Absolute Instabilities of Travelling Wave Solutions in a Keller-Segel Model

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This work is done in conjunction with Peter van Heijster (QUT) and Robert Marangell (USyd)

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Introduction

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Or equivalently set $p_1 := p_z$

$$\begin{pmatrix} p \\ p_1 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ cD^{-1} & D^{-1}(\lambda - D_f(\hat{u})) \end{pmatrix} \begin{pmatrix} p \\ p_1 \end{pmatrix} = A(z,\lambda) \begin{pmatrix} p \\ p_1 \end{pmatrix}$$

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Time: $\sim e^{\lambda t}$ **Space**: $A(z,\lambda)$

The Keller-Segel Model

Chemotaxis of a bacteria $w(z)$ towards an attractant $u(z)$ with logarithmic chemosensitivity in a moving frame $z = x - ct$:

$$u_t = \varepsilon u_{zz} + c u_z - w u^m$$
$$w_t = w_{zz} + c w_z - \beta \left(\frac{w u_z}{u} \right)_z ,$$

with $(z, t) \in (\mathbb{R}, \mathbb{R}^+)$, $\beta > 1 - m$, $0 \le m \le 1$ and $0 \le \varepsilon \ll 1$.

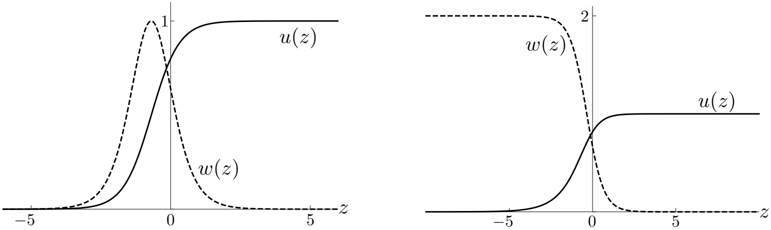


Figure: Travelling wave solution in the moving frame $z = x - ct$

Linearised operator and eigenvalue problem

Set $m = 0$
Perturbation: $u(z,t) = \hat{u}(z) + p(z,t)$, and $w(z,t) = \hat{w}(z) + q(z,t)$
The associated eigenvalue problem:

$$\lambda \begin{pmatrix} p \\ q \end{pmatrix} = \mathcal{L} \begin{pmatrix} p \\ q \end{pmatrix} := \begin{pmatrix} \varepsilon \partial_{zz} + c\frac{\partial}{\partial z} & -1 \\ \mathcal{L}_p & \mathcal{L}_q \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

with $p, q \in H^1(\mathbb{R})$

$$\mathcal{L}_p := \beta \left(\frac{w_z u_z}{u^2} + \frac{w u_{zz}}{u^2} - \frac{2 w u_z^2}{u^3} \right) + \beta \left(\frac{2 w u_z}{u^2} - \frac{w_z}{u} \right) \frac{\partial}{\partial z} - \frac{\beta w}{u} \frac{\partial^2}{\partial z^2}$$

$$\mathcal{L}_q := \beta \left(\frac{u_z^2}{u^2} - \frac{u_{zz}}{u} \right) + \left(c - \frac{\beta u_z}{u} \right) \frac{\partial}{\partial z} + \frac{\partial^2}{\partial z^2}$$

For simplicity we set $\varepsilon = 0$ and transform $\mathcal{L} - \lambda I$ into a 1D system. Introduce variable $s := q_z$;

$$\mathcal{T}(\lambda)\begin{pmatrix} p \\ q \\ s \end{pmatrix} := \begin{pmatrix} p \\ q \\ s \end{pmatrix}' - \underbrace{\begin{pmatrix} \frac{\lambda}{c} & \frac{1}{c} & 0 \\ 0 & 0 & 1 \\ \mathcal{F} & \mathcal{G} & \mathcal{H} \end{pmatrix}}_{A(z,\lambda)} \begin{pmatrix} p \\ q \\ s \end{pmatrix} = 0$$

$$\mathcal{F} = \beta\left(\frac{2wu_z^2}{u^3} - \frac{w_z u_z}{u^2} - \frac{wu_{zz}}{u^2}\right) + \frac{\lambda\beta}{c}\left(\frac{w_z}{u} - \frac{2wu_z}{u^2}\right) + \frac{\lambda^2\beta w}{c^2 u}$$

