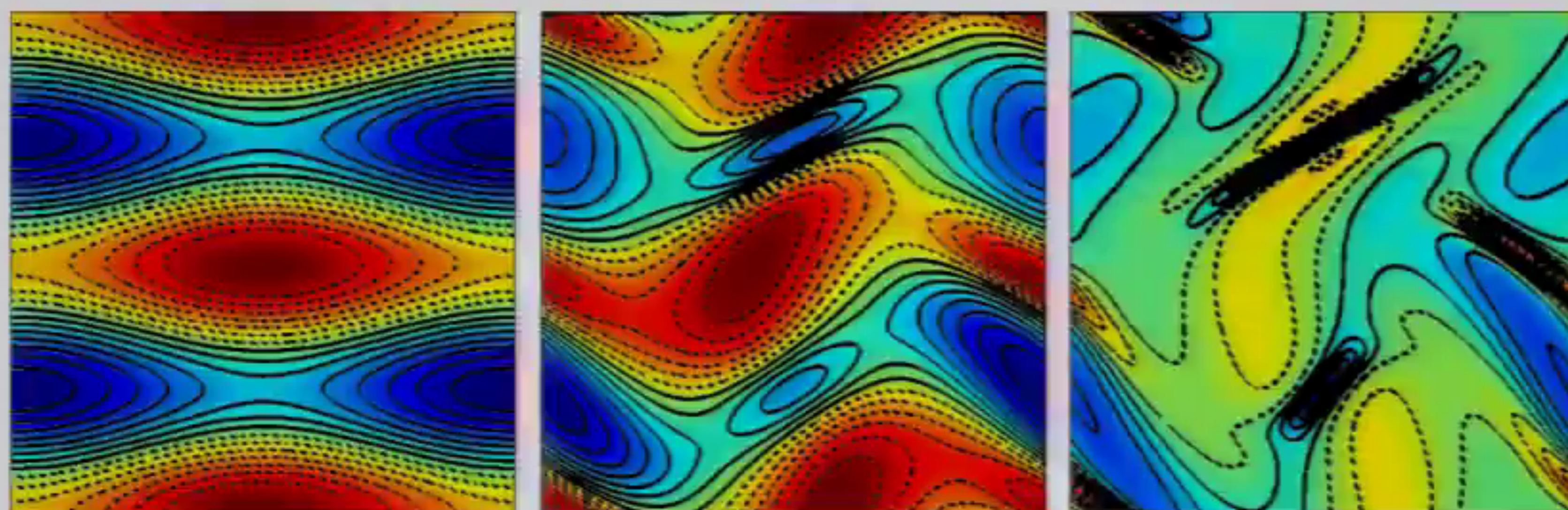


Other Numerical Benefits

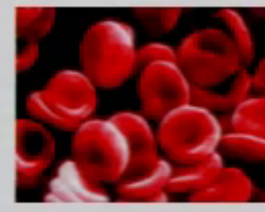
Again, robustness to coarse sampling

- incompressible Orszag-Tang vortex (growth of MHD current sheets)
  - current density contours, 2048^2 vs. 64^2

