Particle Laden Flow: Theory and Experiment

Andrea Bertozzi

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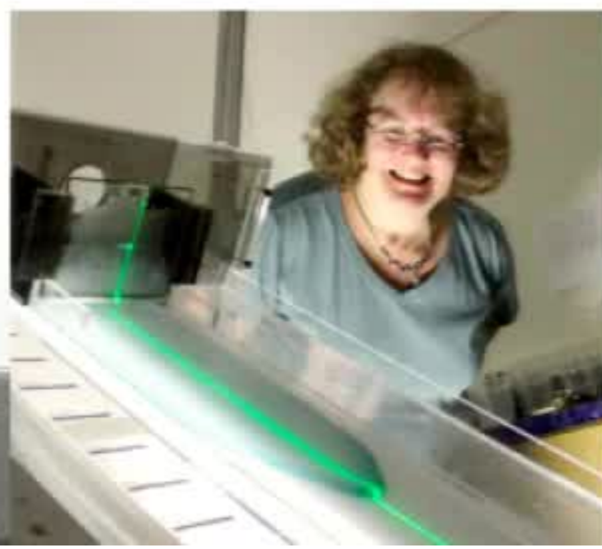


Support and Personnel

NSF grant DMS-1048840, RAPID: Modeling and experiments of oil-particulate mixtures of relevance to the Gulf of Mexico oil spill

NSF DMS-1312543 Particle laden flows - theory, analysis and experiment.

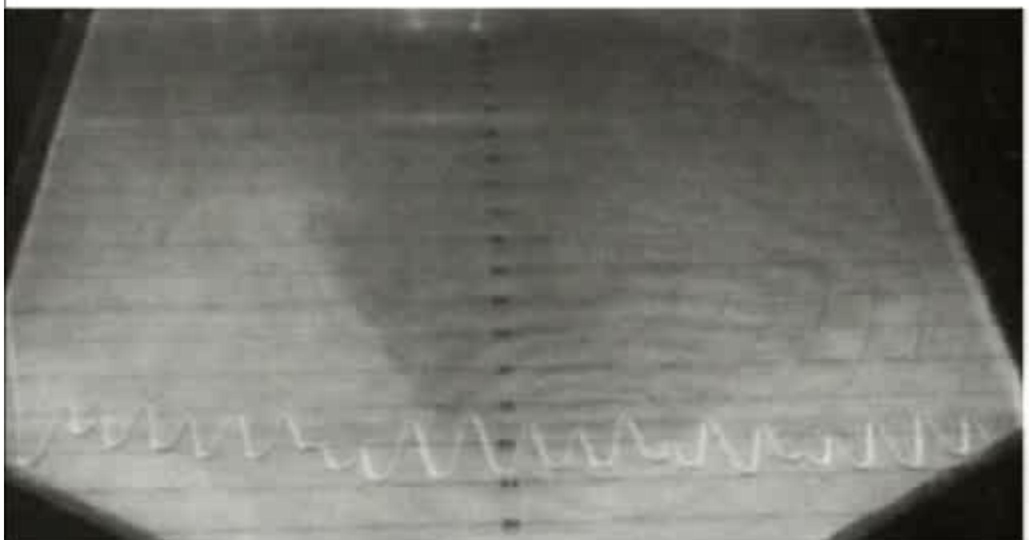
UC Lab Fees Research Grant 09-LR-04-116741-BERA, "Multiscale methods of fracture and multimaterial debris flow

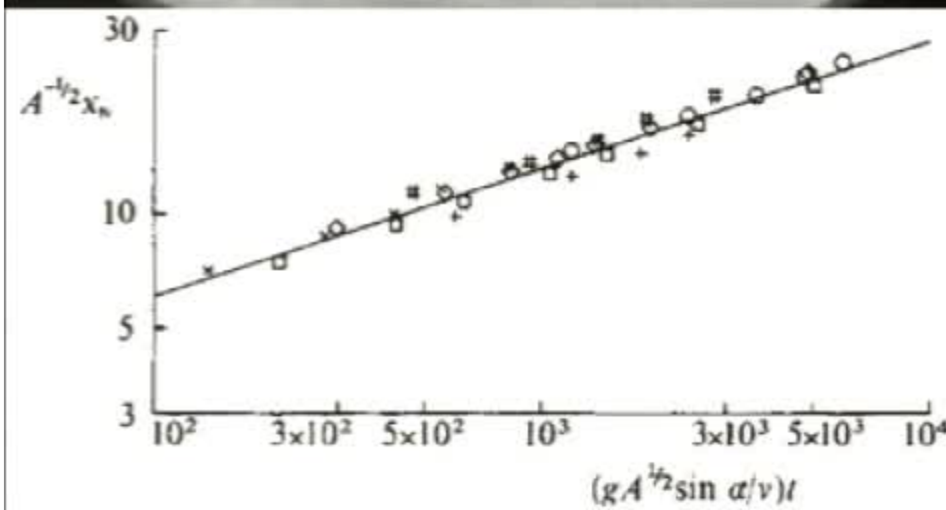
NSF Workforce grants DMS-0601395, DMS-1045536 large summer REU program

Thick oil from the Deepwater Horizon spill is found on a beach in Gulfport, Miss.

Photo from USA Today

Huppert Nature 1982





- Clear Fluid on an Incline
- A is cross sectional area

$$h_t + (g\sin\alpha/\nu)h^2 h_x = 0$$

$$h = (\nu/g\sin\alpha)^{1/2}x^{1/2}t^{-1/2}$$
$$0 \leq x \leq x_N = (9A^2 g\sin\alpha/4\nu)^{1/3}t^{1/3}$$

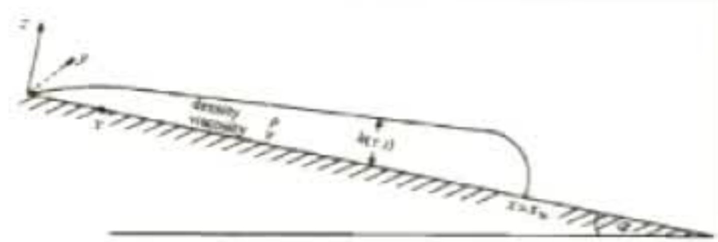


Fig. 2 A sketch of the flow and coordinate system.