$$\mathcal{G} = \beta\left(\frac{u_{zz}}{u} - \frac{u_z^2}{u^2}\right) + \frac{\beta}{c}\left(\frac{w_z}{u} - \frac{2wu_z}{u^2}\right) + \frac{\lambda\beta}{c^2}\left(\frac{w}{u}\right) + \lambda$$

$$\mathcal{H} = \frac{\beta u_z}{u} - c + \frac{\beta}{c}\frac{w}{u}$$

Spectrum

The spectrum of the operator falls into two parts:
The **essential spectrum** indicates stability of perturbations 'at infinity', i.e. of the asymptotic end states when $z \to \pm\infty$.
The **point spectrum** indicates the stability of the nonlinear part of the front and involves the full linearised equation

Definition: Essential Spectrum

Define the matrices $A_\pm(\lambda) := \lim_{z \to \pm\infty} A(z, \lambda)$. If either

- $A_-(\lambda)$ and $A_+(\lambda)$ are both hyperbolic but have a different number of unstable matrix eigenvalues
- $A_-(\lambda)$ or $A_+(\lambda)$ has at least one purely imaginary matrix eigenvalue

then $\lambda \in \sigma_{\text{ess}}$

Spectrum of KS Model

Define the matrices $A_{\pm}(\lambda) := \lim_{z\to\pm\infty} A(z,\lambda)$;

$$A_{+}(\lambda) := \begin{pmatrix} \frac{\lambda}{c} & \frac{1}{c} & 0 \\ 0 & 0 & 1 \\ 0 & \lambda & -c \end{pmatrix}$$

which has characteristic polynomial $(c\mu - \lambda)(\mu^2 + c\mu - \lambda) = 0$

$$A_{-}(\lambda) := \begin{pmatrix} \frac{\lambda}{c} & \frac{1}{c} & 0 \\ 0 & 0 & 1 \\ \frac{\beta\lambda\left((\beta-1)\lambda - c^2\right)}{(\beta-1)^2} & \frac{\left(2\beta^2-3\beta+1\right)\lambda - c^2\beta}{(\beta-1)^2} & \frac{c(\beta+1)}{\beta-1} \end{pmatrix}$$

which has characteristic polynomial

$$\mu^3 - \frac{\mu^2\left((\beta-1)\lambda + (\beta+1)c^2\right)}{(\beta-1)c} - \frac{\mu\left(\left(\beta^2 - 3\beta + 2\right)\lambda - \beta c^2\right)}{(\beta-1)^2} + \frac{\lambda^2}{c} = 0.$$

Essential Spectrum for Keller-Segel

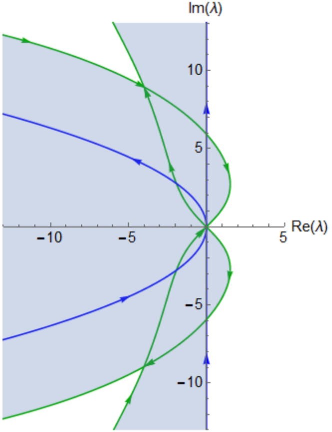


Figure: Essential spectrum (shaded regions). Arrows denote orientation with respect to k

Set $\mu = ik$ in the characteristic equation to obtain dispersion relations

From $A_+(\lambda)$:

$$\lambda = -k^2 + ick$$
$$\lambda = ick$$

From $A_-(\lambda)$:

$$\lambda^2 + \left(k^2 - \frac{i(\beta-2)ck}{\beta-1}\right)\lambda + \frac{(\beta+1)c^2k^2}{\beta-1} + \frac{ick\left(\beta c^2 - (\beta-1)^2k^2\right)}{(\beta-1)^2} = 0$$

For all parameter values the essential spectrum enters the right half plane [Nagai et al. 1991]

Weighted Spaces

Weighting the space restricts the types of perturbations we apply. We make the substitution $\begin{pmatrix} \tilde{p} \\ \tilde{q} \\ \tilde{s} \end{pmatrix} = e^{\nu z} \begin{pmatrix} p \\ q \\ s \end{pmatrix}$ and use the weighted space $H^1_\nu(\mathbb{R})$ defined by the norm

$$\|p\|_{H^1_\nu} = \|e^{\nu z} p\|_{H^1} = \|\tilde{p}\|_{H^1}.$$

Using a two-sided weight

$$\nu = \begin{cases} \nu_- \text{ if } & z < 0 \\ \nu_+ \text{ if } & z > 0. \end{cases}$$

gives the operator

$$\mathcal{T}(\lambda) \begin{pmatrix} \tilde{p} \\ \tilde{q} \\ \tilde{s} \end{pmatrix} = \begin{pmatrix} \tilde{p} \\ \tilde{q} \\ \tilde{s} \end{pmatrix}' - (A(z,\lambda) + \nu I) \begin{pmatrix} \tilde{p} \\ \tilde{q} \\ \tilde{s} \end{pmatrix}$$

