Scale invariant metrics

Dan Raviv
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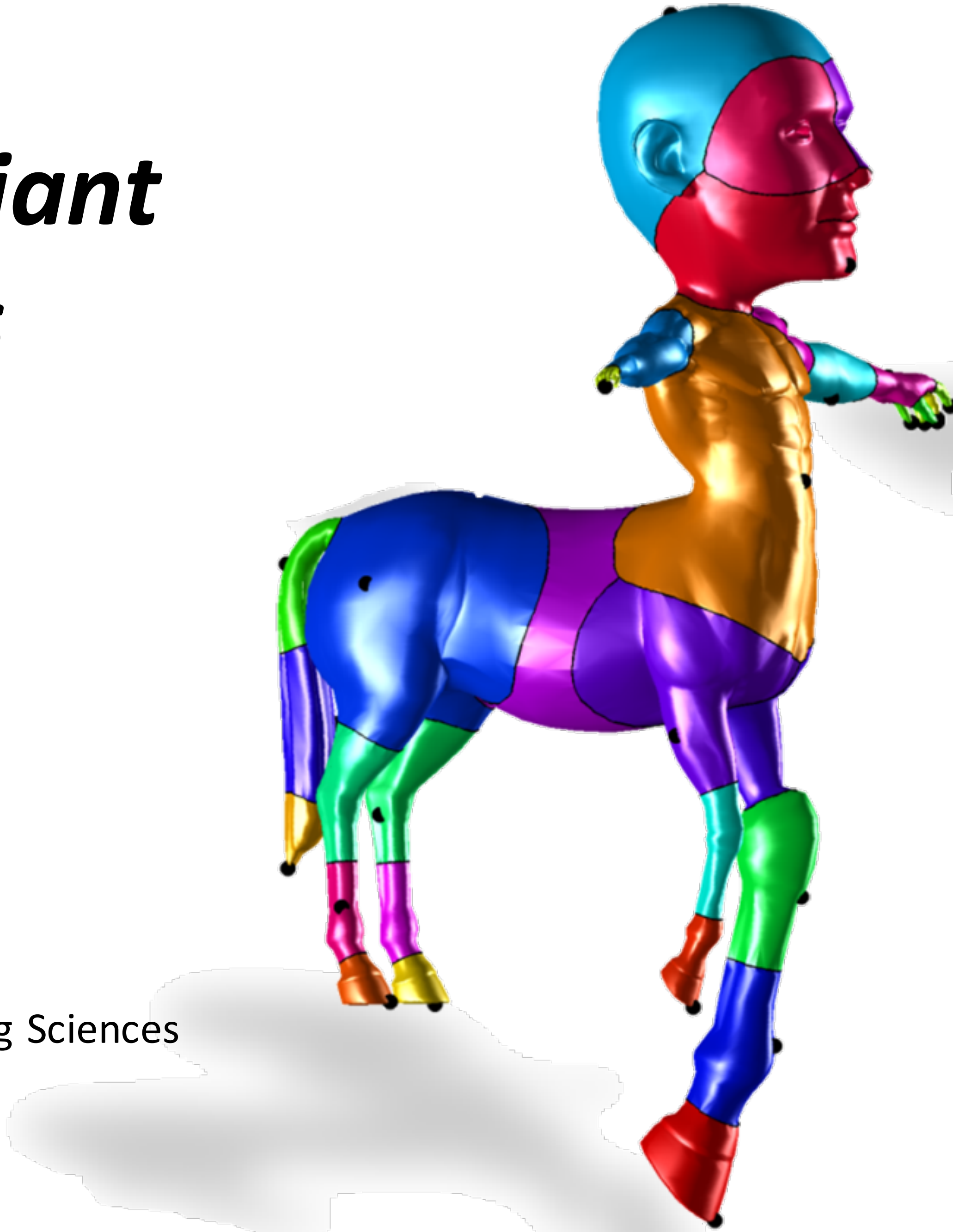

Siam Conference on Imagining Sciences
25 May 2016

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Samsung, Israel

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Scale Invariant Geometry for Nonrigid Shapes

Relevant papers

Scale invariant geometry of non-rigid shapes / Y. Aflalo, R. Kimmel, D. Raviv
SIAM Journal on Imaging Sciences (SIIMS) 2013

Equi-affine Invariant Geometry for Shape Analysis / D. Raviv, A. Bronstein, M. Bronstein, D. Waisman, N. Sochen and R. Kimmel
Jounral of Mathematical Imaging and Vision (JMIV) 2014

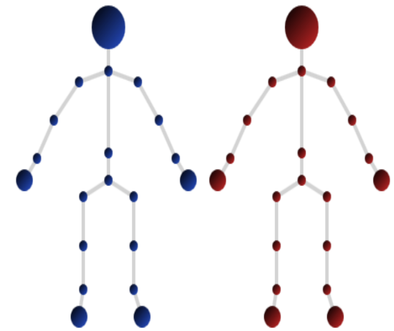
Affine invariant geometry for non-rigid shapes / D. Raviv and R. Kimmel
International Journal of Computer Vision (IJCV) 2015

Scale invariant metrics of volumetric datasets / D. Raviv, and R. Raskar
SIAM Journal on Imaging Sciences (SIIMS) 2015

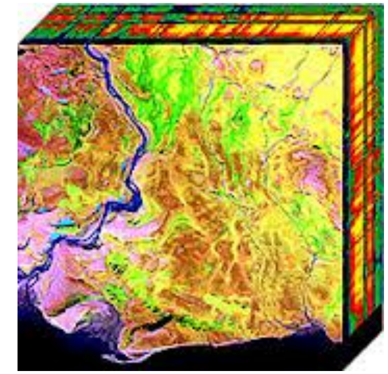
Outline

- Introduction to **non-rigid** shapes
- Scale invariant **arc-length**
- Scale invariant metric in **surfaces**
- **Applications** and algorithms
- Scale invariant Riemannian **tensor**
- From equi-affine to **affine** invariant metrics

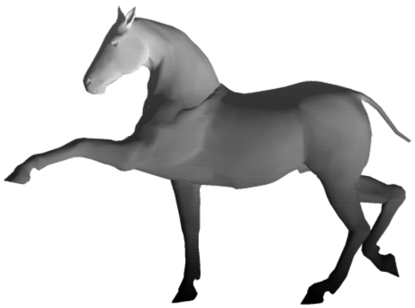
Why non-rigid ?



Skeleton (1D)



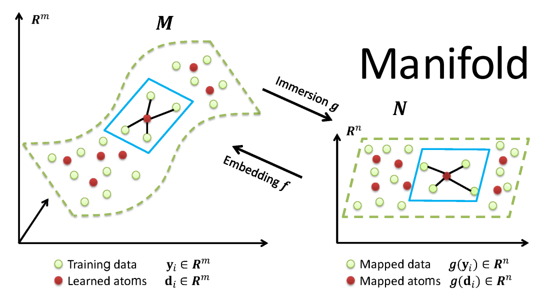
Multi-spectral

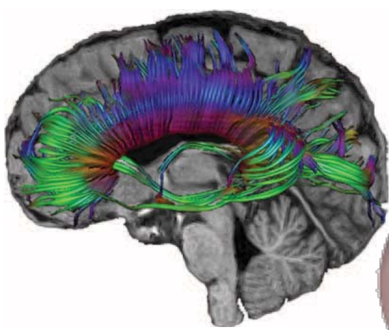


Shapes (2D)

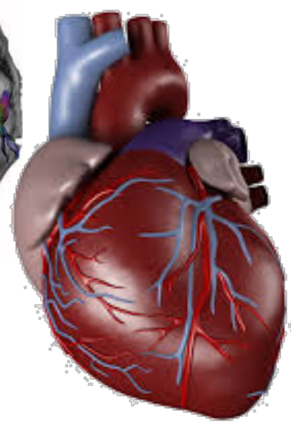
2.5D

Manifold learning



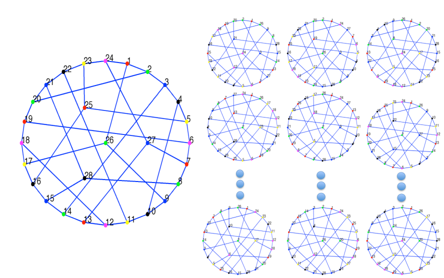


Brain
3D / 6D



Heart
3D

Graphs



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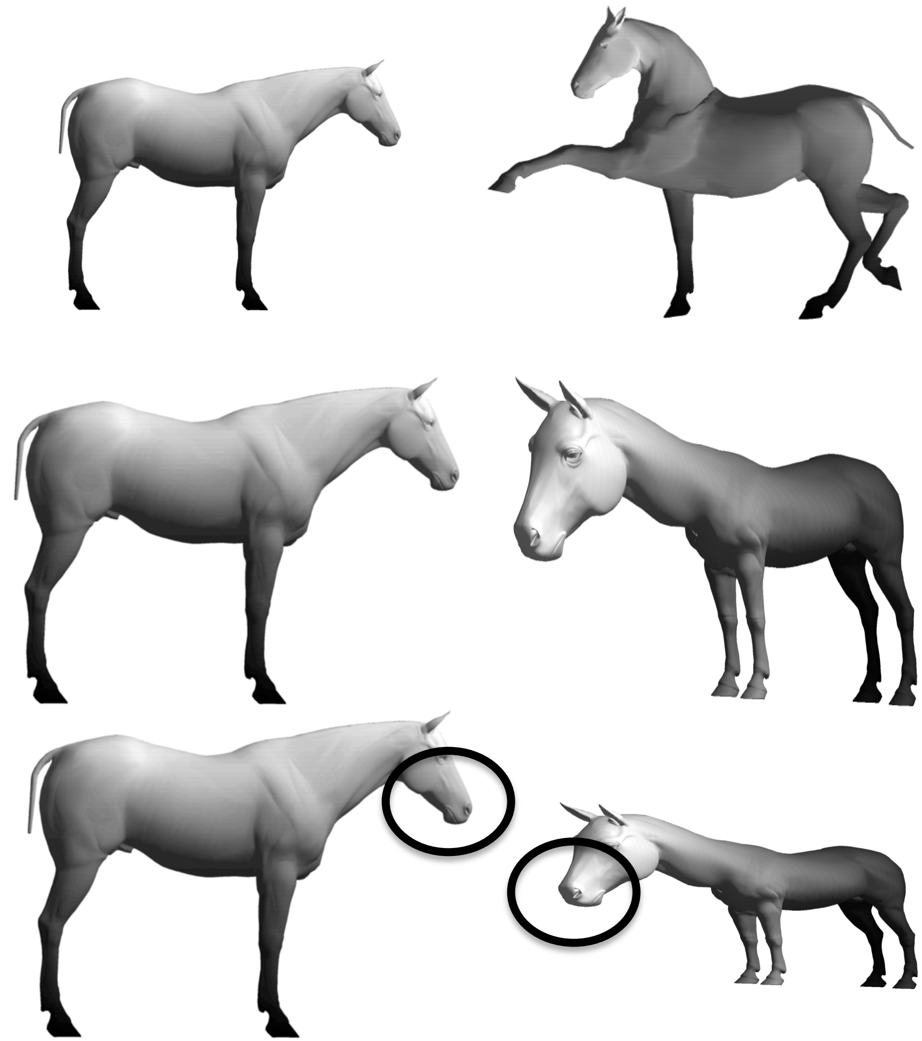
Why ***stretchable*** non-rigid ?

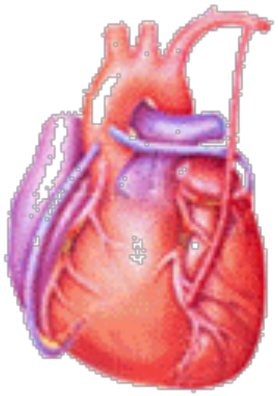


Isometry

Local scale

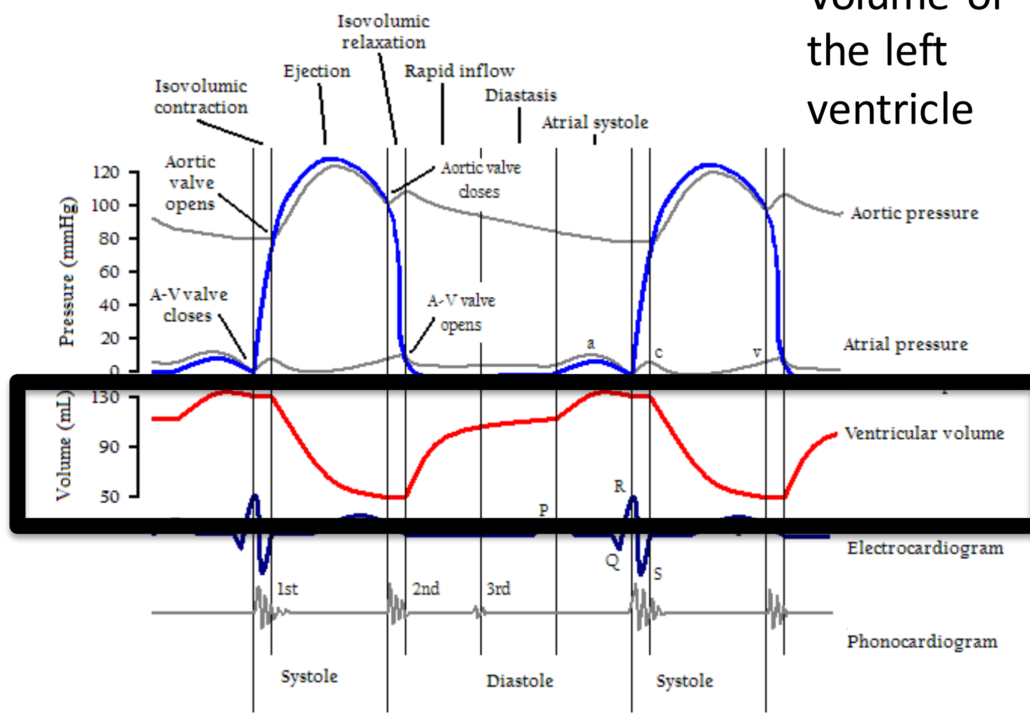
Affine

Why ***stretchable*** non-rigid ?