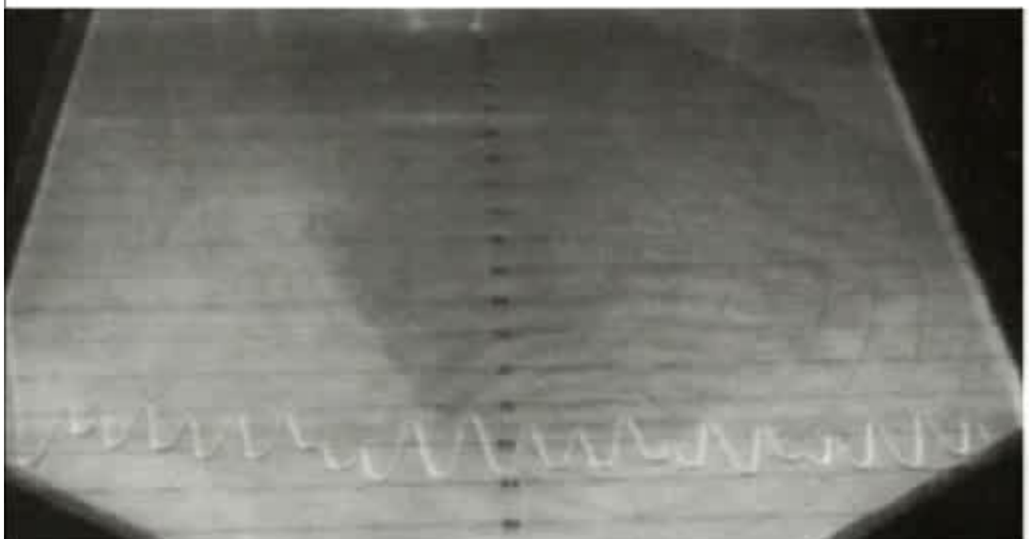
Model derivation

- Flux equations
  - $div\ \Pi + \rho(\phi) g = 0,\ div\ j = 0$
  - $\Pi = -pI + \mu(\phi)(grad\ j + (grad\ j)^T)$ *stress tensor*
  - j = *volume averaged flux,*
  - ρ=*effective density*
  - μ = *effective viscosity*
  - p = *pressure*
  - ϕ = *particle concentration*
  - $j_p = \phi v_p,\ j_f = (1-\phi)\ v_f,\ j = j_p + j_f$

J. Zhou et al PRL 2005

Original model

Huppert Nature 1982



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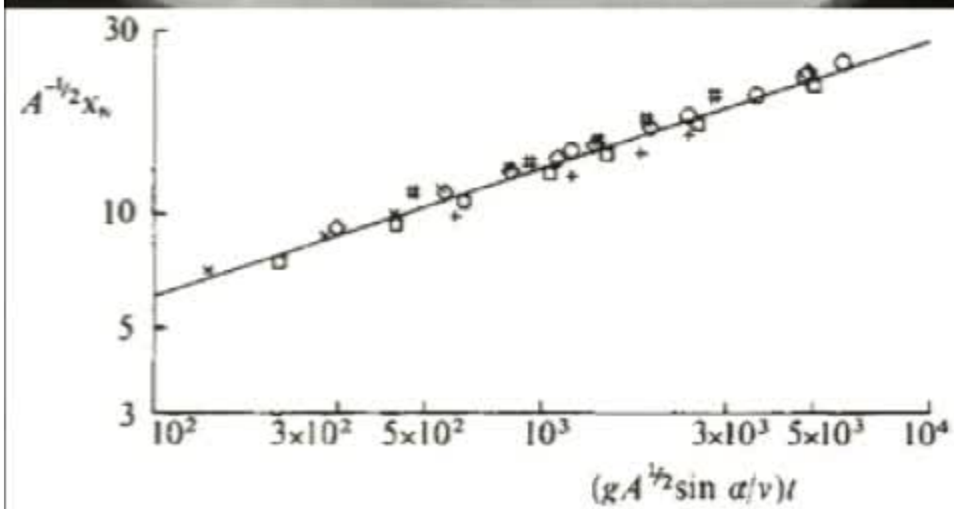

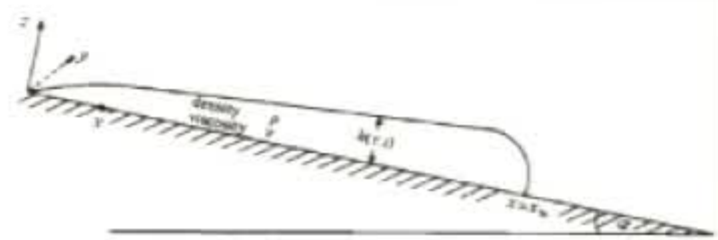

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J. Zhou et al PRL 2005 Original model

Model Derivation II

- *Particle velocity v_R relative to fluid*

$$v_R = \frac{2}{9}\frac{(\rho_p - \rho_f)a^2}{\mu_f}\overline{f}(\phi)w(h)g$$

$$w(h) = \frac{ah^2}{\sqrt{1 + (Ah^2)^2}}$$

- *w(h) wall effect*

- *Richardson-Zaki correction m=5.1*

$$\overline{f}(\phi) = (1 - \phi)^m$$

- *Flow becomes solid-like at a critical particle concentration*

$$\mu(\phi) = (1 - \phi/\phi_{\max})^{-2}$$

$\mu(\phi)$ = viscosity, a = particle size

ϕ = particle concentration

Lubrication approximation

*dimensionless variables as in clear fluid**

$$\frac{\partial(\rho(\phi)h)}{\partial t} + \left\{ \cancel{\frac{\rho(\phi)}{\mu(\phi)} h^3 h_{xxx}} - D(\beta)\left[\cancel{\frac{\rho(\phi)}{\mu(\phi)} h^3 (\rho(\phi)h)_x} - \cancel{\frac{5}{8}\frac{\rho(\phi)}{\mu(\phi)} h^4 (\rho(\phi))_x} \right] + \frac{\rho(\phi)^2}{\mu(\phi)} h^3 \right\}_x = 0$$

$$\frac{\partial(\phi h)}{\partial t} + \left\{ \cancel{\frac{\phi}{\mu(\phi)} h^3 h_{xxx}} - D(\beta)\left[\cancel{\frac{\phi}{\mu(\phi)} h^3 (\rho(\phi)h)_x} - \cancel{\frac{5}{8}\frac{\phi}{\mu(\phi)} h^4 (\rho(\phi))_x} \right]\right\}_x + \left\{ \frac{\phi\rho(\phi)}{\mu(\phi)} h^3 + \frac{2}{3} V_s \phi h f(\phi) w(h) \right\}_x = 0$$

$$V_s = \frac{\rho_p - \rho_f}{\rho_f}\frac{a^2}{H^2} \qquad f(\phi) = (1-\phi)\bar{f}(\phi)$$

Dropping higher order terms

$*D(\beta) = (3Ca)1/3\cot(\beta)$, $Ca=\mu_f U/\gamma$,
- Bertozzi & Brenner Phys. Fluids 1997

Reduced model

Remove higher order terms

$$\frac{\partial(\rho(\phi)h)}{\partial t}+\left\{\frac{\rho(\phi)^2}{\mu(\phi)}h^3\right\}_x=0$$

$$\frac{\partial(\phi h)}{\partial t}+\left\{\frac{\phi\rho(\phi)}{\mu(\phi)}h^3+\frac{2}{3}V_s\phi h f(\phi)w(h)\right\}_x=0$$