Absolute Spectrum

- Absolute spectrum gives an indication of how far the essential spectrum can be shifted
- If the absolute spectrum moves into the right half plane it indicates the onset of an absolute instability

Definition-Absolute Spectrum

For a given $\lambda \in \mathbb{C}$ we rank and label the spatial eigenvalues by their real part, i.e.

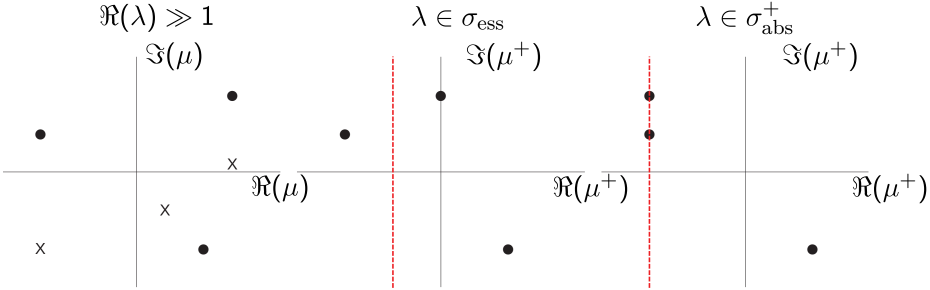
$$\Re(\mu_1^{\pm}) \geq \Re(\mu_2^{\pm}) \geq \Re(\mu_3^{\pm}).$$

The absolute spectrum consists of the values λ such that either $\Re(\mu_2^{+}) = \Re(\mu_3^{+})$ or $\Re(\mu_2^{-}) = \Re(\mu_3^{-})$, in this case.

Essential and Absolute Spectrum

Spatial Eigenvalues of $A_-(\lambda)$: $\times$

Spatial Eigenvalues of $A_+(\lambda)$: $\bullet$



To the Right of the Essential Spectrum

Essential Spectrum

Absolute Spectrum

Image adapted from Kapitula et al 2013

Absolute Spectrum-Plus Infinity

Characteristic polynomial of $A_+(\lambda)$ is $(c\mu - \lambda)(\mu^2 + c\mu - \lambda) = 0$ which has roots

$$\mu_1 = \frac{\lambda}{c}, \qquad \mu_2 = \frac{-c + \sqrt{c^2 + 4\lambda}}{2}, \qquad \mu_3 = \frac{-c - \sqrt{c^2 + 4\lambda}}{2}$$

Absolute spectrum corresponds to $\Re(\mu_2) = \Re(\mu_3)$ when $\Re(\lambda) > -c^2/2$ and $\Re(\mu_1) = \Re(\mu_3)$ if $\Re(\lambda) < -c^2/2$, i.e.

$$\sigma^+_{\text{abs}} = \left\{ \lambda \in \mathbb{R} \middle| -\frac{c^2}{2} \leq \lambda \leq \frac{-c^2}{4} \right\} \cup \left\{ \lambda = \lambda_1 + i\lambda_2,\ \lambda_1, \lambda_2 \in \mathbb{R} \middle| \lambda_1 < -\frac{c^2}{2};\ \lambda_2 = \pm\lambda_1 \left(1 + \frac{2\lambda_1}{c^2}\right) \right\}.$$