Cordoba & Marliani '00

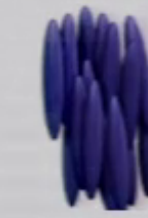


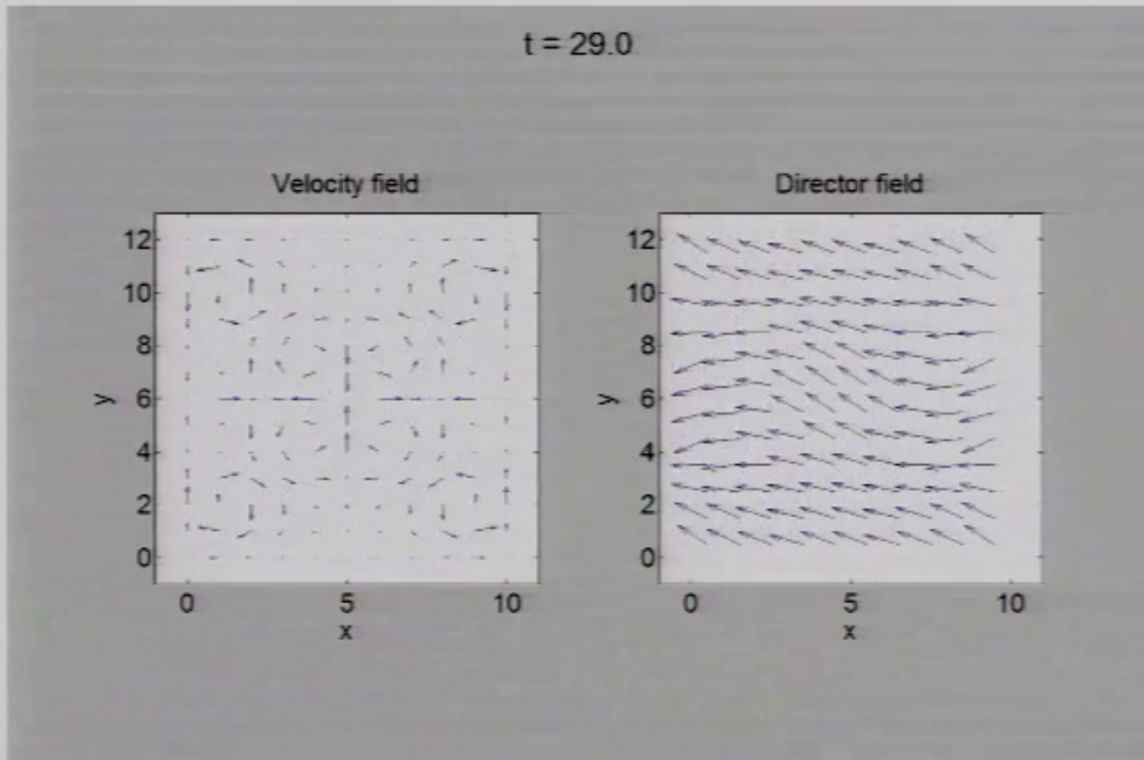
Complex Fluids

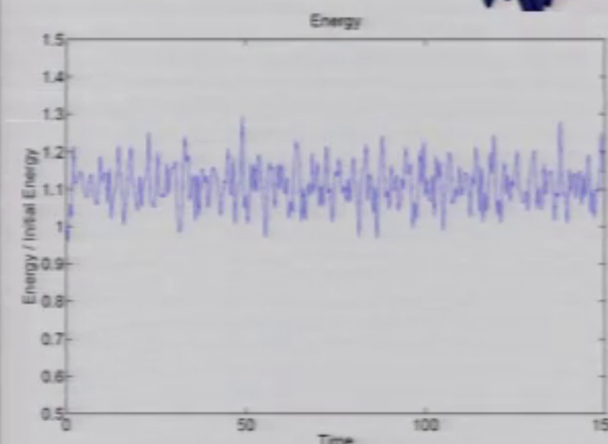


2D Simulation

- nematic liquid crystal, microstretch continua
  - based on recent formulation by Gay-Balmaz & Ratiu







Even Geophysical Flows

General circulation of atmosphere or ocean

- based on the Navier–Stokes equations
- on a rotating sphere
- with thermodynamic terms for energy sources

Stratified and/or rotating fluids

- 2.5D Boussinesq approximation for stratification
- one advected parameter (buoyancy)
  or two (geostrophic momentum)

