The Dynamic Likelihood Filter

Filtering for Sparsely Observed, Hyperbolic Systems

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IMPROVING HURRICANE PREDICTIONS

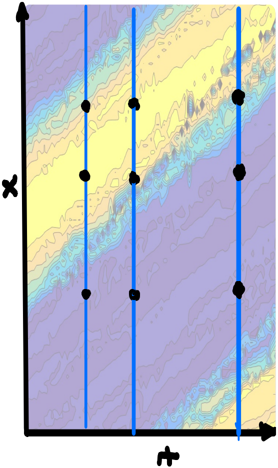
Property Damage ($USD)

- Maria $ 102B ?
- Harvey/Irma $190B
- Katrina $108B
- Sandy $65B
- Ike $30B
- Andrew $27B
- . . .

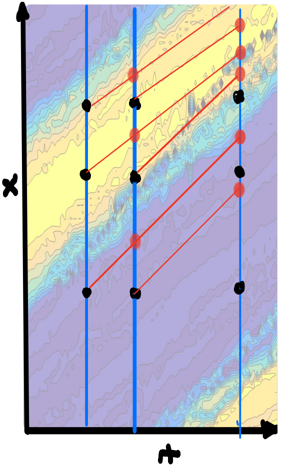
- L_1 SHARPENING
- DISPLACEMENT CORRECTION

See S. Rosenthal, S. Venkataramani, J.M.R., A. Mariano, *Displacement Data Assimilation*, J. Comp. Phys. 2016

Improving Predictions using Sparse Observations



Improving Predictions using Sparse Observations



DYNAMIC LIKELIHOOD DATA ASSIMILATION

Stochastic One-Way Wave Equation:

$$\begin{aligned} u_t - C(x,t)u_x &= F(x,t), \quad t > 0, x \in [0,L], \\ u(x,0) &= \mathcal{U}(x), \quad x \in [0,L], \end{aligned}$$

$$F(x,t) = f(x,t) + N_f(t), \quad C(x,t) = c(x,t) + N_c(t)$$

$$\begin{aligned} \Phi_\ell(0) &= \mathcal{U}(x_\ell), \quad \ell = 1,2,...,N \\ d\Phi &= f(\Phi)dt + A(t)dW_t^{(f)}, \\ \Phi(0) &= \mathcal{U}(x_\ell), \\ dx &= c(x,t)dt + B(t)dW_t^{(c)}, \\ x(0) &= x_\ell, \end{aligned}$$

J.M.R., *Dynamic Likelihood Approach to Filtering*, Q. J. Roy. Met. Soc, 2017,
P. Krause, J.M.R. *Using the Diffusion Kernel Filter in Lagrangian Data Assimilation*, Mon. Wea. Rev, 2009

Data Assimilation Statement

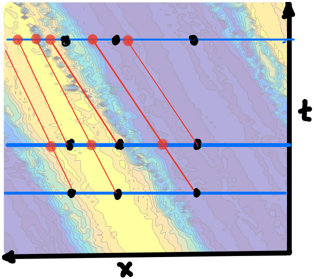
Given MODEL outcomes $\varphi(t_n)$*,*
Model: $\varphi_{n+1} = M\varphi_n + N_f(t_n)[0, A], n = 0, 1, 2, \ldots$
and DATA $Y(t_n)$
Observations: $Y_m = H_m\varphi_m + N_d(t_m)[0, R], m = 0, 1, 2, \ldots$
Propagated: $Y(\zeta_n, t_n)$

Find the time dependent mean and the uncertainty of the posterior $P(\varphi|Y)(t_0 \leq t_n \leq t_f)$. *Found by minimizing trace of the posterior covariance in model space* .

DATA:

OBSERVED ●

PROPAGATED ●

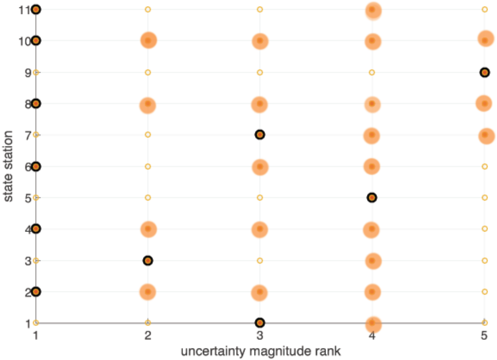


$$\zeta_{n+1} = \Delta t c(\zeta_n, t_n) \quad + \quad \zeta_n,$$

$$Y(\zeta_{n+1}, t_{n+1}) = Y(\zeta_n, t_n),$$

$$R_m^{n+1} \approx A_n(t)[A_n(t)]^T \Delta t \quad + \quad R^n,$$

Observations and Projected Data



Pick projected data with the least uncertainty

Formulating the DLF

In KF: at measuring times t_m:

$$\mathbf{Y}_m = H_m \mathbf{V}_n + \boldsymbol{\epsilon}_m,$$

In DLF: at times $t_n \geq t_m$,

$$\boldsymbol{\mathcal{H}}_m^n \mathbf{Y}_m^n = \mathbf{V}_n + \boldsymbol{\mathcal{H}}_m^n \boldsymbol{\epsilon}_m^n,$$

where $\boldsymbol{\epsilon}_m^m$ is equal to $\boldsymbol{\epsilon}_m$, with covariance

$$R_m^{n+1} \approx A_n(t)[A_n(t)]^T \Delta t + R^n, \quad t_n \geq t_m.$$

Formulating the DLF

The multi-analysis stage in DLF is

$$\langle \mathbf{V} \rangle_n = \tilde{\mathbf{V}} + \boldsymbol{\mathcal{K}}_m(\boldsymbol{\mathcal{H}}_m \mathbf{Y}_m - \tilde{\mathbf{V}}).$$

Compare to $\langle \mathbf{V} \rangle_n = \tilde{\mathbf{V}} + \mathbf{K}_m(\mathbf{Y}_m - \mathbf{H}_m \tilde{\mathbf{V}})$, for KF.