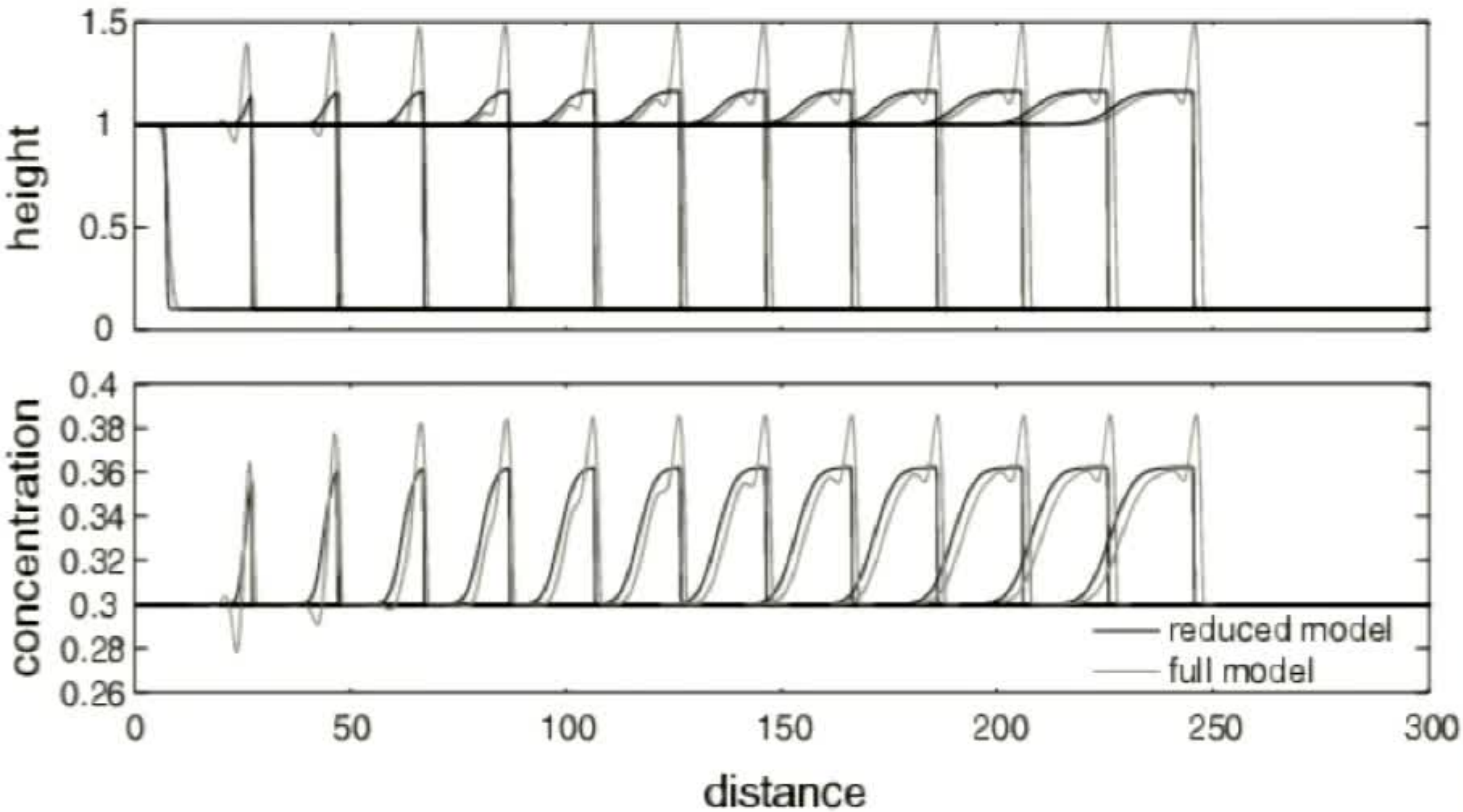
System of conservation laws for u=$\rho(\phi)$h and v=ϕh

$$\frac{\partial u}{\partial t}+\left[F(u,v)\right]_x=0$$

$$\frac{\partial v}{\partial t}+\left[G(u,v)\right]_x=0$$

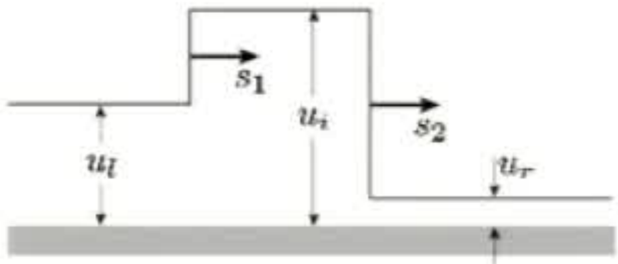
Comparison between full and reduced models

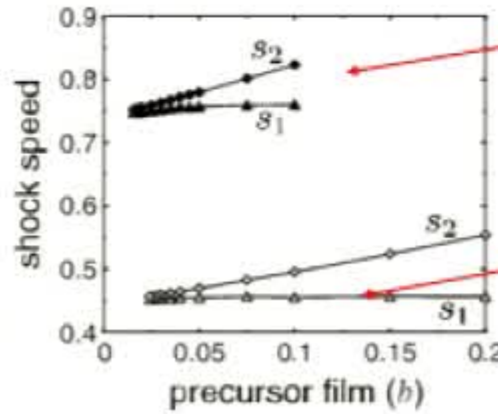
macroscopic dynamics well described by reduced model



Double shock solution

- Riemann problem can have double shock solution





ϕ=15%

ϕ=30%

- Four equations in four unknowns (s_1, s_2, u_i, v_i)

$$s_1 = \frac{F(u_i, v_i) - F(u_l, v_l)}{u_i - u_l} = \frac{G(u_i, v_i) - G(u_l, v_l)}{v_i - v_l}$$

$$s_2 = \frac{F(u_r, v_i) - F(u_r, v_r)}{u_i - u_r} = \frac{G(u_r, v_i) - G(u_r, v_r)}{v_i - v_r}$$