Heart
3D x time

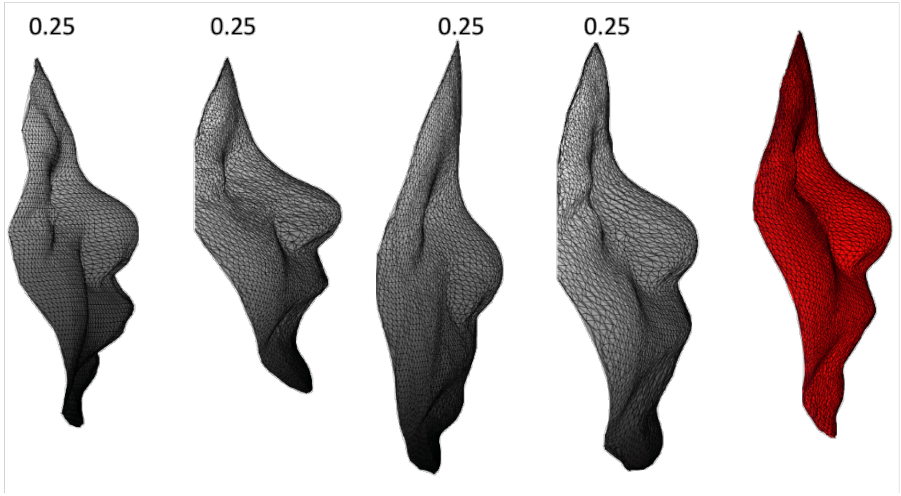
Volume of the left ventricle



End Systole

End Diastole

Why ***stretchable*** non-rigid ?



W / Israel Amirav, M.D. & Ron Kimmel
Ziv Medical Center Technion

An ‘eye’ for an ‘eye’
A ‘nose’ for a ‘nose’




How did we tackle alignment until now?

- Claimed the models are isometric
- Claimed the models are 'almost' isometric
- Forced 1:1 constraints (e.g. diffeomorphisms)
- Used models (e.g. Bsplines)
- Added regularization (e.g. Total variation)
- Changed regularization (e.g. TVL1)

⋮

What are we suggesting to ***improve***?

- Deformation constraints remain the same
- Data term should be **metric dependent**
- Build ***local*** (metric) invariants which generate **global** invariance

Curves

Parameters:

$$p \in P \subset \mathbb{R}$$

Mapping:

$$C(p) : P \rightarrow \mathbb{R}^2$$

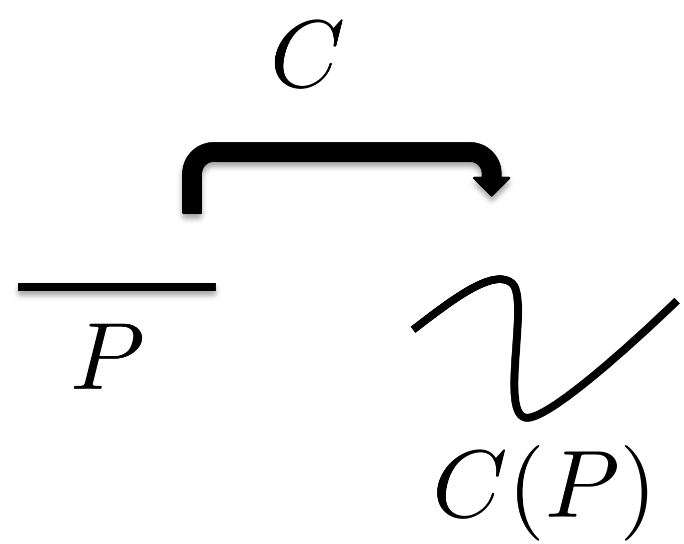
Derivatives:

$$C'(p) = \frac{\partial C(p)}{\partial p}$$

$$C''(p) = \frac{\partial C'(p)}{\partial p}$$

Distance:

$$ds = |C'(p)|dp$$



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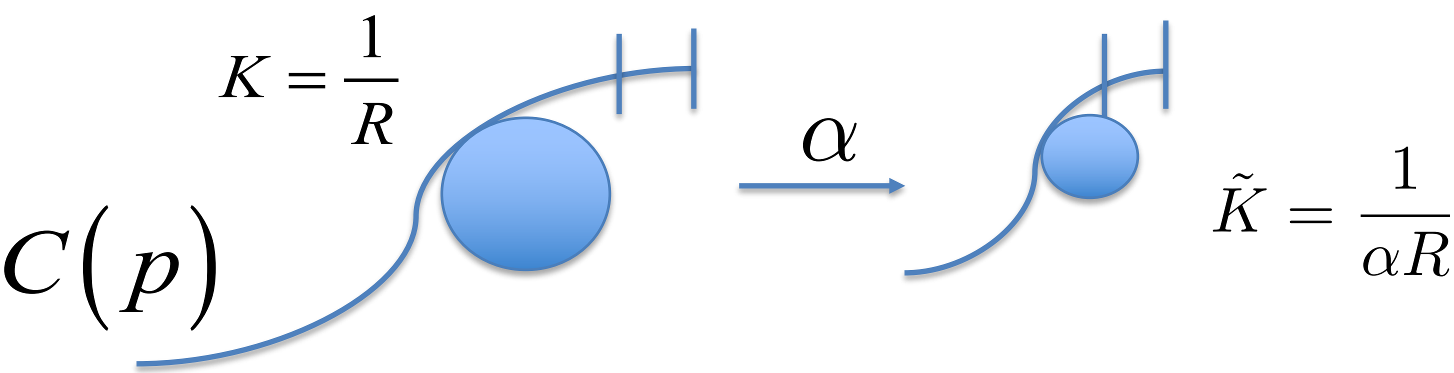
Scale invariant curves and surfaces

Theorem 1: $ds = |K| \cdot |C'(p)|dp$ is scale invariant arc-length



$$ds = |C'(p)|dp \longrightarrow ds = |K| \cdot |C'(p)|dp$$

$$d\tilde{s} = |\tilde{K}| \cdot |\tilde{C}'(p)|dp = \left|\frac{K}{\alpha}\right| \cdot |\alpha C'(p)| = |K| \cdot |C'(p)| = ds$$

Aflalo, Kimmel, Raviv. SIIMS'13 *SIAM (SIAG/IS)*
Raviv and Raskar, SIIMS'15

Surfaces

Parameters:

$$U = \{u^1, u^2\} \in \mathbb{R}^2$$

Mapping:

$$X : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

Derivatives:

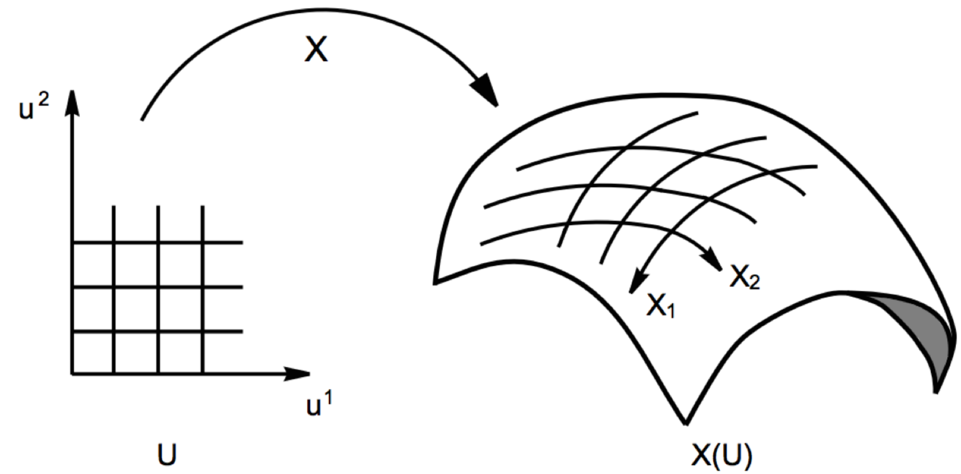
$$X_i = \frac{\partial X}{\partial u^i}$$

Metric:

$$g_{ij} = X_i \cdot X_j$$

Distance:

$$ds = \sqrt{g_{ij} du^i du^j}$$



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Scale invariant curves and surfaces

Theorem 2: $q_{ij} = |K_G| g_{ij}$ is scale invariant metric

Sketch of proof:

$$b_{ij} = \langle S_{ij}, n \rangle$$

$$K_G = \frac{\det b}{\det g} = \frac{b_{11} b_{22} - b_{12}^2}{g_{11} g_{22} - g_{12}^2}$$

$$n = \frac{S_1 \times S_2}{||S_1 \times S_2||}$$

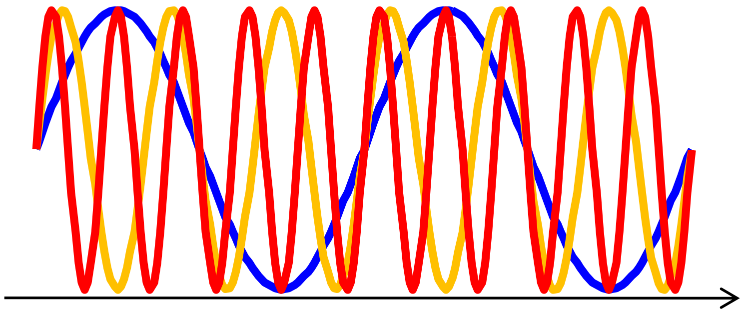
$$\tilde{g}_{ij} = \langle \alpha S_i, \alpha S_j \rangle = \alpha^2 \langle S_i, S_j \rangle = \alpha^2 g_{ij}$$

$$\tilde{b}_{ij} = \langle \alpha S_{ij}, n \rangle = \alpha \langle S_{ij}, n \rangle = \alpha b_{ij}$$

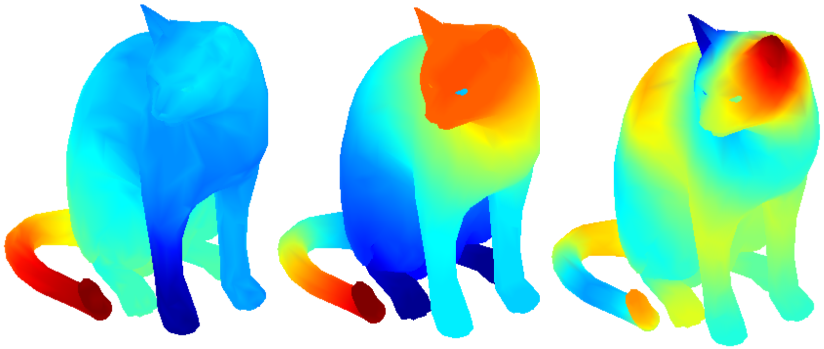
$$|\tilde{K}_G| \tilde{g}_{ij} = \left| \frac{\det \tilde{b}}{\det \tilde{g}} \right| \tilde{g}_{ij} = \frac{\alpha^2}{\alpha^4} |K_G| \alpha^2 g_{ij} = |K_G| g_{ij}$$

Aflalo, Kimmel, Raviv. SIIMS'13 *SIAM (SIAG/IS)*
Raviv and Raskar, SIIMS'15

From Local to Global



$$-\frac{d^2}{dx^2}e^{jnx} = n^2 e^{jnx}$$



$$-\Delta_g \phi_n(x) = \lambda_n \phi_n(x)$$

$$\Delta_g \phi = \text{div grad } \phi = \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j \phi\right)$$

What can we extract from the Laplacian ?