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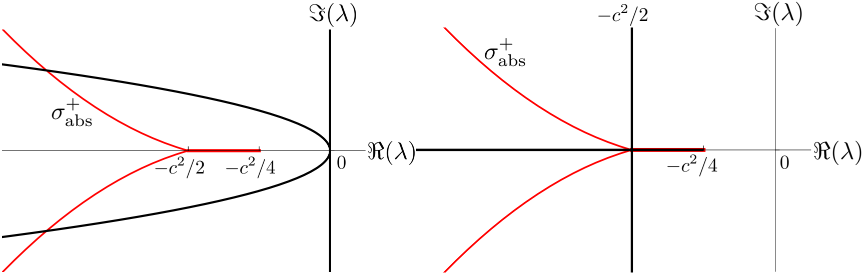
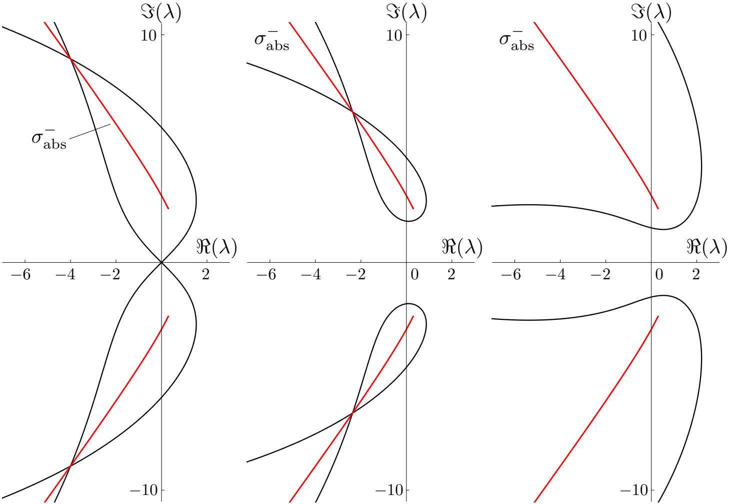


Figure: Absolute spectrum associated with the state $z \to \infty$. Left: $\nu_+ = 0$, Right $\nu_+ = c/2$

Absolute Spectrum-Minus Infinity



Summary of Results, $m = 0$

Absolutely unstable

$\beta > \beta_{crit}$

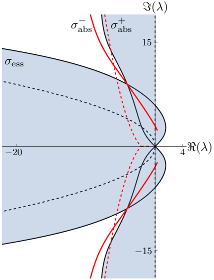


Figure: Unweighted function space

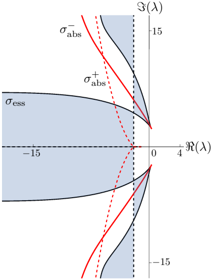


Figure: Ideally weighted function space

Transiently unstable

$\beta < \beta_{crit}$

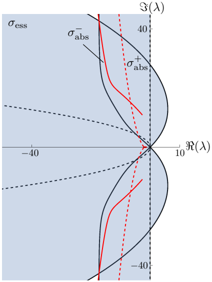


Figure: Unweighted function space

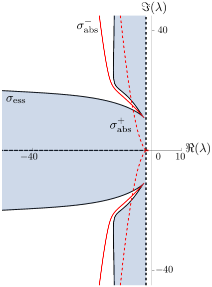


Figure: Ideally weighted function space

The branch point is a second order root of the characteristic polynomial of $A_{-}(\lambda)$ for some purely imaginary λ.

Theorem

There exists a value $\beta = \beta_{crit}$, independent of speed, such that the essential spectrum can be weighted so $Re(\lambda) < 0$ for all $\lambda \in \sigma_{ess}$ when $\beta < \beta_{crit}$. This value β_{crit} is found as the largest root of

$$310\beta^{10} - 3234\beta^{9} + 17112\beta^{8} - 49101\beta^{7} + 76180\beta^{6} - 58398\beta^{5} + 10056\beta^{4} + 15040\beta^{3} - 9680\beta^{2} + 1716\beta - 4 = 0.$$

For $\varepsilon = 0$ we have $\beta \approx 1.619$

This value is a bifurcation from a conditionally spectrally stable to an absolutely unstable regime.