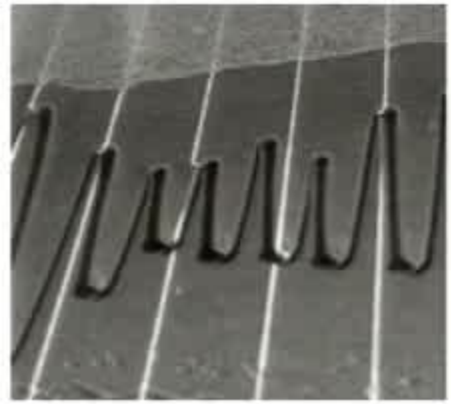
Singular behavior at contact line

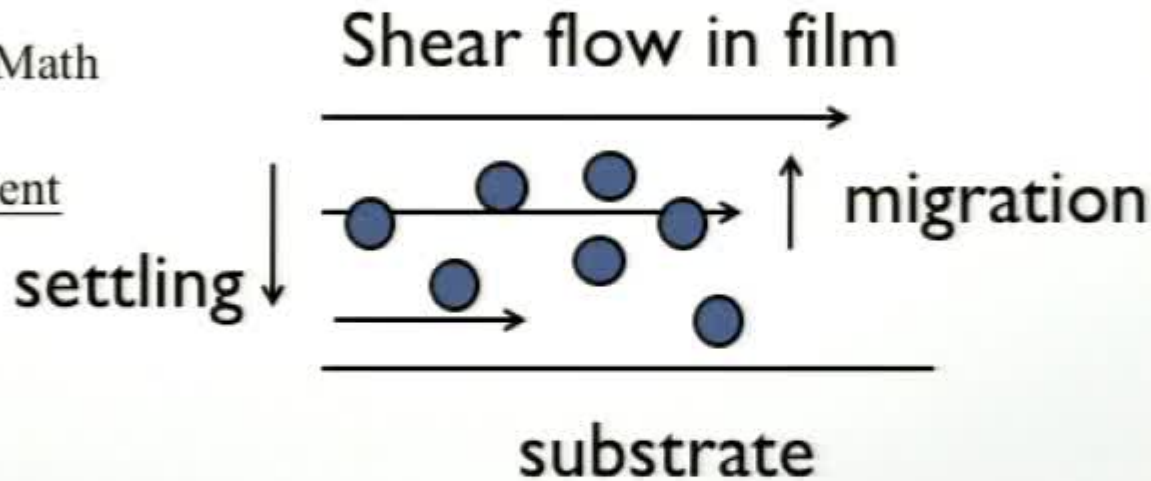
How to model changes in settling behavior?

Motivation: older experiments (Hosoi MIT, 05)

New theory proposed by *Cook PRE 2008* balances shear induced migration with hindered settling. Agrees well with old and limited data- promising, but it is universally observed?

-Our experiments: carried out at Applied Math Lab, UCLA, summer/fall 2009
-Confirm Cook's theory excellent agreement
-Paper published in *Physica D* 2011.





Undergraduate REU: Joyce Ho, Paul Latterman, Stephen Lee, Kanhui Lin, Vincent Hu, mentor: Nebo Murisic

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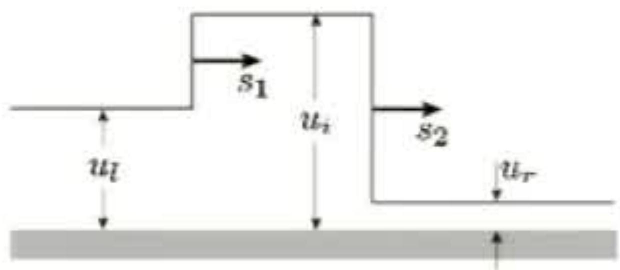

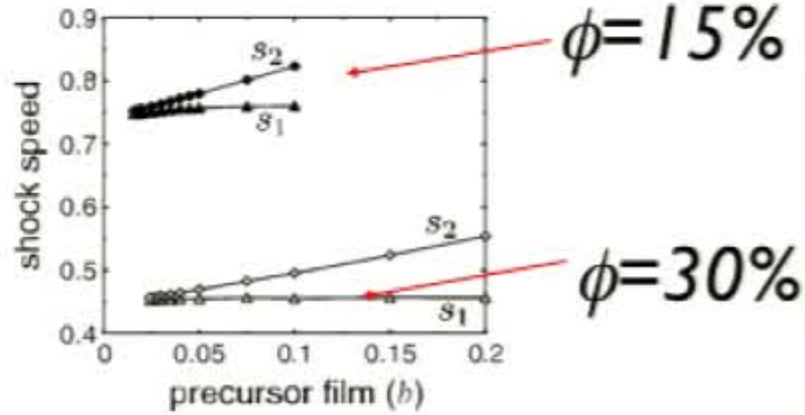

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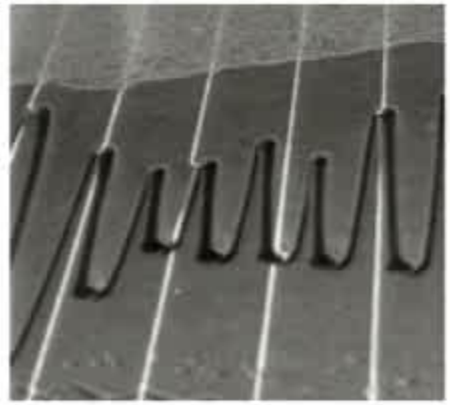

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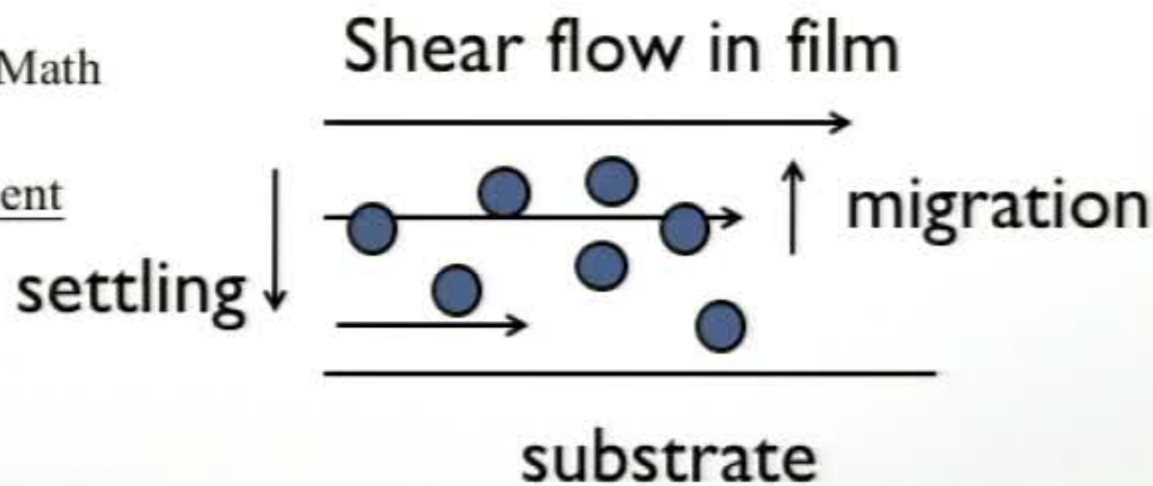