Mappings, Distances, Features

$$d^2(x,y) = \left\| K_t(x,\cdot) - K_t(y,\cdot) \right\|^2_{L^2(X)}$$

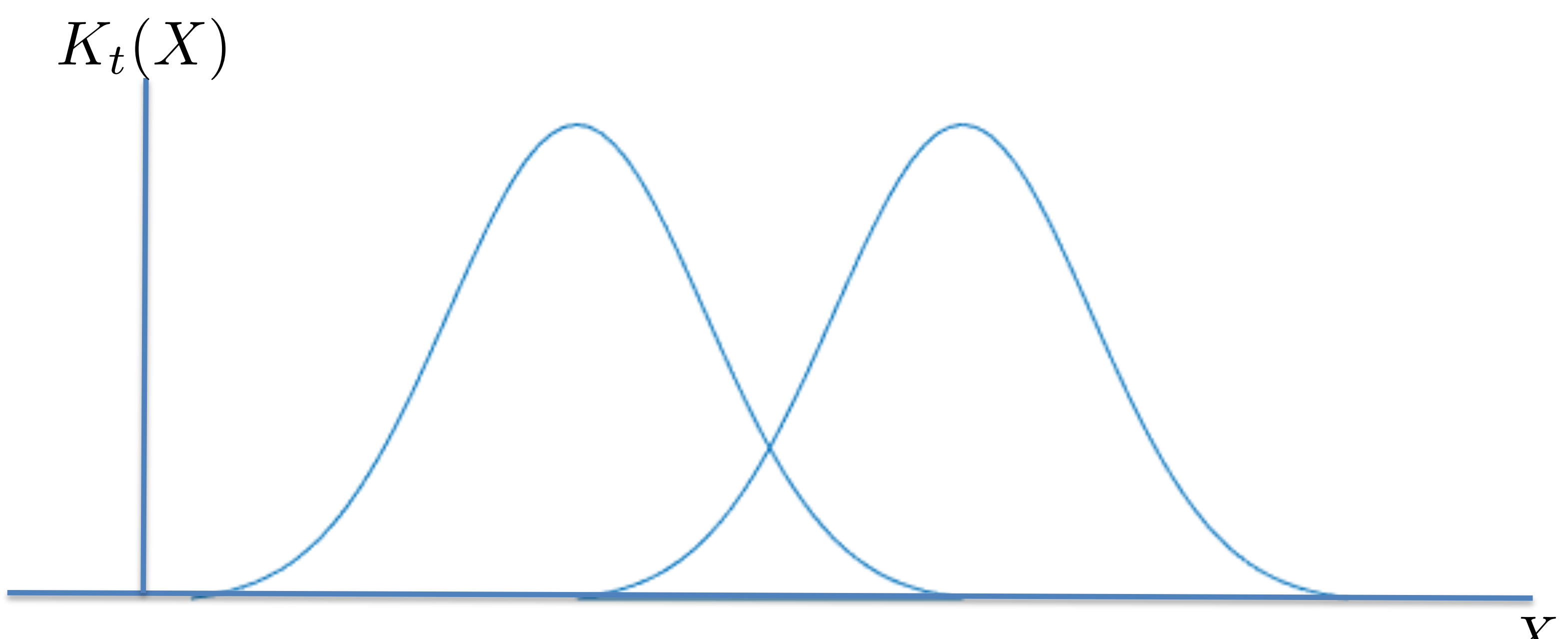
$$= \int_X \left| K_t(x,z) - K_t(y,z) \right|^2 dz$$

Invariant to:

Translation

Rotation

Isometry



Coifman, Nadler, Singer, Guibas and many more

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What can we extract from the Laplacian ?

Mappings, Distances, Features

$$d^2(x,y) = \left\| K_t(x,\cdot) - K_t(y,\cdot) \right\|^2_{L^2(X)}$$

$$= \int_X \left| K_t(x,z) - K_t(y,z) \right|^2 dz$$

Invariant to:

Translation

Rotation

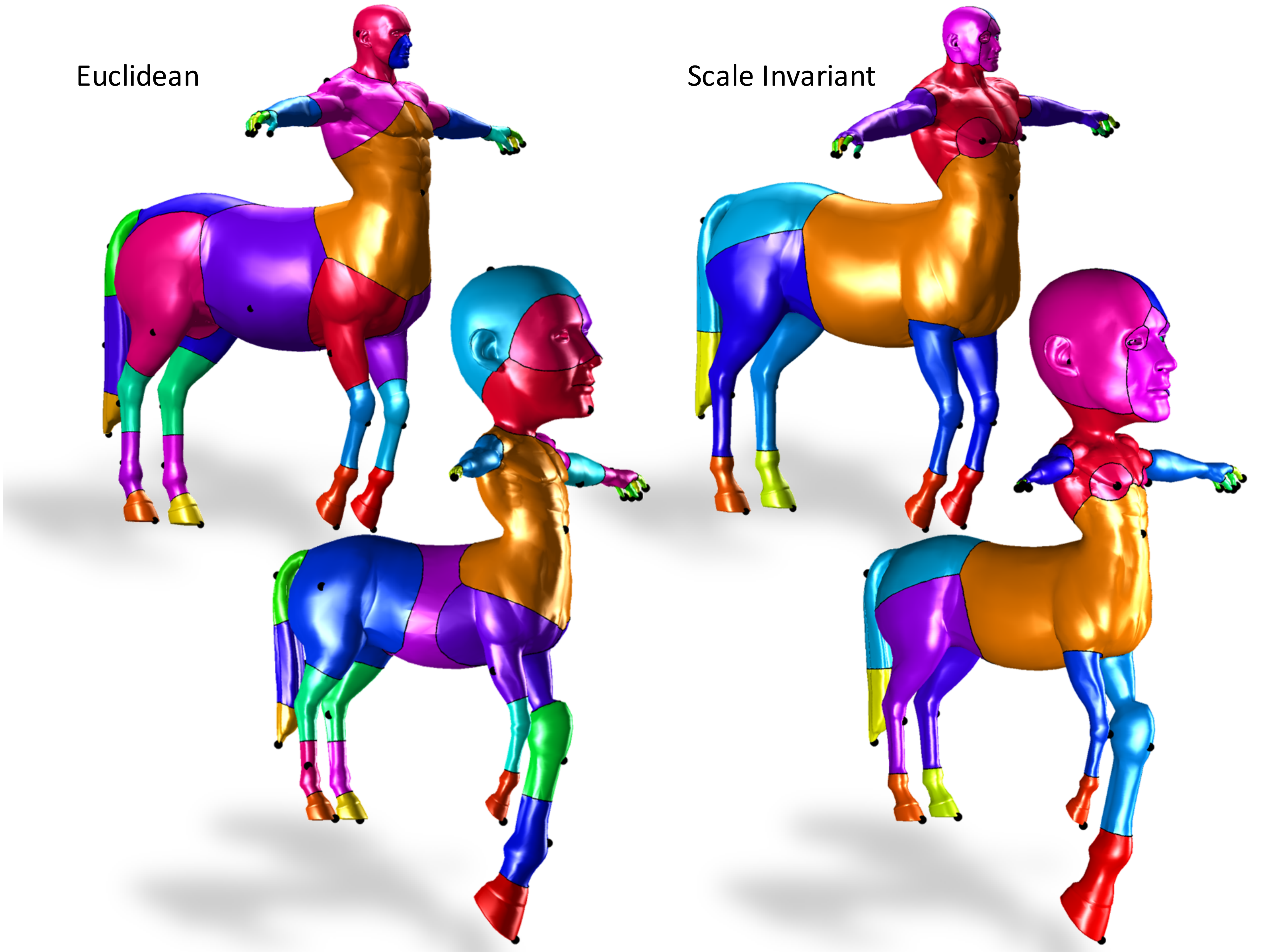
Isometry

$$\int_X \phi_i(z)\phi_j(z)dz = \delta_{ij}$$

$$K_t(x,y) = \sum_{i\geq 0} e^{-\lambda_i t}\phi_i(x)\phi_i(y)$$

$$d^2(x,y) = \sum_{i,j\geq 0} e^{-(\lambda_i+\lambda_j)t}\left(\phi_i(x)-\phi_i(y)\right)\left(\phi_j(x)-\phi_j(y)\right)\delta_{ij}$$

$$= \sum_{i\geq 0} e^{-2\lambda_i t}\left(\phi_i(x)-\phi_i(y)\right)^2$$


Euclidean
Scale Invariant

Heat kernel Signatures:

$k << n$

$$K_t(x,y) = \sum_{i \geq 0}^{k} e^{-\lambda_i t} \phi_i(x) \phi_i(y)$$

$$HKS(x,t) = K_t(x,x)$$

$$HKS(x) = [K_t(x,x)]_{t \in T}$$

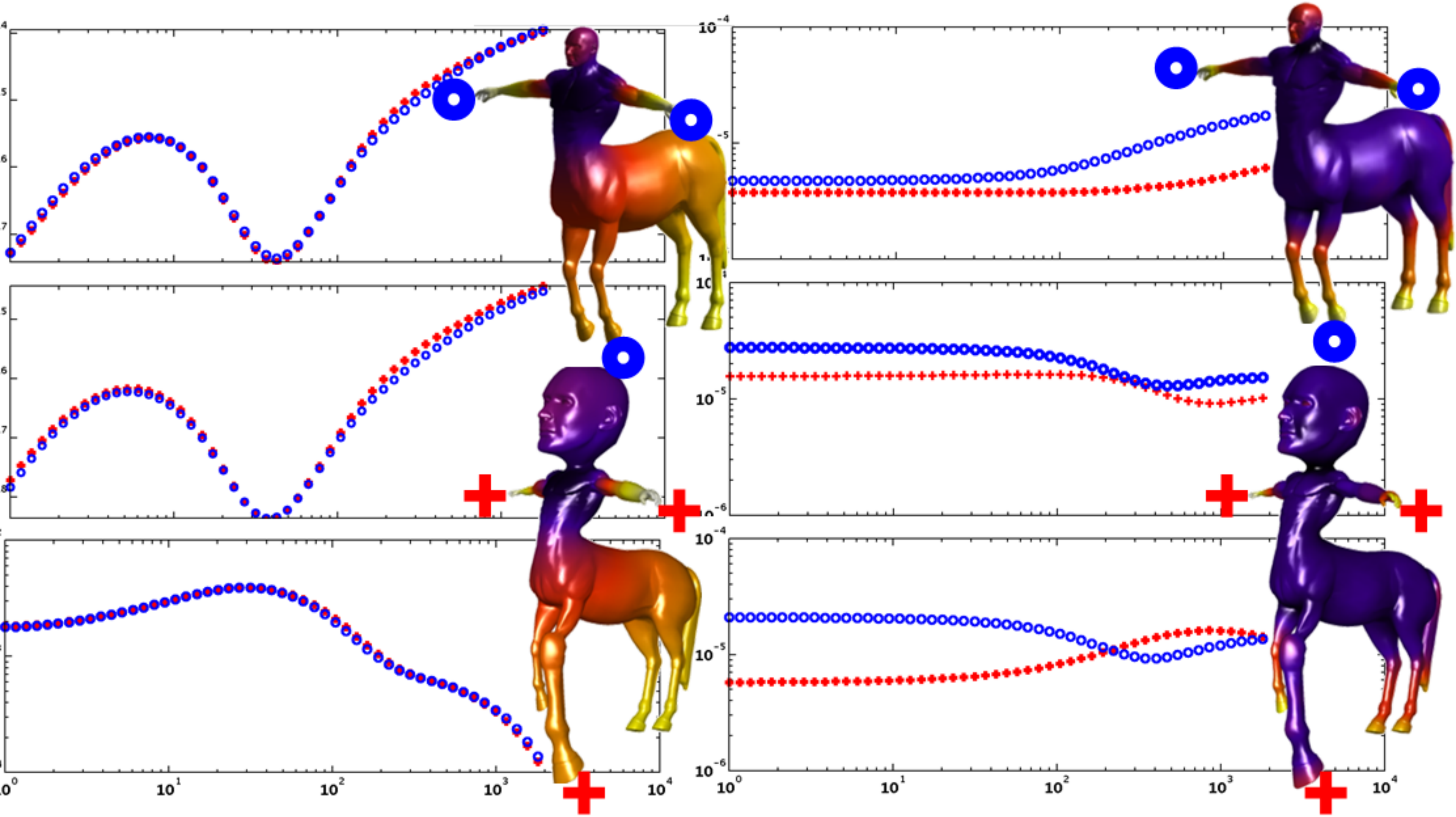
Logarithmic time table

$|T| << n$

Sun, Ovsjanikov, Guibas 09

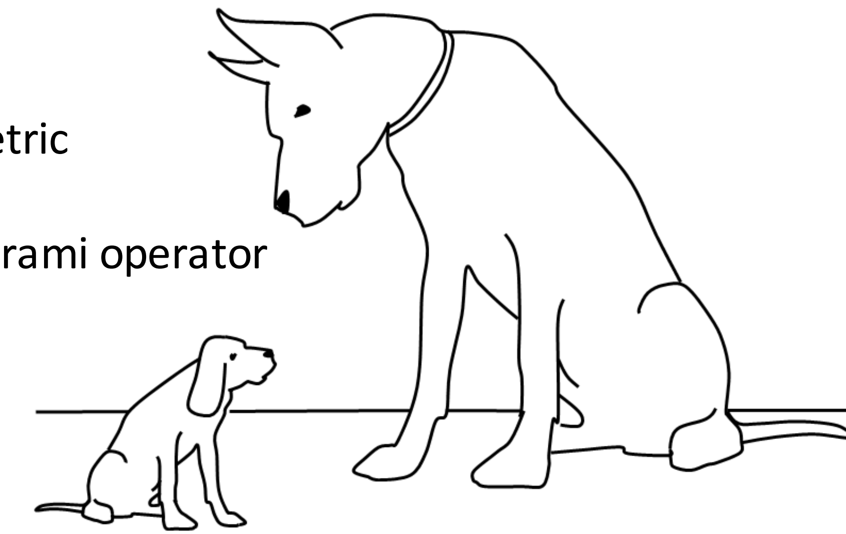
Scale Invariant

Euclidean



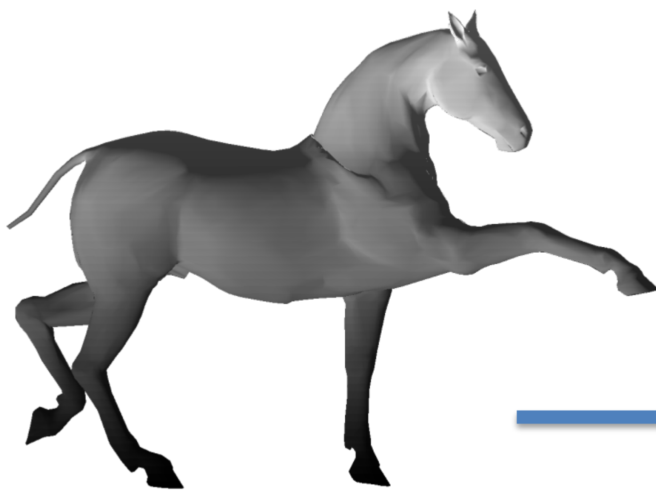
(quick) Summary

1. Define a new (scale invariant) local metric
2. Build a metric-dependent Laplace Beltrami operator
3. Eigen-decomposition of LBO
4. Construct Kernel
5. Evaluate distances and features