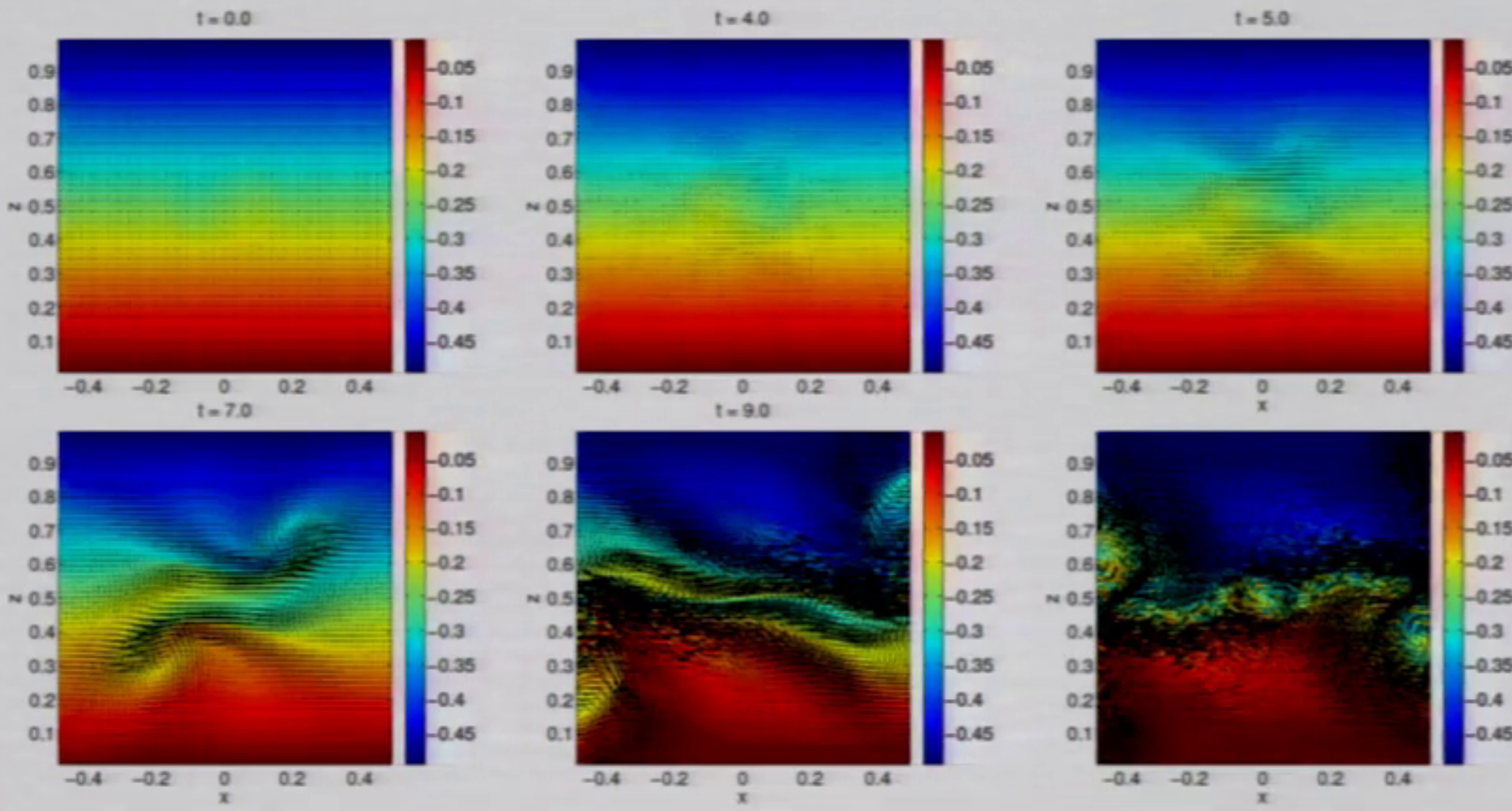
Captures hydrostatic/geostrophic adjustments

- Emitted gravity waves with correct spectrum

Barotropic Anticyclonic Shear

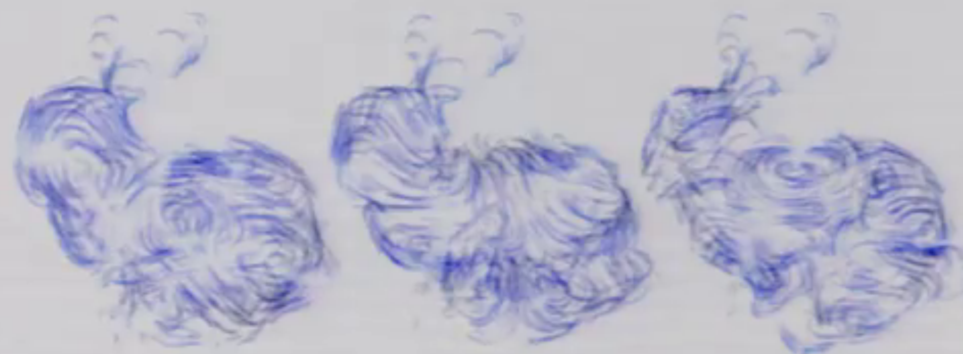
Inertially stable/unstable flows (based on Rossby number)

- emission of inertia-gravity waves vs vorticity ejection
  - captures fine details usually smeared by dissipation