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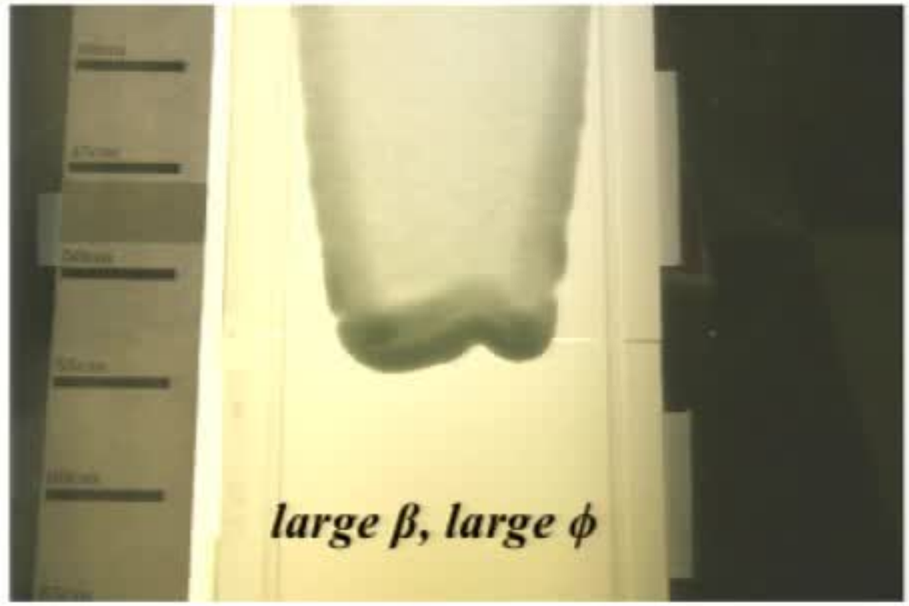
Experiments (cont.)

small β, small ϕ



large β, large ϕ

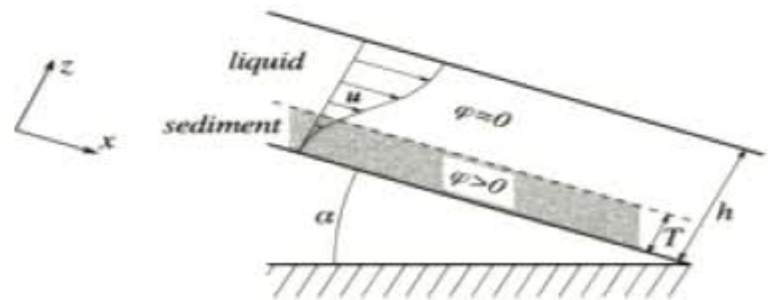
intermediate values of β & ϕ

Particle Volume Fraction Model

- Main question: *Will particle settle out of the flow or remain in the suspension?*

- Simple model: equilibrium balance of particle settling against shear induced migration normal to substrate (Ben Cook PRE 2008, tested against old data from MIT). New experiments varying bead size and viscosity of oil.

-Particle volume fraction model:



$$\frac{D\phi}{Dt} = -\nabla \cdot \mathbf{J}$$

$$K_c a^2 \left(\phi^2 \nabla\dot\gamma + \phi\dot\gamma\nabla\phi\right) + K_v \dot\gamma \phi^2 \left(\frac{a^2}{\mu(\phi)}\right)\frac{d\mu}{d\phi}\nabla\phi = -\frac{1}{18}\phi\frac{a^2 g(\rho_p - \rho_l)}{\mu_l} f(\phi) w(h) \begin{vmatrix} \sin\beta \\ \cos\beta \end{vmatrix}$$

$$\dot\gamma = \nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \qquad f(\phi) = \frac{\mu_l (1-\phi)^M}{\mu(\phi)} \qquad w(h) = \frac{A\left(\frac{h}{a}\right)^2}{\sqrt{1+\left[A\left(\frac{h}{a}\right)^2\right]^2}}$$

Particle Volume Fraction Model (cont.)

- Concentrate on z-direction (cross-section of the film) & after some manipulation (*Cook*)
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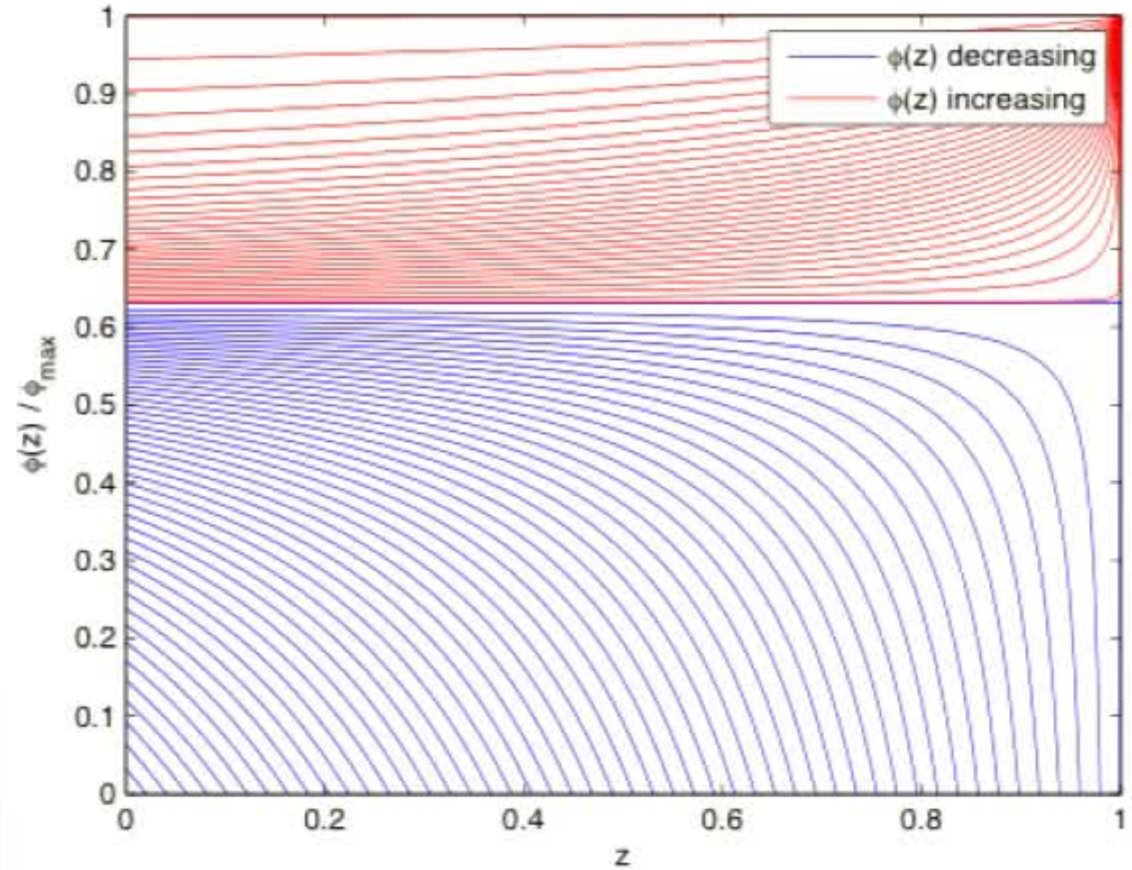
- ***May be solved for particle volume fraction, shear stress σ*** *given inclination angle, height of liquid column, and hight-averaged particle volume fraction in the column (R-K & shooting)*

Bifurcation: Two types of solutions.