Essential and absolute spectrum for $0 \leq m < 1$

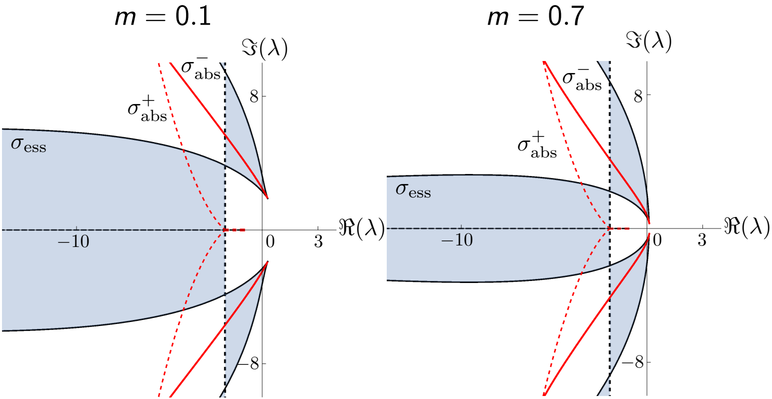


Figure: Ideally weighted essential spectrum for $\beta = 2$

Critical Parameters

For $0 \leq m < 1$ we have

$$\beta^m_{crit} := \beta_{crit}(1 - m)$$

Essential and absolute spectrum for $m = 1$

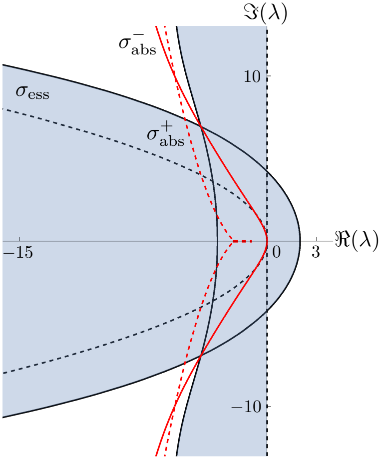


Figure: Ideally weighted essential spectrum for $\nu_- = 0$ (left) $\nu_- = -c/\beta$ (right)

Results

The absolute spectrum contains $\lambda = 0$ for all $\beta > 1$

Small diffusivity of attractant $0 < \varepsilon \ll 1$

Keller-Segel Model:

$$u_t = \varepsilon u_{zz} + c u_z - w u^m$$
$$w_t = w_{zz} + c w_z - \beta \left(\frac{w u_z}{u} \right)_z ,$$

• For $|\lambda| = \mathcal{O}(1)$ the results are to leading order the same.

• The weighted essential spectrum does not cross into the right half plane except for in the region $|\lambda| = \mathcal{O}(1)$ as long as $\nu_- > -\frac{c(\beta+m)}{\beta+m-1}$

Results

For $0 < \varepsilon \ll 1$ there exists

$$\beta^{\varepsilon}_{crit} := \beta_{\text{crit}} + \mathcal{O}(\varepsilon)$$

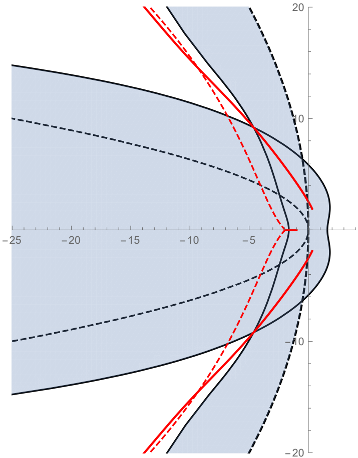


Figure: Essential and absolute spectrum $0 < \varepsilon \ll 1$ and $\beta > \beta_{\text{crit}}$

Point Spectrum

The eigenvalue associated with translation invariance ($\lambda = 0$) is contained in the essential spectrum in the unweighted space. The eigenvalue is order 2 for all parameter values and we have an eigenvector $(u_z, w_z)^T$;

$$\mathcal{L}\begin{pmatrix} u_z(z) \\ w_z(z) \end{pmatrix} = 0$$

as a result of the translation invariance and a generalised eigenvector

$$\mathcal{L}\begin{pmatrix} u_c(z) \\ w_c(z) \end{pmatrix} = -\begin{pmatrix} u_z(z) \\ w_z(z) \end{pmatrix}$$

It has been shown numerically that there are no eigenvalues in a large region of the complex plane-$|\lambda| \sim \mathcal{O}(10^9)$ in the right half plane (discluding the absolute spectrum)

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References

Absolute Instabilities of Travelling Wave Solutions in a Keller-Segel Model (Submitted) P. N. Davis, P. van Heijster, R. Marangell

Keller-Segel Stability

Traveling Waves in a Chemotactic Model, T. Nagai and T. Ikeda, J. Math. Biology, 1991