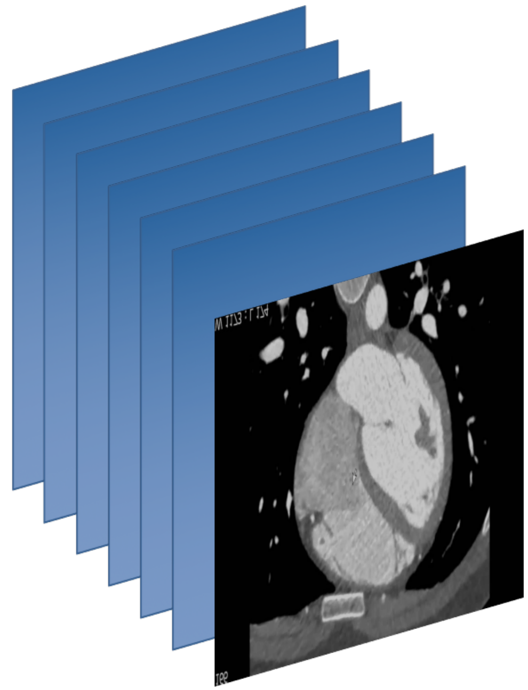


We get a global measurement which is invariant to piecewise constant deformations (without knowing their strength and location)

From surfaces to volumes



2D manifold



3-manifold

Scale invariant nD manifolds

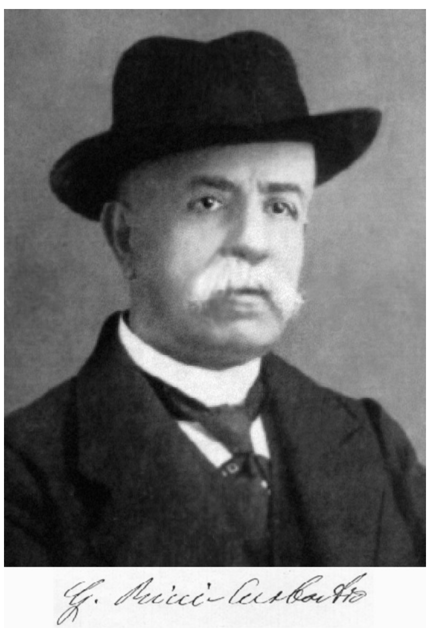
- Concept:

Curvature (1D), Gaussian curvature (Surfaces)

↓

Scalar curvature

(Trace of Ricci curvature)



Wikipedia

Ricci Curvature :Amount by which the volume of a geodesic ball in a curves Riemannian manifolds deviates from that of the standard ball in Euclidean space.

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Theorem 3:

Aflalo, Kimmel and Raviv, SIIMS 2013

$|Sc|g_{ij}$ Is scale invariant metric

Sketch of Proof:

scaling of the manifold will not change the Christoffel symbols of the second kind

$$
\begin{aligned}
\tilde{\Gamma}^k_{ij} &= \frac{1}{2}\tilde{g}^{km}\left(\frac{\partial \tilde{g}_{jm}}{\partial i} + \frac{\partial \tilde{g}_{im}}{\partial j} + \frac{\partial \tilde{g}_{ij}}{\partial m}\right) \\
&= \frac{1}{2\alpha^2}g^{km}\left(\alpha^2\frac{\partial g_{jm}}{\partial i} + \alpha^2\frac{\partial g_{im}}{\partial j} + \alpha^2\frac{\partial g_{ij}}{\partial m}\right) \\
&= \frac{1}{2}g^{km}\left(\frac{\partial g_{jm}}{\partial i} + \frac{\partial g_{im}}{\partial j} + \frac{\partial g_{ij}}{\partial m}\right) = \Gamma^k_{ij}.
\end{aligned}
$$

Raviv and Raskar, SIIMS 2015

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Theorem 3: Raviv and Raskar, SIIMS 2015

$$|Sc|g_{ij}$$ Is scale invariant metric

Sketch of Proof:

$$\tilde{R}^l_{ijk} = R^l_{ijk}$$

$$\tilde{R}_{ijkl} = \tilde{R}^m_{ijk}\tilde{g}_{lm} = R^m_{ijk}g_{lm}\alpha^2$$

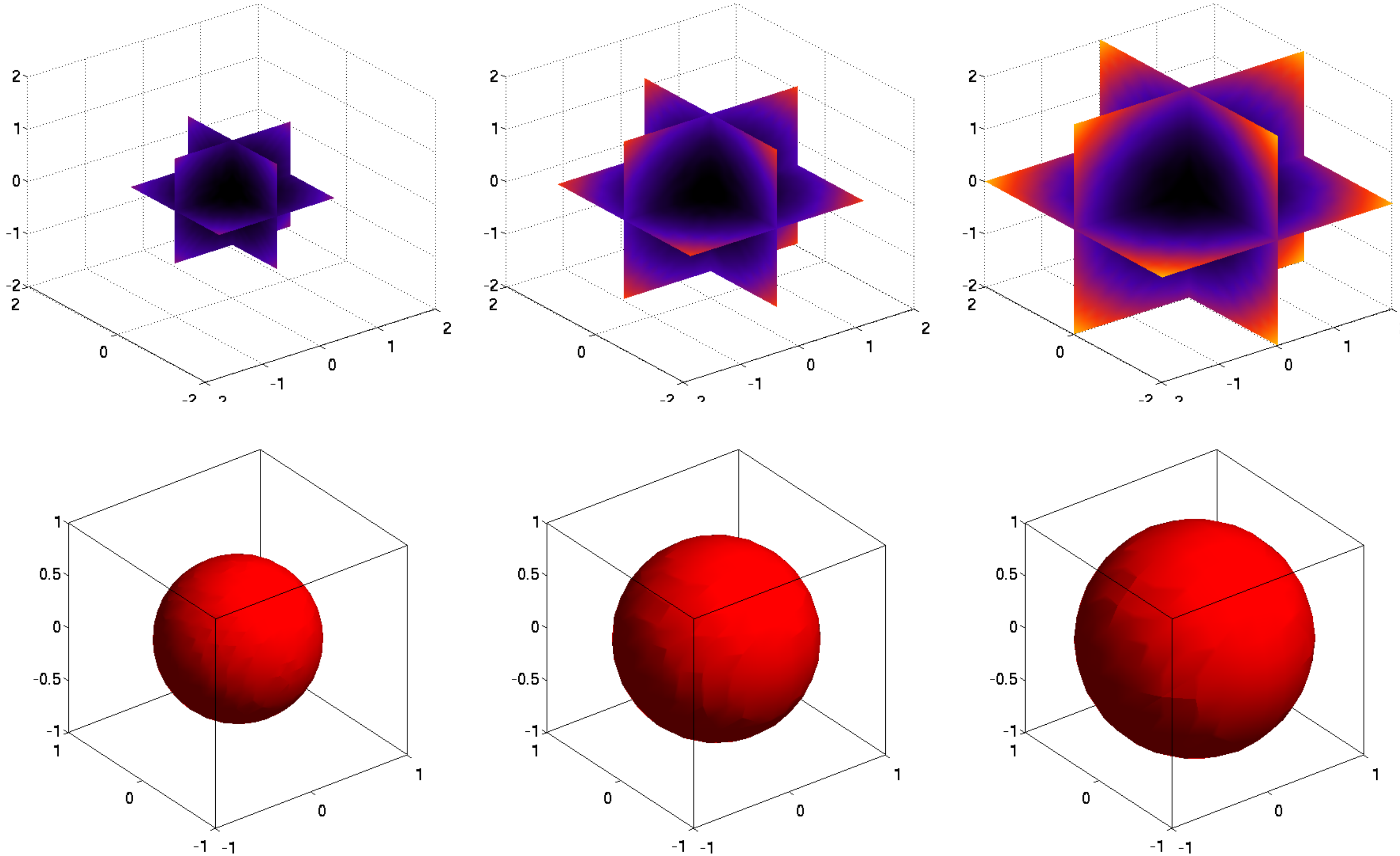
$$\tilde{R}_{ij} = \tilde{g}^{kl}R_{kijl} = \frac{1}{\alpha^2}g^{kl}R_{ijkl}\alpha^2 = R_{ij}$$

$$\tilde{S}c = \tilde{g}^{ij}\tilde{R}_{ij} = \frac{1}{\alpha^2}g^{ij}R_{ij} = \frac{1}{\alpha^2}Sc.$$

Q. How does it relate to Aflalo, Kimmel and Raviv (SIIMS 2013) for surfaces ?
A. It is a generalization. Since $S_c = 2K_g$

3-manifold

$$V : \mathbb{R}^3 \to \mathbb{R}^4$$
$$V(u, v, w) = (u, v, w, I(u, v, w))$$



3-manifold

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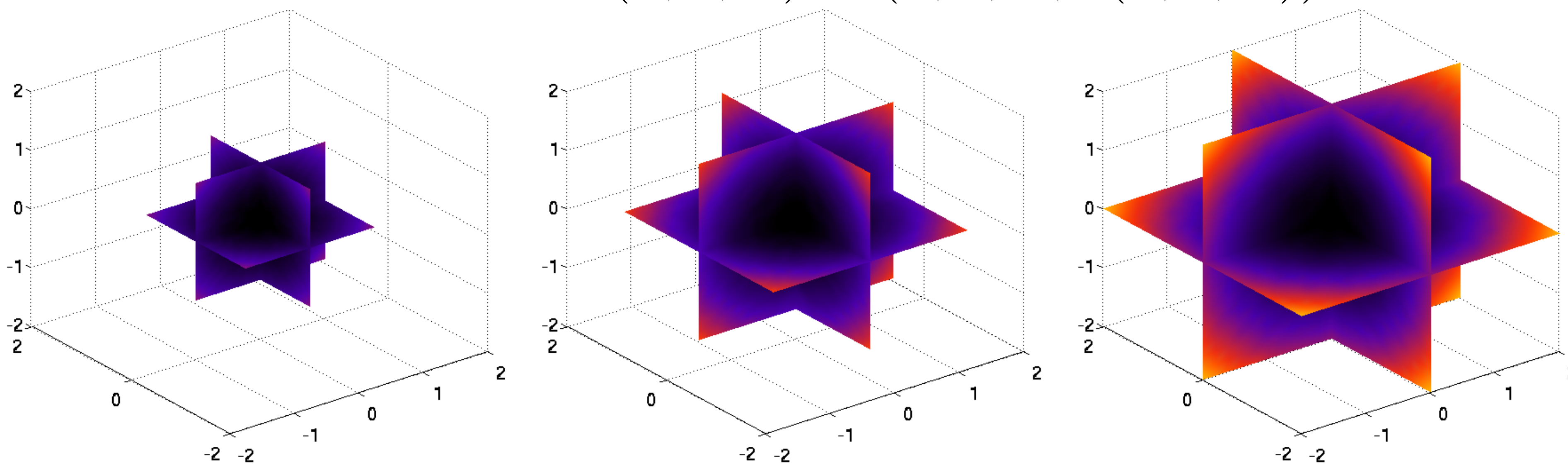
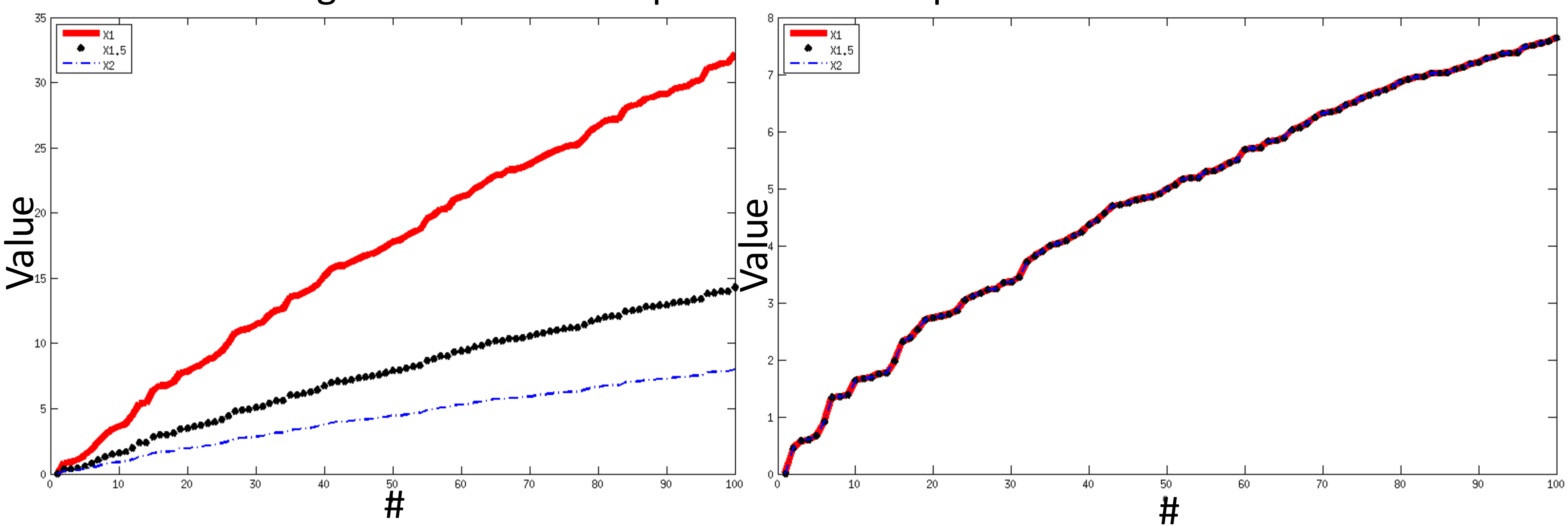
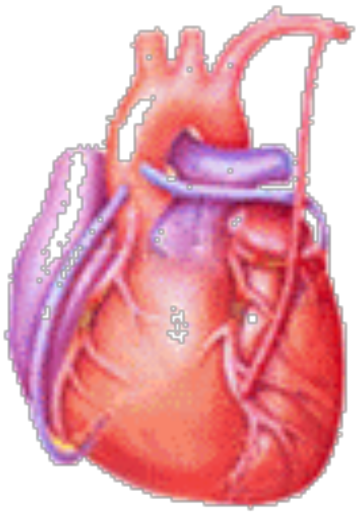