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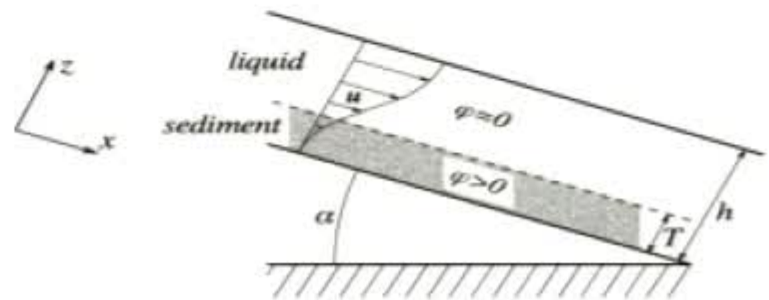

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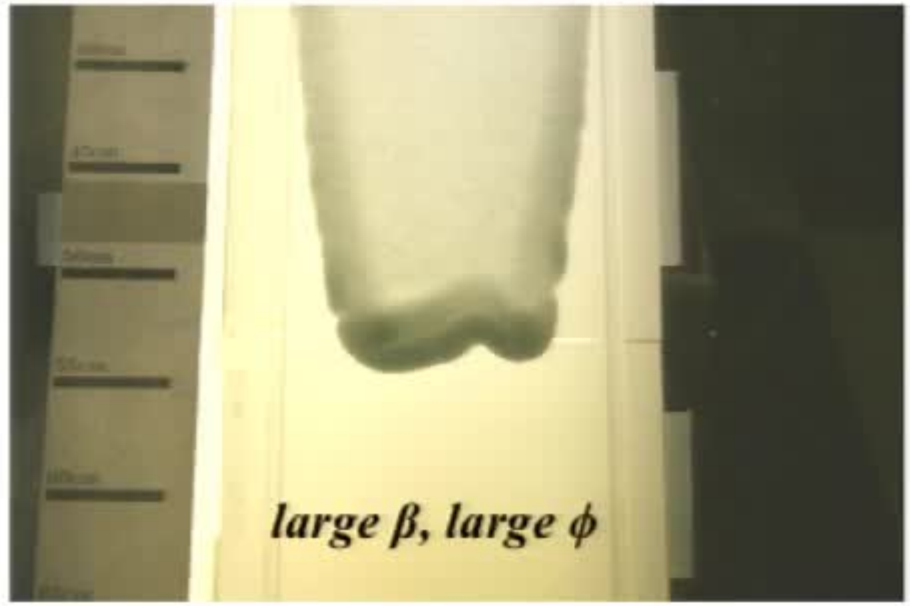

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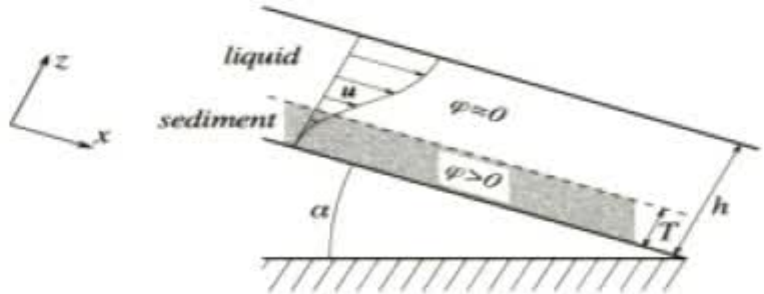

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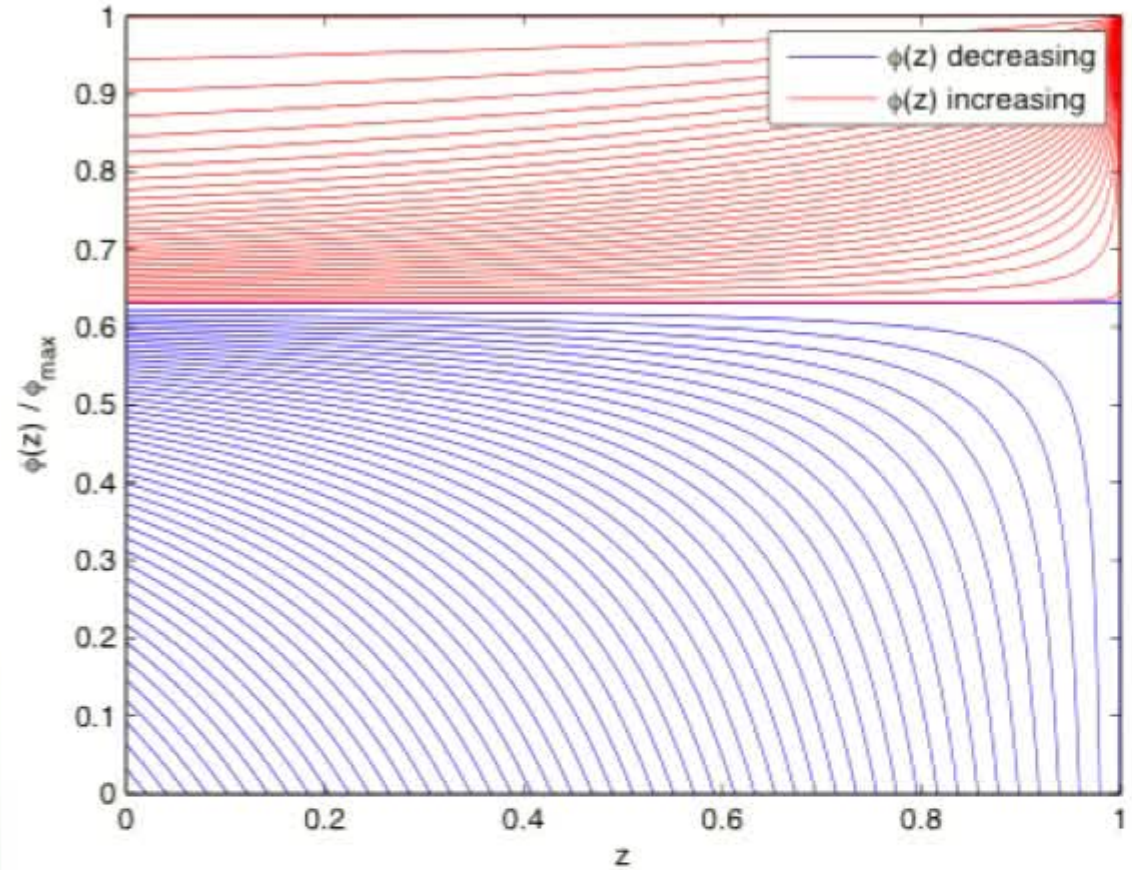