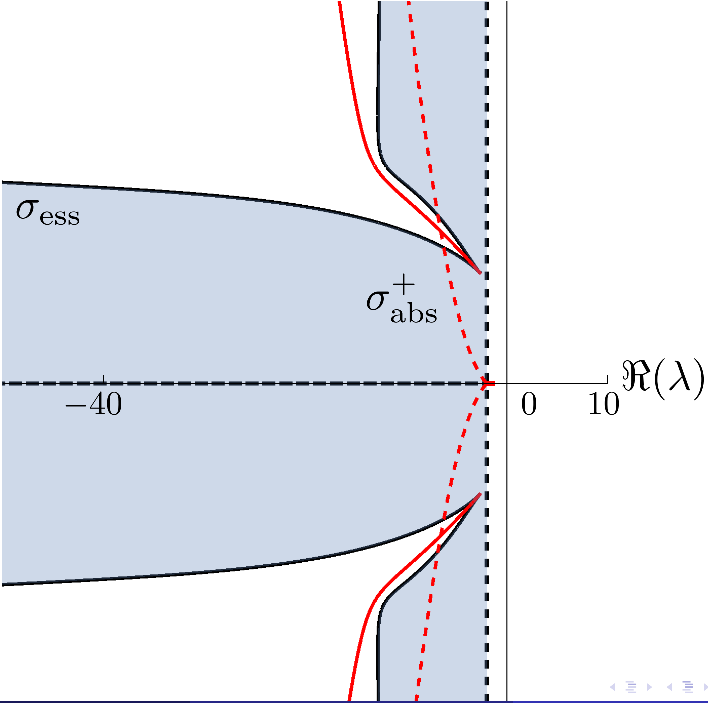
Mathematics of traveling waves in chemotaxis, Z. Wang, Discrete Contin. Dyn. Syst., 2013

Stability of traveling waves of the KellerSegel system with logarithmic sensitivity, J. Li, T. Li, and Z. Wang, Math. Models Methods Appl. Sci.

Stability Theory

Stability of travelling waves, B. Sandstede, Handb. Dyn. Syst., 2002

Spectral and dynamical stability of nonlinear waves, T. Kapitula and K. Promislow, Springer, 2013



σ_{ess}
σ_{abs}^{+}
$\Re(\lambda)$
-40
0
10

$\beta \to \infty$

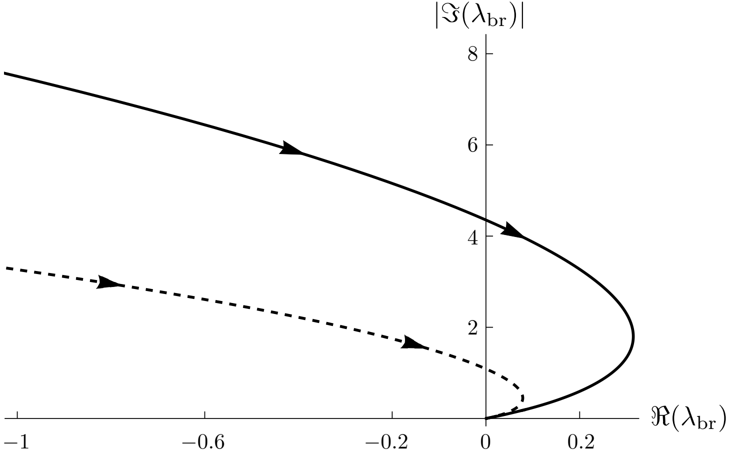


Figure: Location of branchpoints with increasing β and $m = 0$.

Range of admissable weights

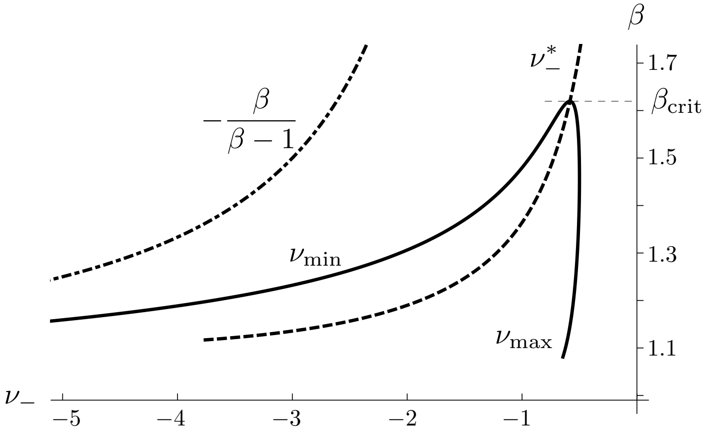


Figure: Range of weights such that essential spectrum is contained in the left half plane (solid line)