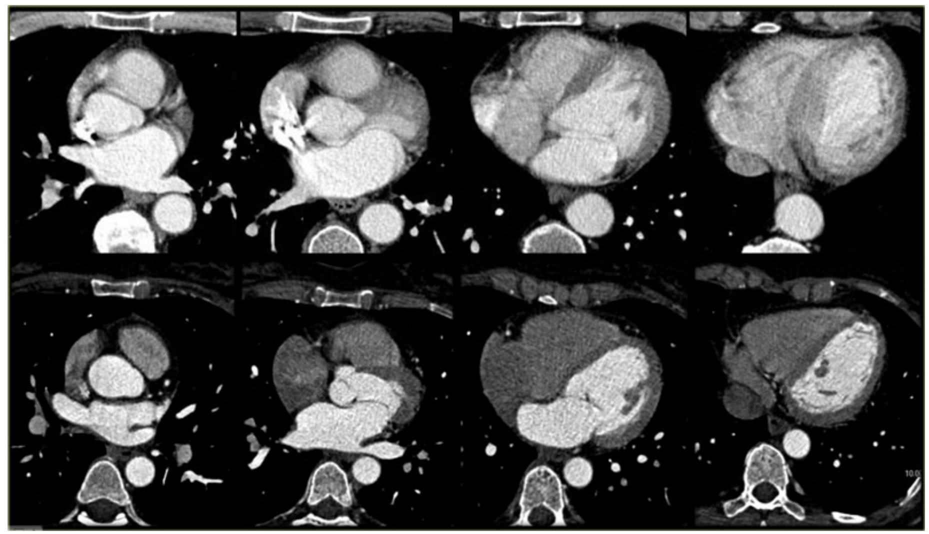
Eigenvalues of the Laplace Beltrami operator



Q. Is 3-manifold scale invariant metric useful for medical data?



Cardiac CT

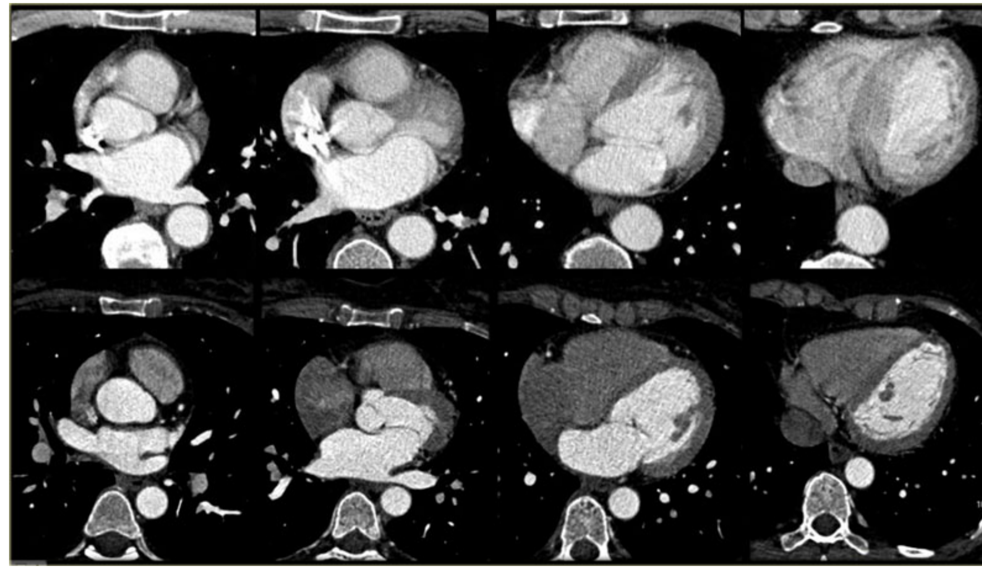


A. Probably not, as the intensity is fixed

Raviv and Raskar, SIIMS 2015

Q. So, scaling of 'what' ?

A. The level-sets!
   In images: Curves
   In volumes: Surfaces

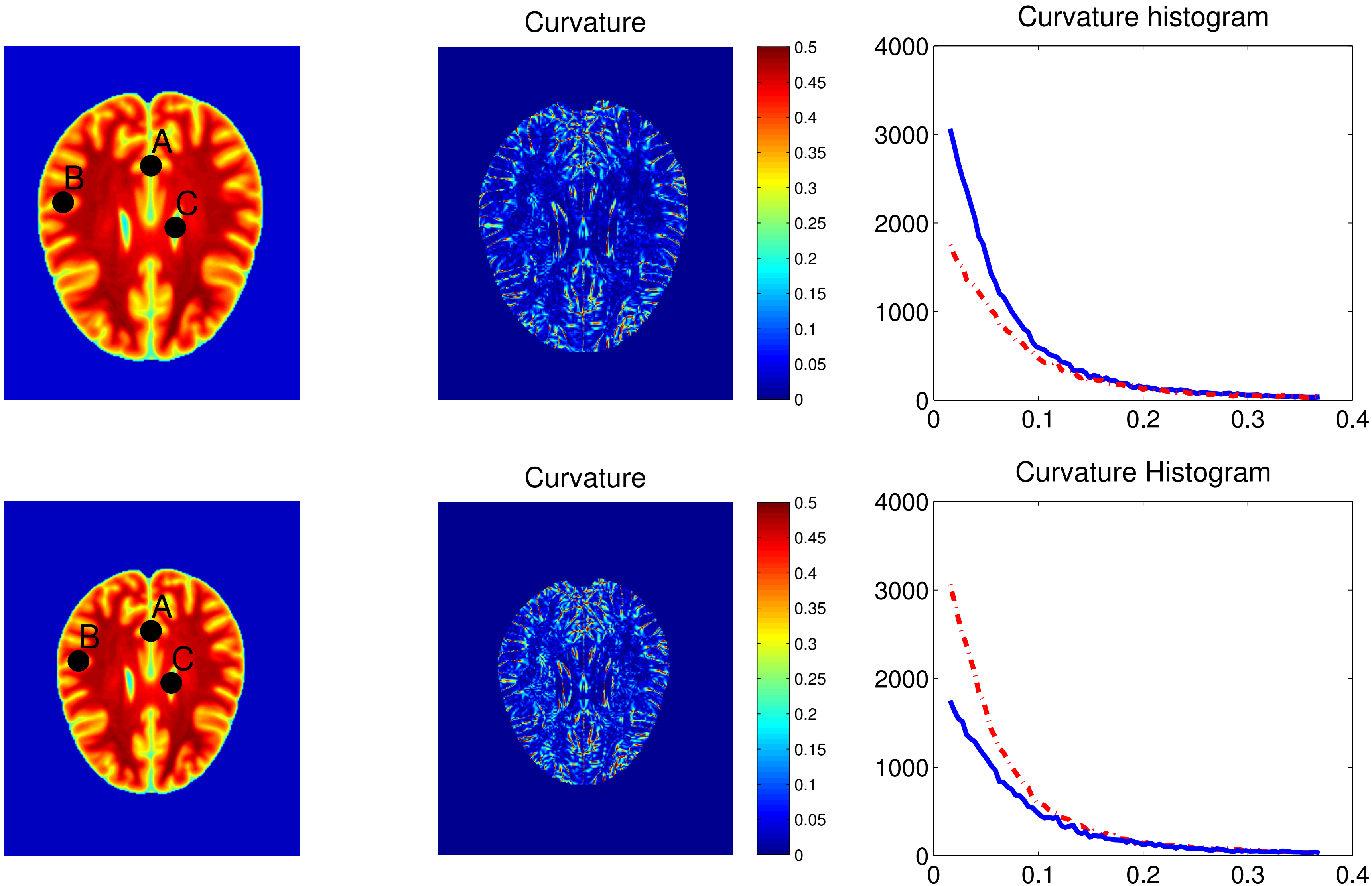


Theorem 4: Raviv and Raskar, SIIMS 2015

$|K_g|\mathcal{I}_{3\times 3}$ Is a level set scale invariant metric

Details in the paper

Raviv and Raskar, SIIMS 2015


A
B
C
Curvature
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Curvature histogram
4000
3000
2000
1000
0
0
0.1
0.2
0.3
0.4
A
B
C
Curvature
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Curvature Histogram
4000
3000
2000
1000
0
0
0.1
0.2
0.3
0.4

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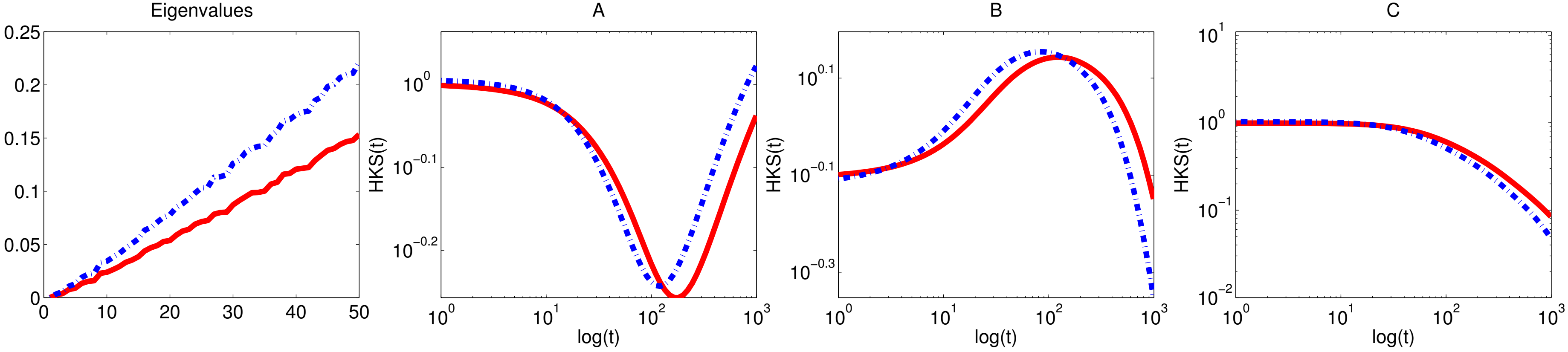
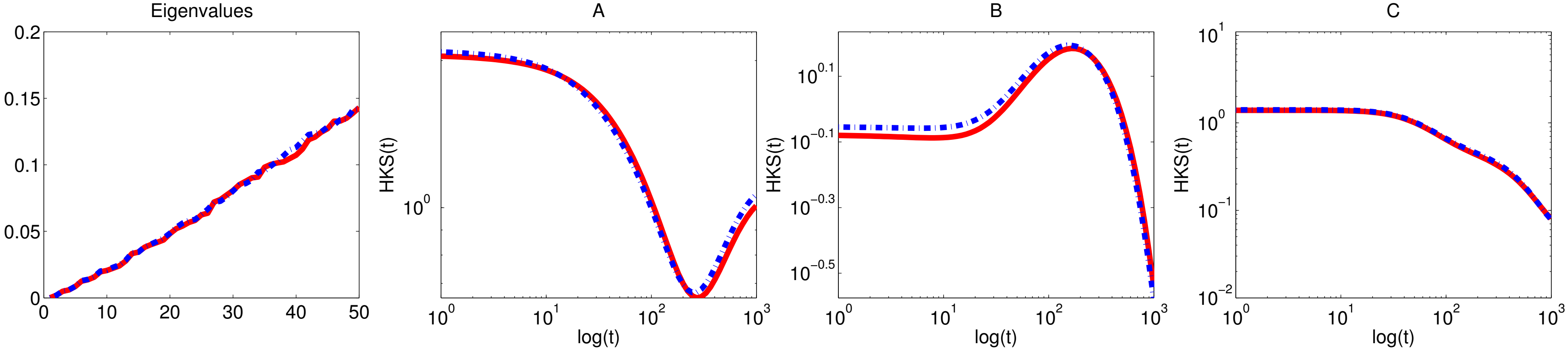
Heat kernel signatures – on volumes