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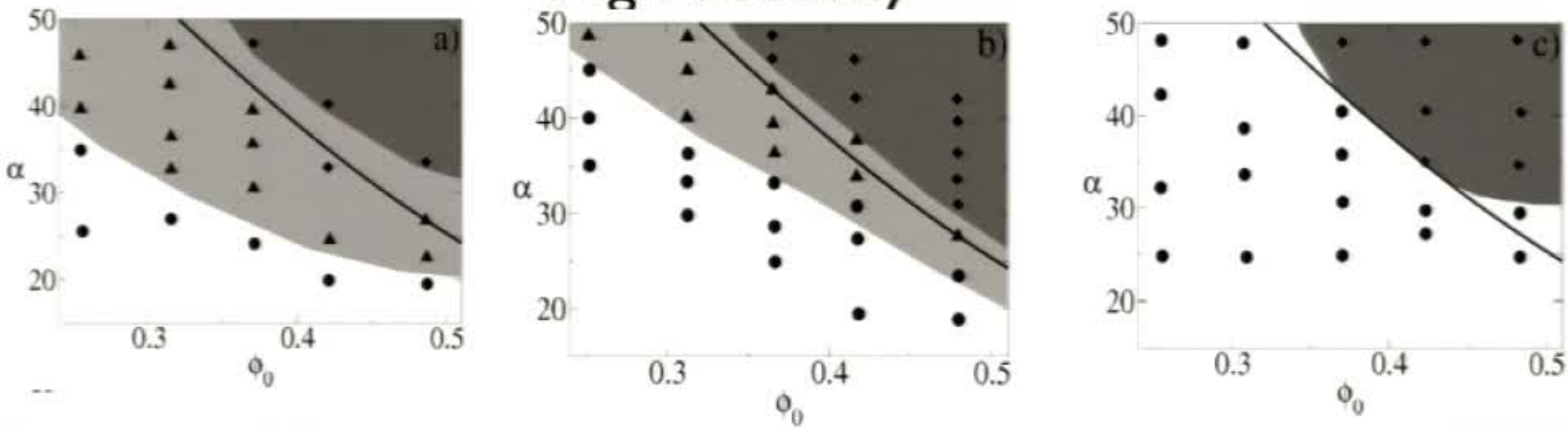
- Consider a solution of the system where there is no variation of ϕ in z-direction ($\phi` = 0$; well mixed case):

$$\alpha = \arctan\left[\frac{2\Delta}{9K_c(\phi)}\frac{f(\phi)}{\phi(1+\Delta\phi)}\right]$$

High viscosity



Small beads-143um   Medium beads-337um   Large beads-625um

Murisic et al *Physica D* 2011 special Childress Issue

Dynamics:

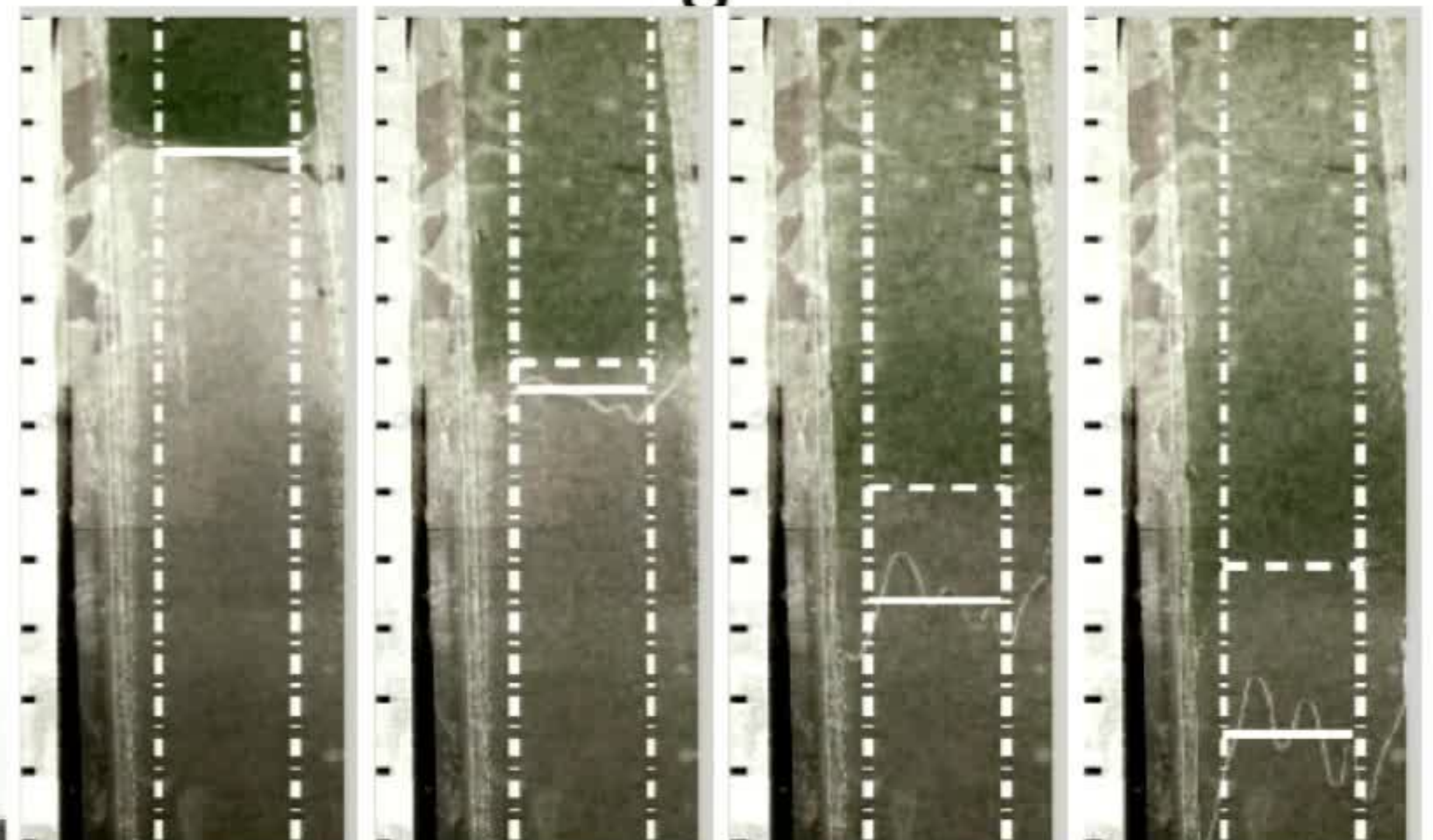
thin film limit formal asymptotics

N. Murisic et al JFM 2013

$$\partial_t h + \partial_x F(h,n) = 0$$
$$\partial_t n + \partial_x G(h,n) = 0,$$

$$F(h,n) = \int_0^h u(t,x;z)\mathrm{d}z = h^3 \int_0^1 \tilde{u}(t,x;s)\,\mathrm{d}s = h^3 f(\phi_0)$$
$$G(h,n) = \int_0^h \phi(t,x;z)u(t,x;z)\mathrm{d}z = h^3 \int_0^1 \tilde{\phi}(t,x;s)\tilde{u}(t,x;s)\,\mathrm{d}s = h^3 g(\phi_0),$$