Finding the Kalman Gain:

The covariance $\mathbf{P}_n = \langle (\langle \mathbf{V}_n - \langle \mathbf{V} \rangle_n)(\langle \mathbf{V}_n - \langle \mathbf{V} \rangle_n)^\top \rangle$.
with

$$\mathrm{Tr}[\mathbf{P}_m] = \mathrm{Tr}[\tilde{\mathbf{P}}] - 2\mathrm{Tr}[\boldsymbol{\mathcal{K}}_m \tilde{\mathbf{P}}] + \mathrm{Tr}[\boldsymbol{\mathcal{K}}_m(\tilde{\mathbf{P}} + \boldsymbol{\mathcal{H}}_m \mathbf{R}_m \boldsymbol{\mathcal{H}}_m^\top)\boldsymbol{\mathcal{K}}_m^\top].$$

Differentiating with respect to $\mathcal{K}_m$ and setting the derivate to zero, the minimizer of the trace is

$$\boldsymbol{\mathcal{K}}_m = \tilde{\mathbf{P}}(\tilde{\mathbf{P}} + \boldsymbol{\mathcal{H}}_m \mathbf{R}_m \boldsymbol{\mathcal{H}}_m^\top)^{-1}$$

Hence, the update to the covariance is

$$\mathbf{P}_n = (\mathbf{I} - \boldsymbol{\mathcal{K}}_n)\tilde{\mathbf{P}}.$$

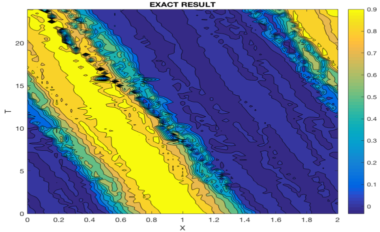
Forecast (Like Kalman Filter):

$$\begin{aligned}\tilde{V} &= L_{n-1}\langle V\rangle_{n-1} + \Delta t f_{n-1}, \quad n = 1, 2, \ldots, N_f - 1.\\ \tilde{P} &= L_{n-1}P_{n-1}L_{n-1}^T + Q_{n-1}, \quad n = 1, 2, \ldots, N_f - 1.\end{aligned}$$

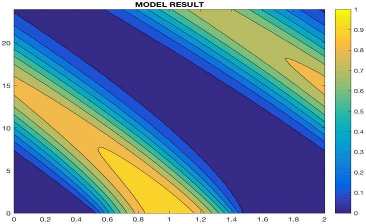
Multi-analysis (Dynamic Likelihood): project onto state space...

$$\begin{aligned}\langle V\rangle_n &= \tilde{V}_n + \mathcal{K}_n \sum_{m'\in m} (\mathcal{H}_{m'}^n Y_{m'}^n - \tilde{V}_n)\delta_{m',n},\\ \mathcal{K}_n &= \tilde{P}_n[\tilde{P}_n + \sum_{m'\in m} \mathcal{H}_{m'}^n R_{m'}^n [\mathcal{H}_{m'}^n]^T \delta_{m',n}]^{-1},\\ P_n &= (I - \mathcal{K}_n)\tilde{P}_n.\end{aligned}$$

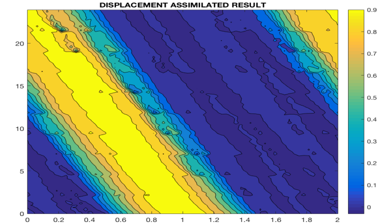
Dynamic Likelihood Data Assimilation



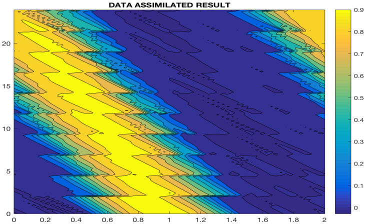
Exact



Model

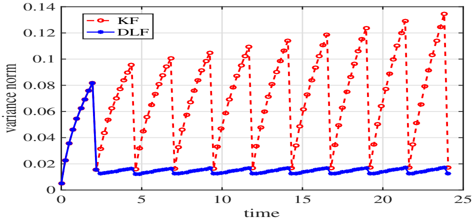


Dynamic Likelihood

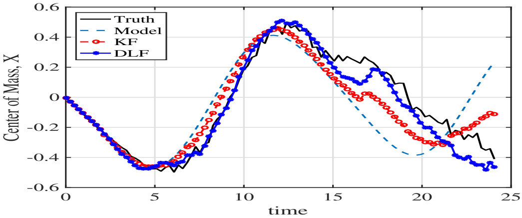


Kalman

Evolution of Uncertainty:



Center of Mass Estimate:



The DLF Method's Features

- Applies to hyperbolic/strongly advective dynamics
- Can be incorporated into any Bayesian assimilation scheme
- Works best when data has low uncertainty
- Better than conventional assimilation when data is sparse
- Shaper posterior uncertainties possible
- Makes use of data between observation times
- Can project observations into the future

Further Information

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GULF OF MEXICO RESEARCH INITIATIVE

NSF National Science Foundation
WHERE DISCOVERIES BEGIN