Euclidean



Scale Invariant



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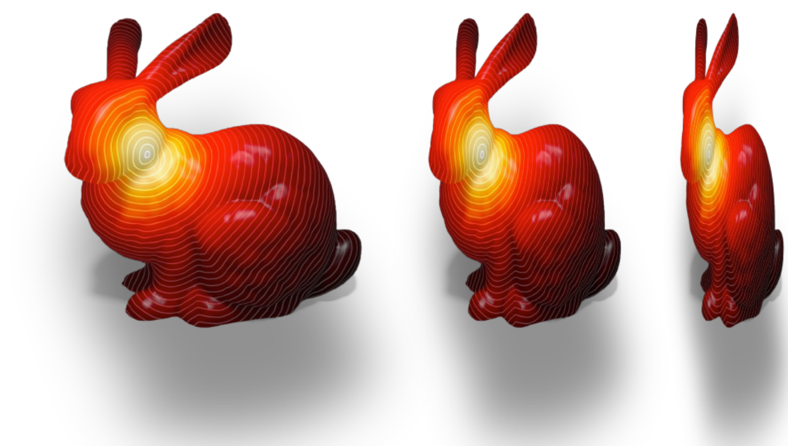
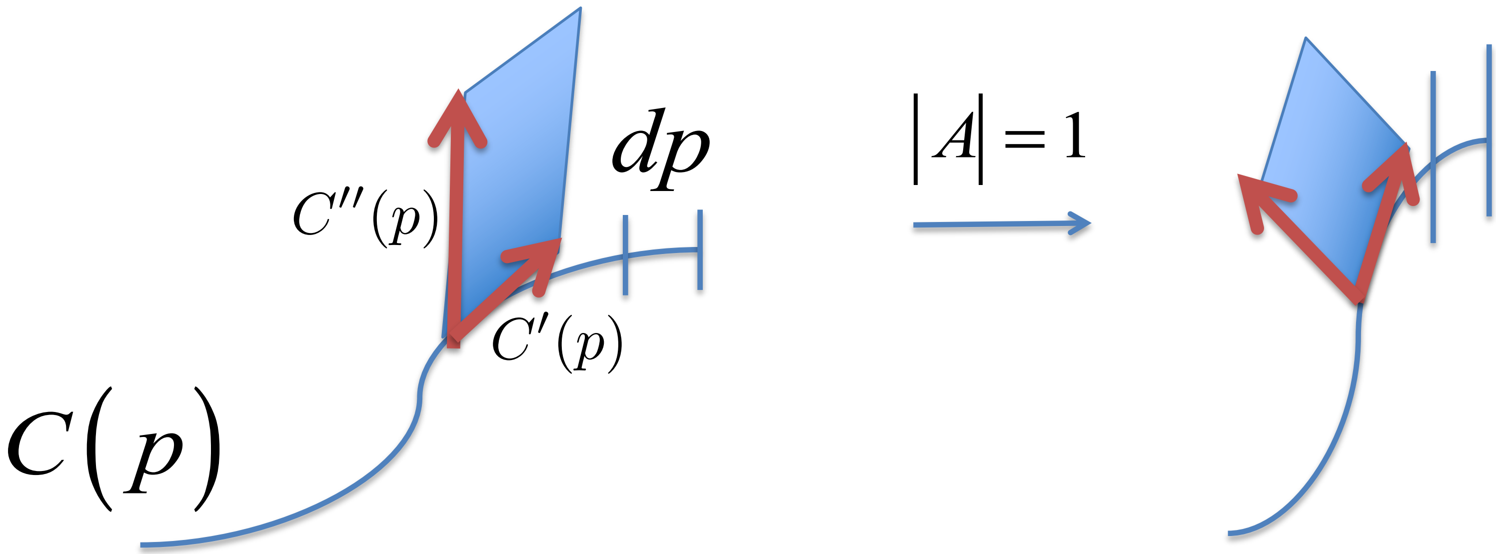
From scale to affine
and
Decomposition of invariants



Concepts – Equi-affine invariant



$$ds = |C'(p)|\,dp \longrightarrow ds = |C'(p) \times C''(p)|^{\frac{1}{3}}\,dp$$

Blaschke'23, Su'84
Raviv JMIV'15

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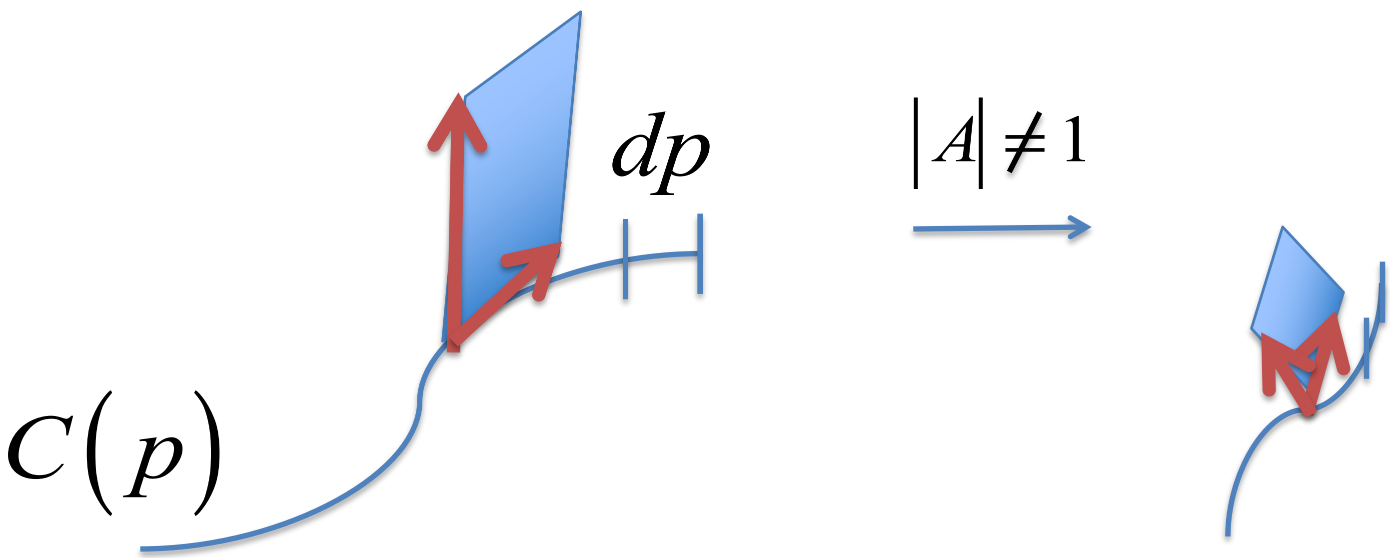
Concepts – Equi-affine invariant



$$ds = |C'(p)|\,dp \longrightarrow ds = \left|K^{EA}\right|^{\frac{1}{2}} \left|C'(p) \times C''(p)\right|^{\frac{1}{3}} dp$$

Equi-affine invariant curvature

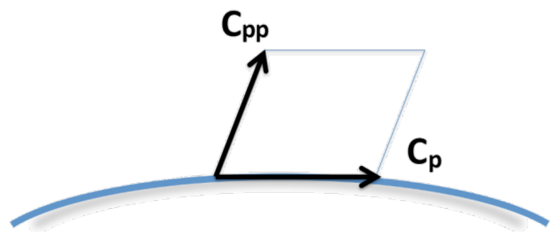
Raviv and Kimmel, IJCV'15

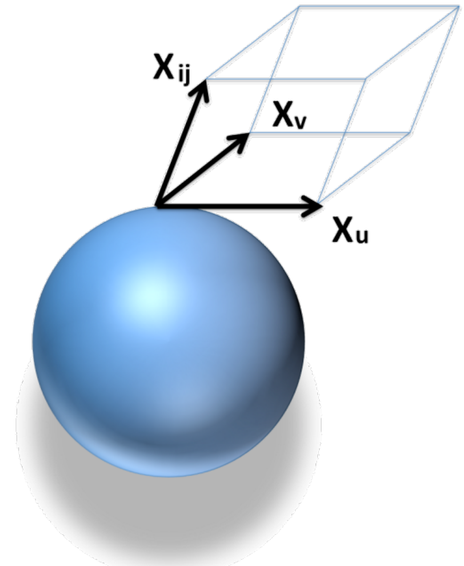
Equi-affine metric in 2D

From curves to surfaces:

Curvature → Gaussian curvature

Parallelogram → parallelepiped





new area form

$$\tilde{q}_{ij} = (X_1, X_2, X_{ij}) = det[X_1\, X_2\, X_{ij}]$$

$$q_{ij} = \frac{\tilde{q}_{ij}}{det^{\frac{1}{4}}\tilde{q}}$$

Blaschke'23, Su'84, Raviv'15

Theorem 5:

$$h_{ij} = \left|K^q\right| \frac{\tilde{q}_{ij}}{\det^{\frac{1}{4}}\left(\tilde{q}\right)} = \left|K^q\right| q_{ij}$$ is affine invariant metric

Proof(sketch)

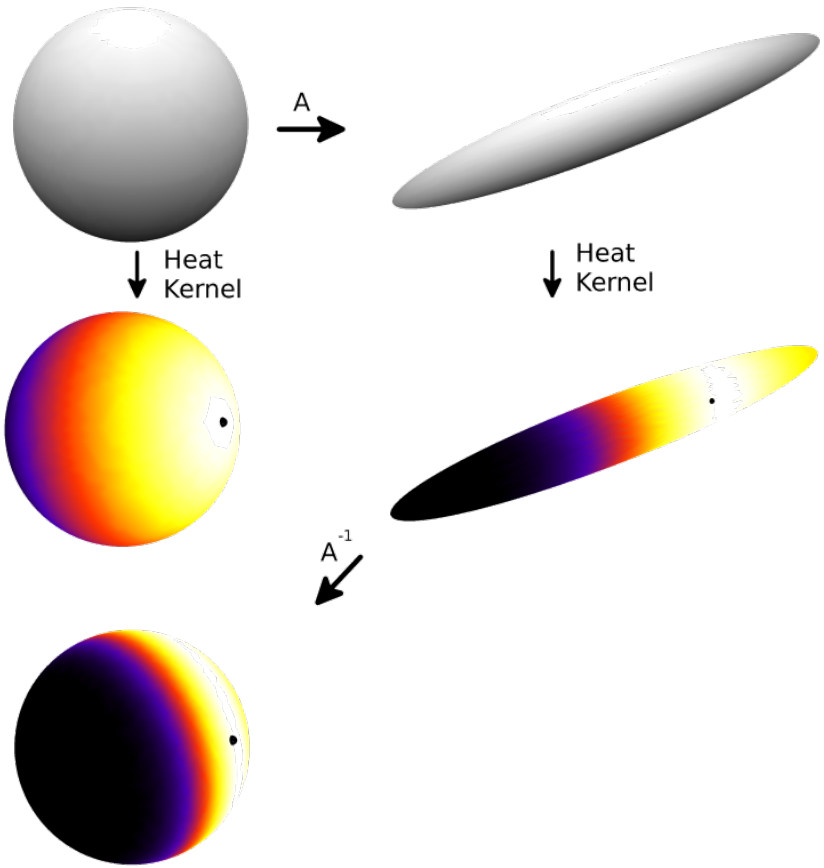
Using Brioschi $K^q = f\left(q_{ij}, q_{ij,m}, q_{ij,mn}\right)$

$$q_{ij} \overset{\alpha}{\rightarrow} \alpha^{1.5} q_{ij} \qquad \det\left(q\right) \overset{\alpha}{\rightarrow} \alpha^{3} \det\left(q\right) \qquad K^q \overset{\alpha}{\rightarrow} \alpha^{-1.5} \kappa^q$$

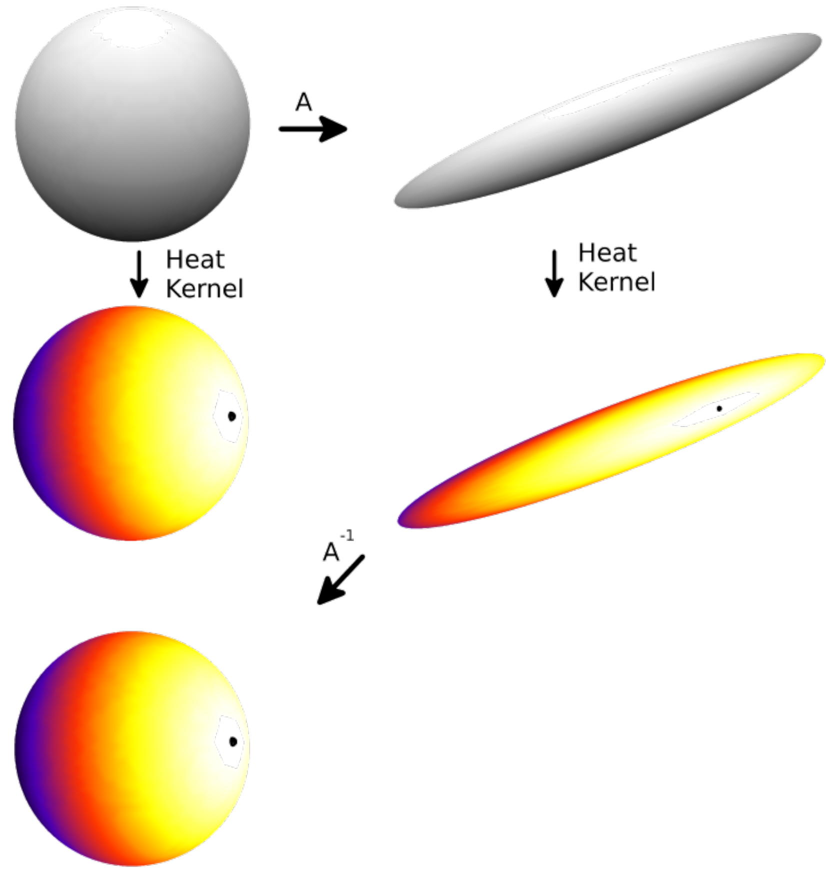
$$\left|K^q\right| q_{ij} \overset{\alpha}{\rightarrow} \left|\alpha^{-1.5} K^q\right| \alpha^{1.5} q_{ij} = \left|K^q\right| q_{ij}$$