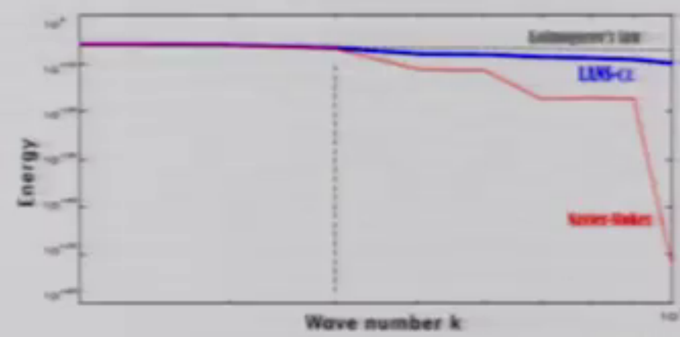
Followups & Future Work

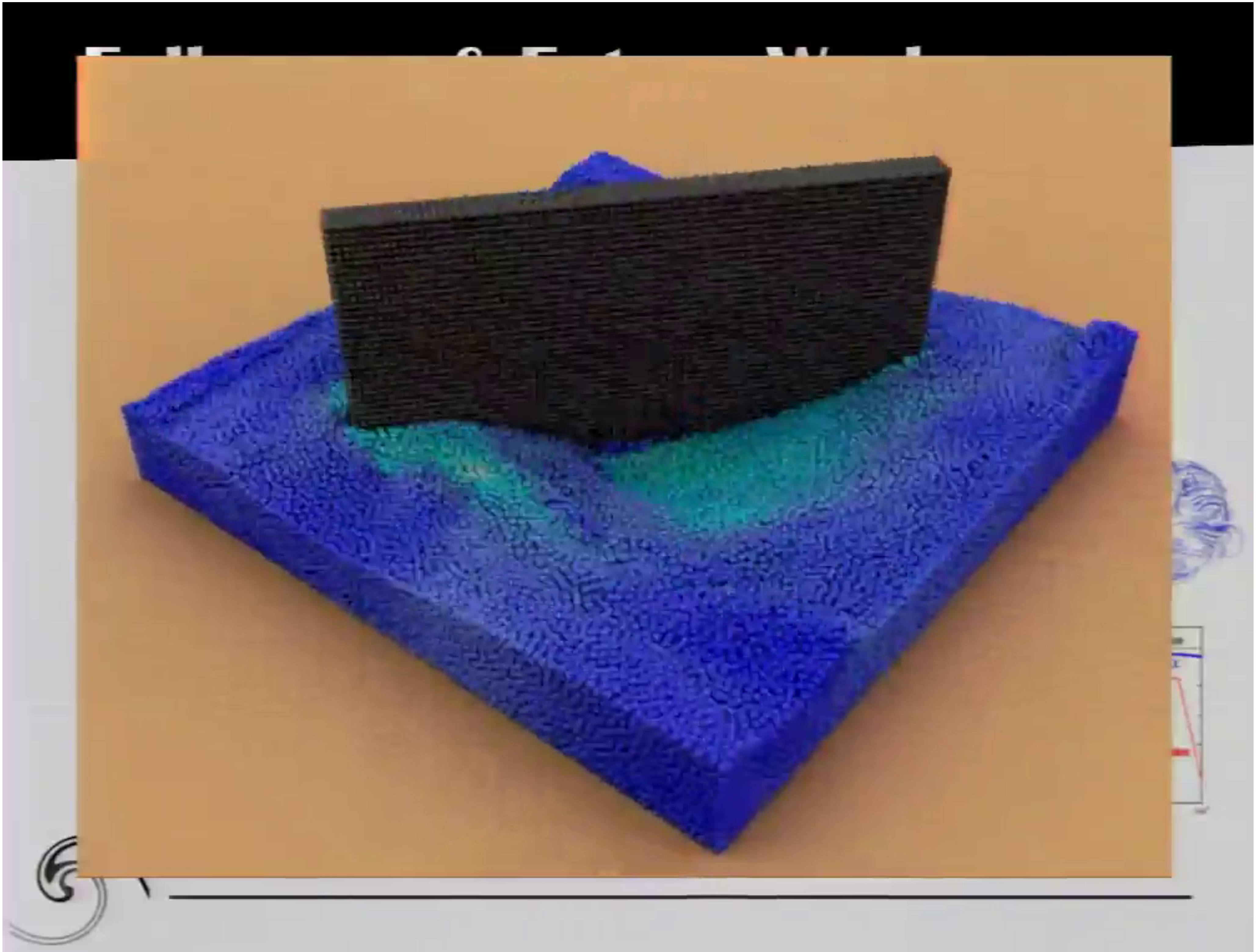
- Hamiltonian view of incompressible fluids
  - see Gemma Mason's work
- Model-reduced fluids
  - low count basis
  - work by Liu & Mason
- Turbulence models
  - tests on LANS-α seem good



Flow analysis (LCS)?

Lagrangian setup?





Geometry-guided Computations

Computations that *respect the geometry™*

- computational science guided by geometry
- leverage geometric structures of mechanics
  - while recognizing the realities of computation

Discretize the geometric, variational principles

- NOT THE RESULTING PDES
- importance of spatial discretization
- coordinate-free computations
  - DEC [Hirani], FEEC [Arnold]—perfect for E&M [Stern]

Lots more to explore