Raviv and Kimmel, IJCV'15

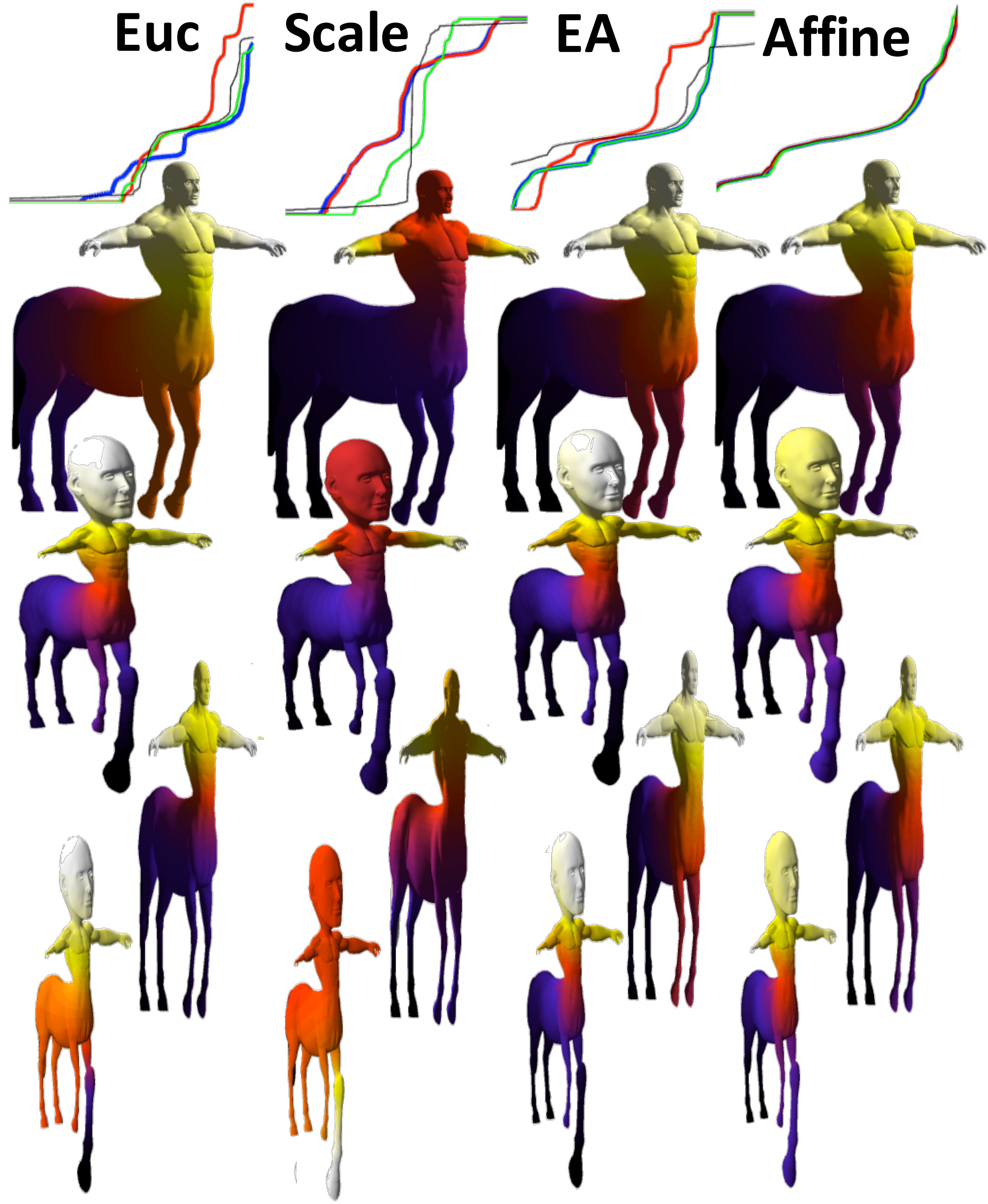
Heat kernel



Equi-affine heat kernel



9'th eigenfunction



Raviv and Kimmel, IJCV'15

Non-rigid alignment

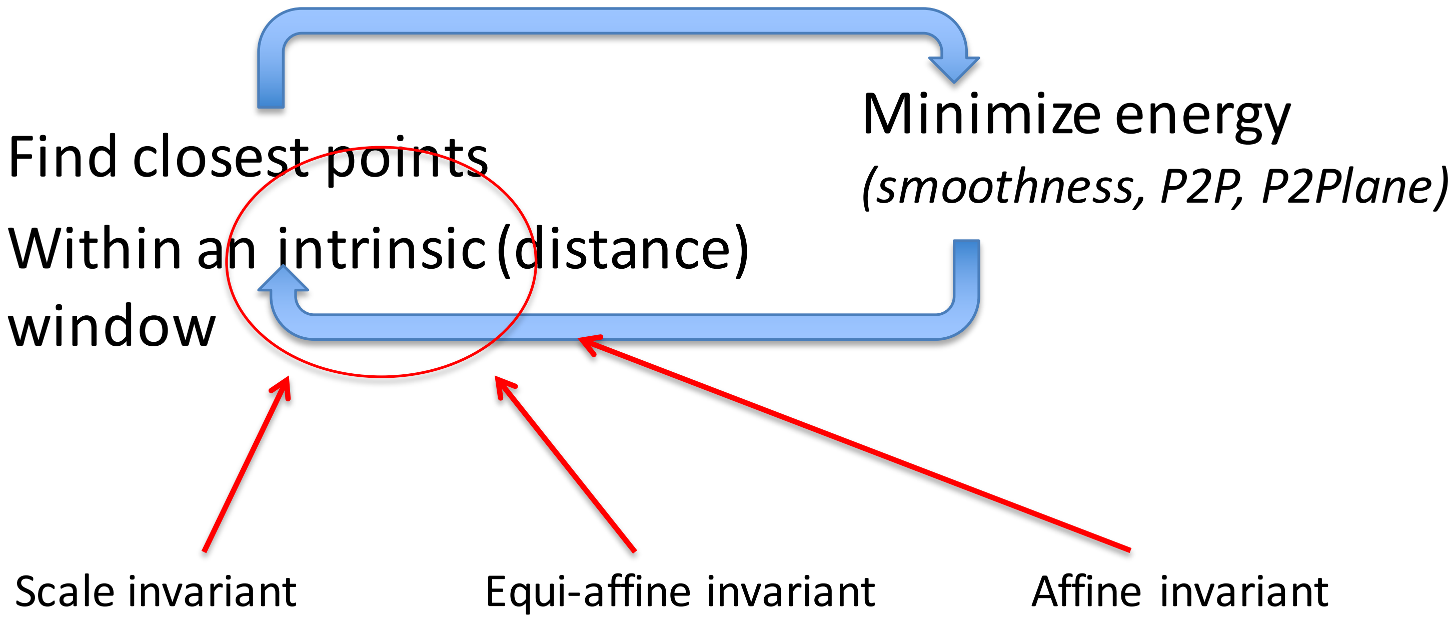
- Non-rigid ICP
- Shape based (metric/conformal)
- Diffeomorphism
- Optical-flows
- B-splines

⋮

Too many people to cite / (extremely) active area

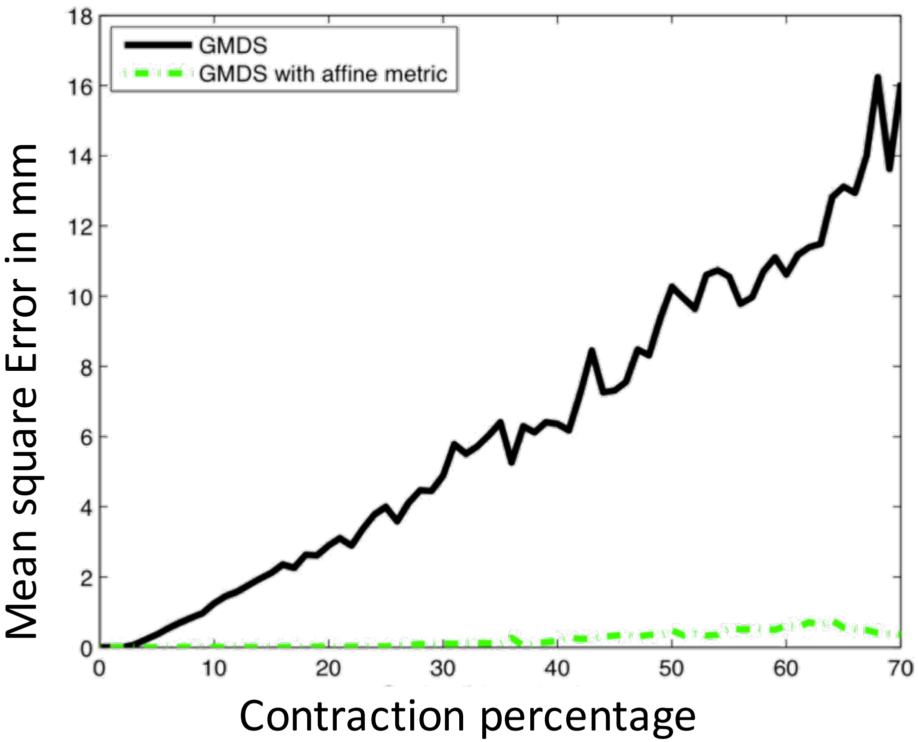
(one example for upgrading a known method)

Intrinsic Non-rigid ICP

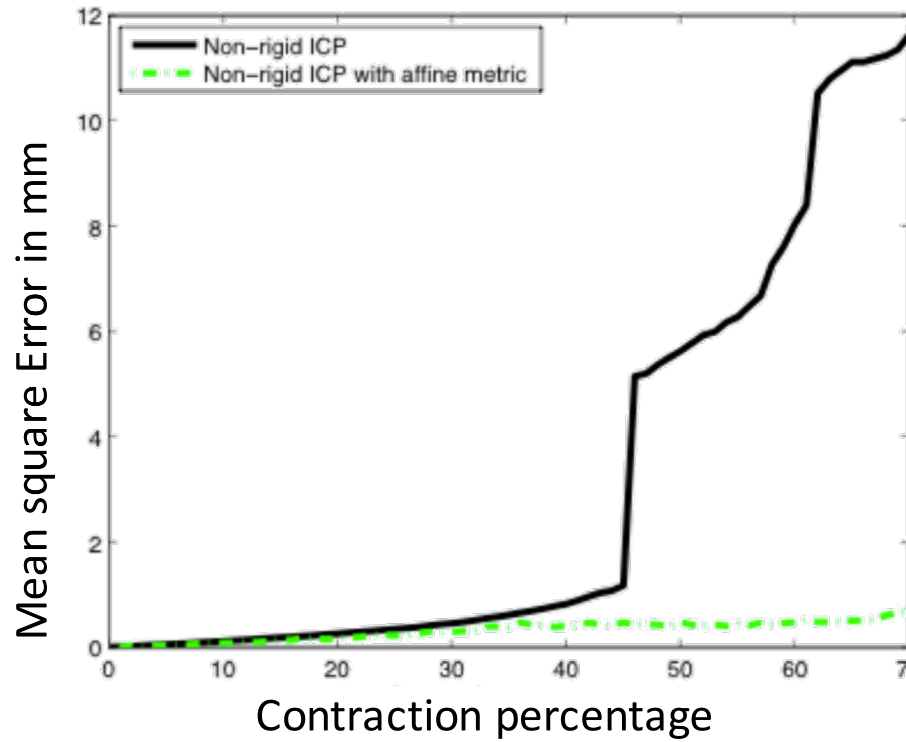


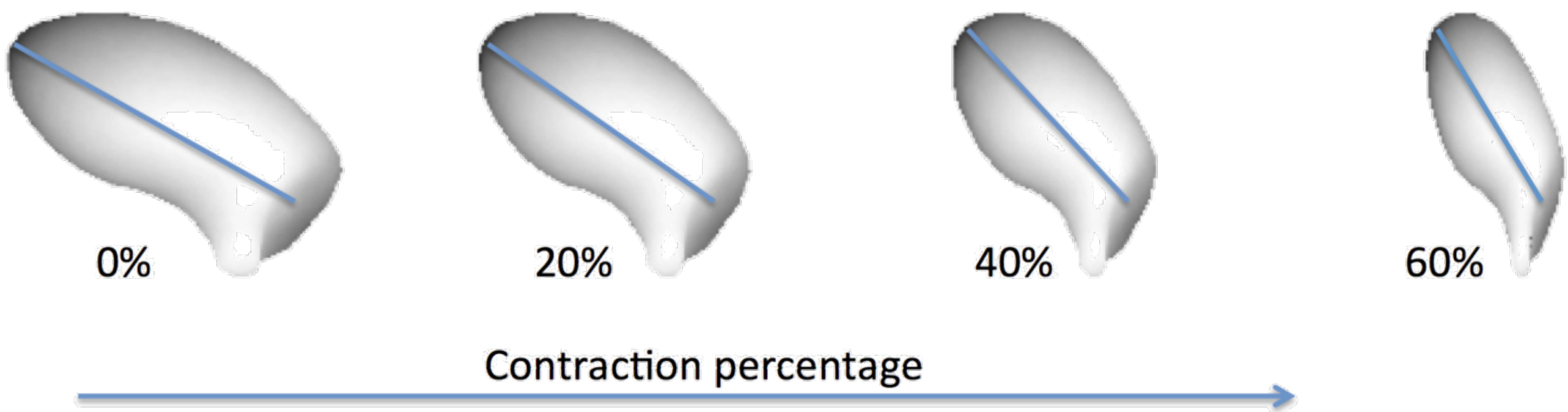
Raviv and Raskar, STACOM (MICCAI) 2015

Generalized Multi-Dimensional Scaling (GMDS)



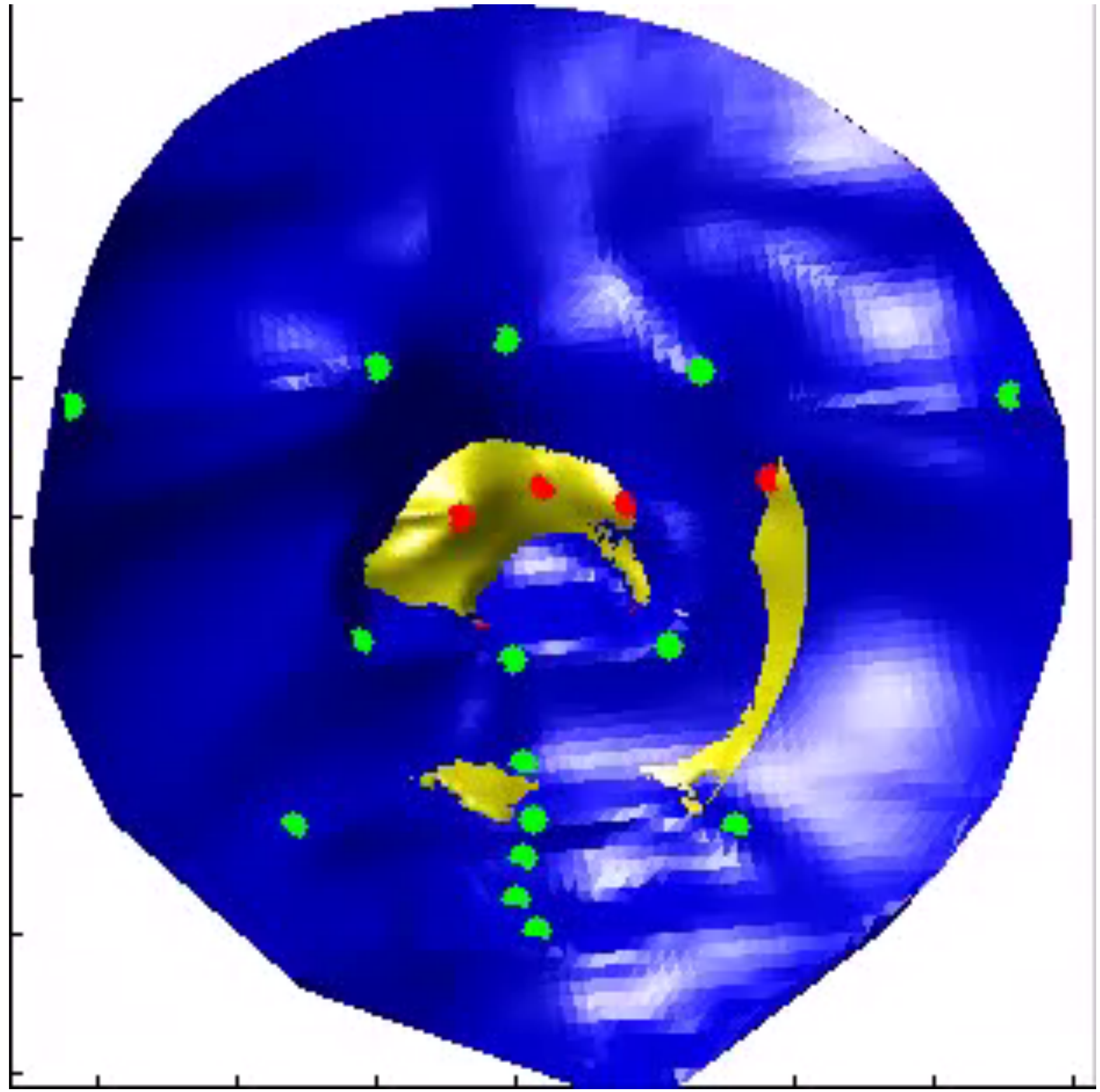
Non-rigid ICP



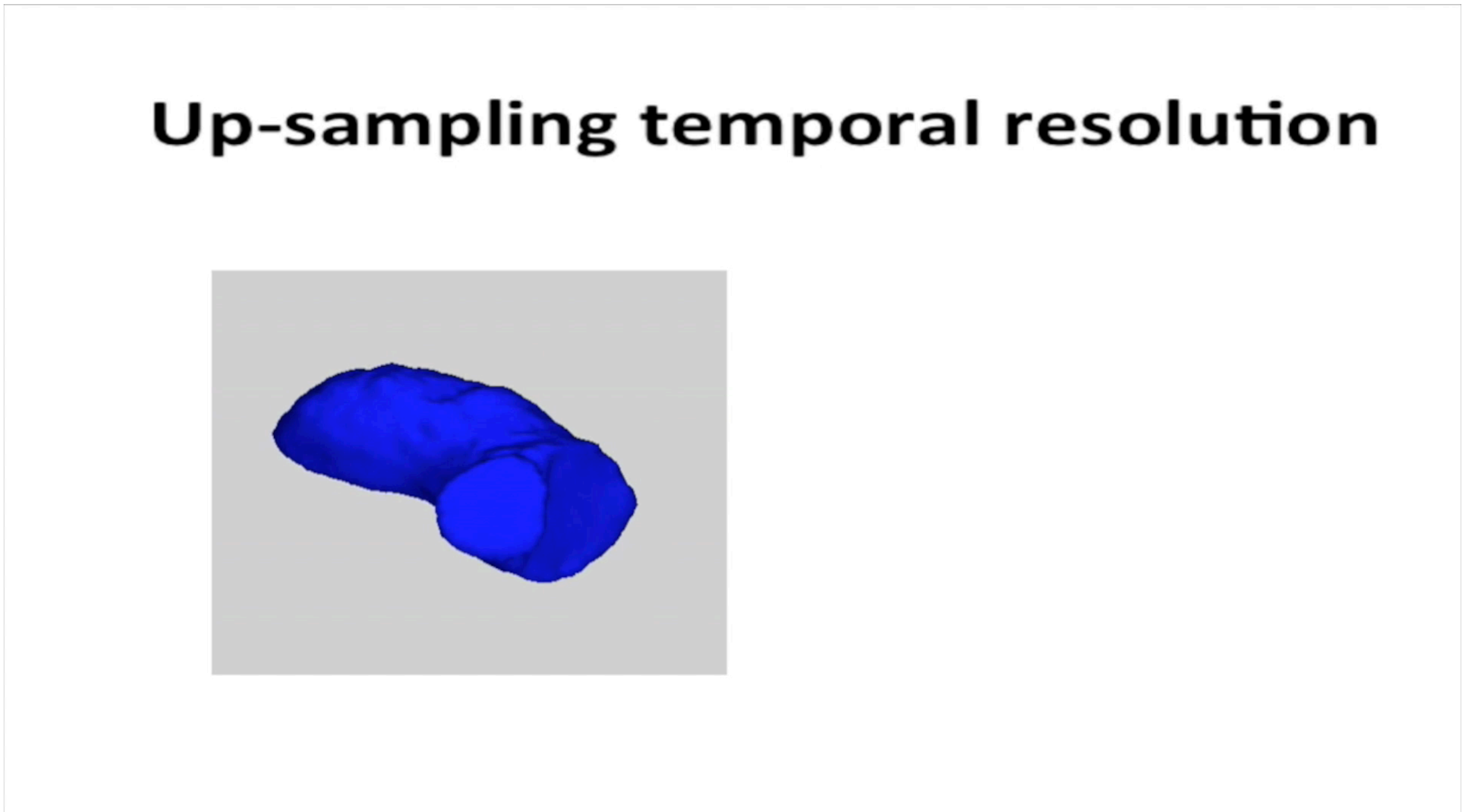


Raviv and Raskar, STACOM (MICCAI) 2015

Slow motion mapping

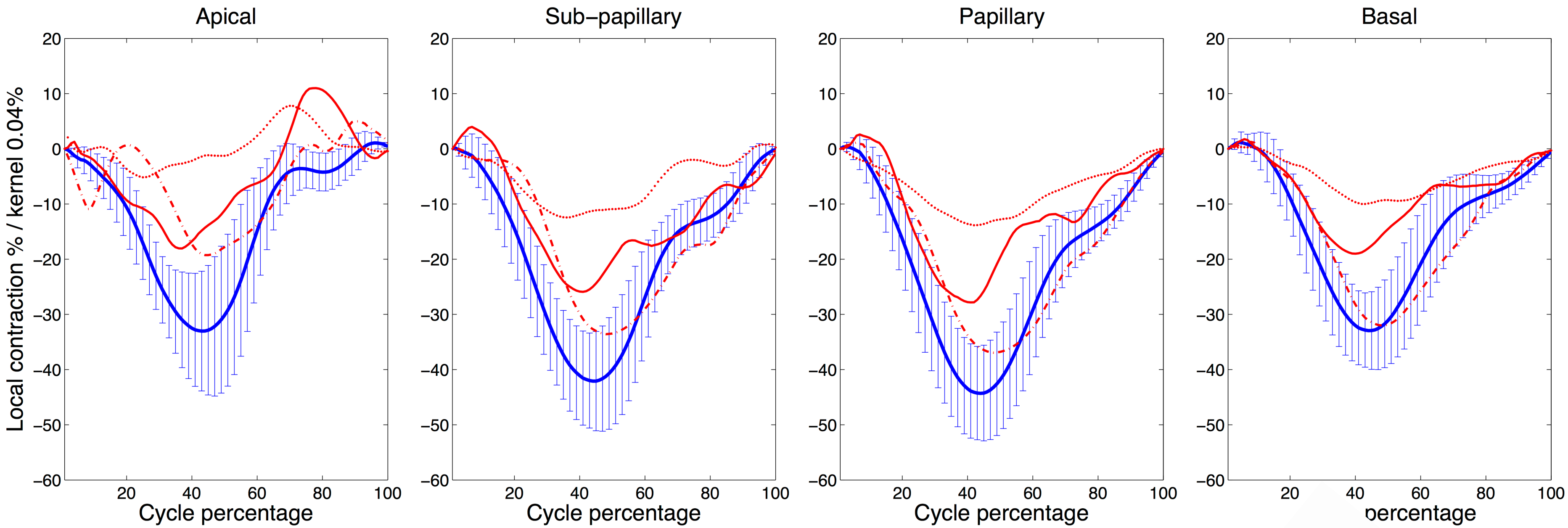


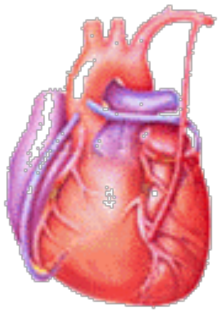
Local Contractions of the left ventricle

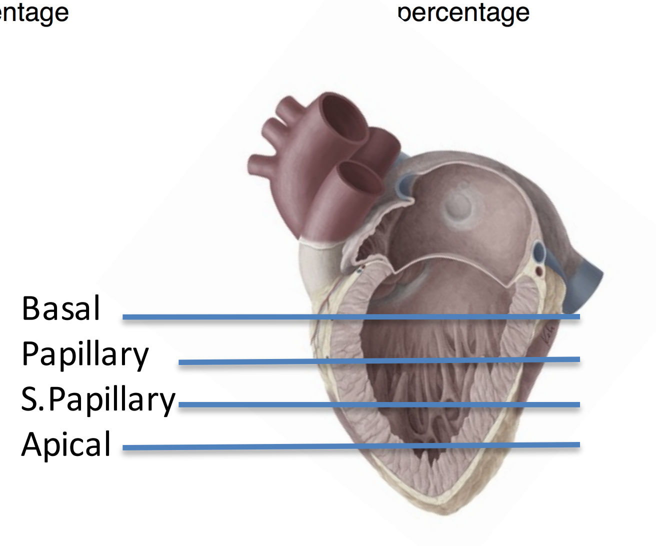


Raviv et. al. STACOM (part of MICCAI) 2014

Atrial Fibrillation



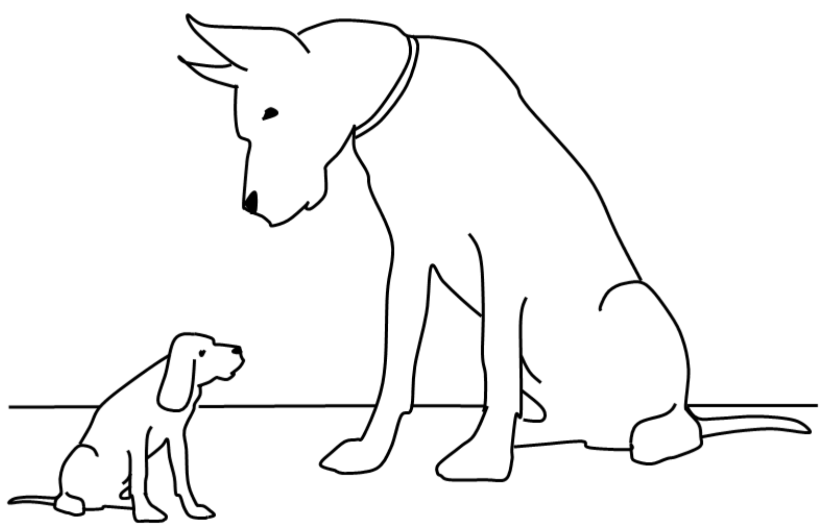




Raviv et. al. STACOM (part of MICCAI) 2014

Summary

- Build metrics which are invariant to $X \in$ {Scale, Equi-affine, Affine}
- Encapsulate those primitives within a non-rigid framework
- Enhance known algorithms (alignment / inference)



Thank you