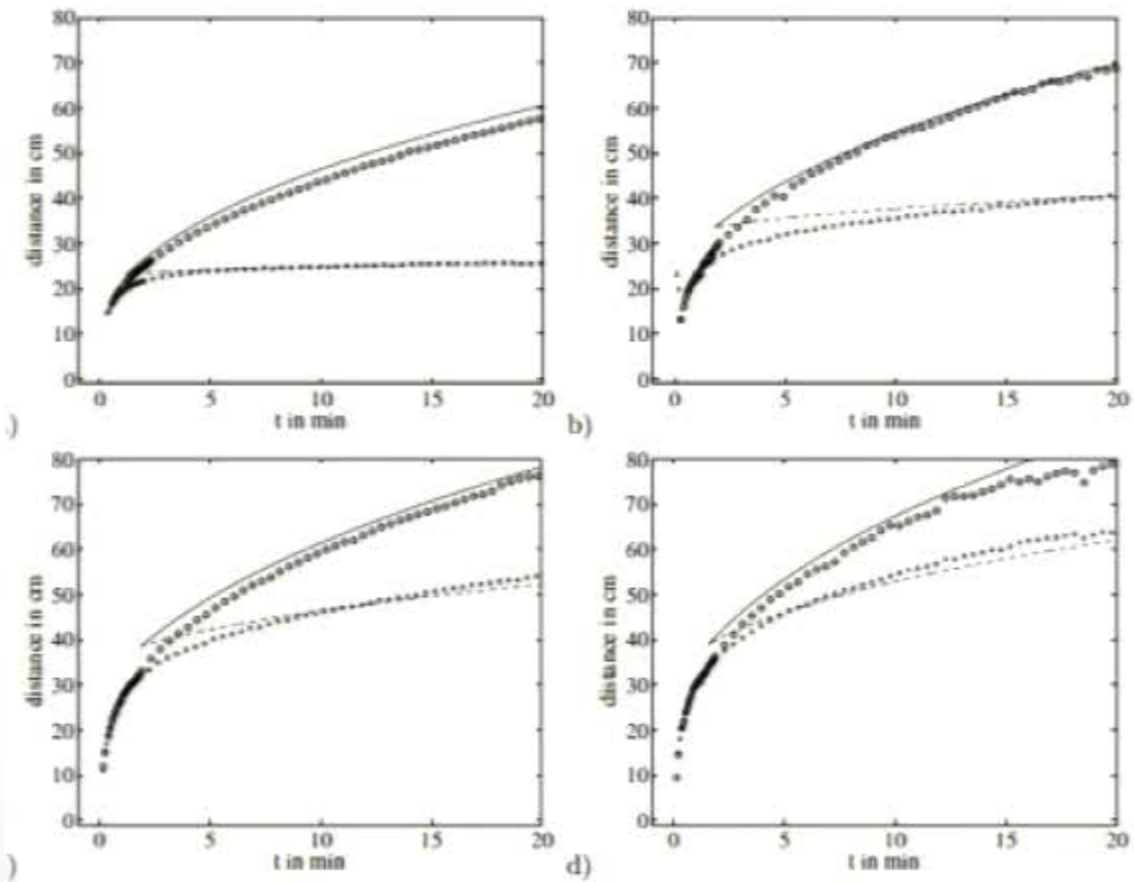
Dynamic Experiments – Settled Regime



Theory vs Experiment 30 % volume fraction

10 degrees

15 degrees



20 degrees

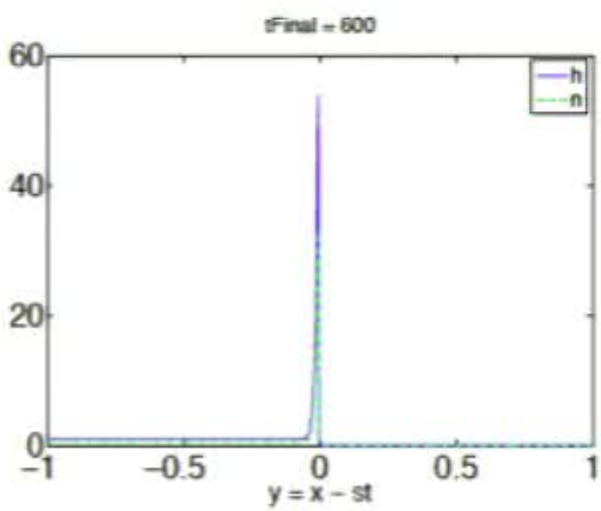
25 degrees

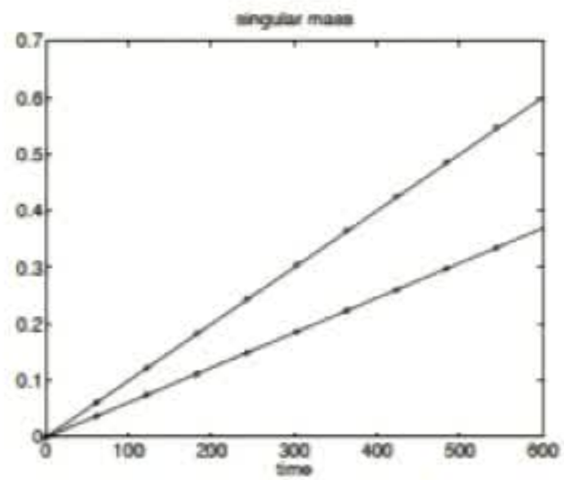
Shock Solutions

- Settled regime described by a double shock – Mavromoustaki and ALB, 2013
- Particle ridge regime described by a singular shock- Li Wang and ALB, 2013- mass concentration in the shock

$$\frac{dM_h}{dt} = s[h] - \left[h^3 f\left(\frac{n}{h}\right)\right], \quad \frac{dM_n}{dt} = s[n] - \left[h^3 g\left(\frac{n}{h}\right)\right]$$

$$s = \frac{(h_L^2 + h_R^2 + h_L h_R)\left(\phi_{max} f(\phi_L) - g(\phi_L)\right)}{\phi_{max} - \phi_L},$$





Bidisperse flow

Conservation equation for particles:

- Shear-induced migration

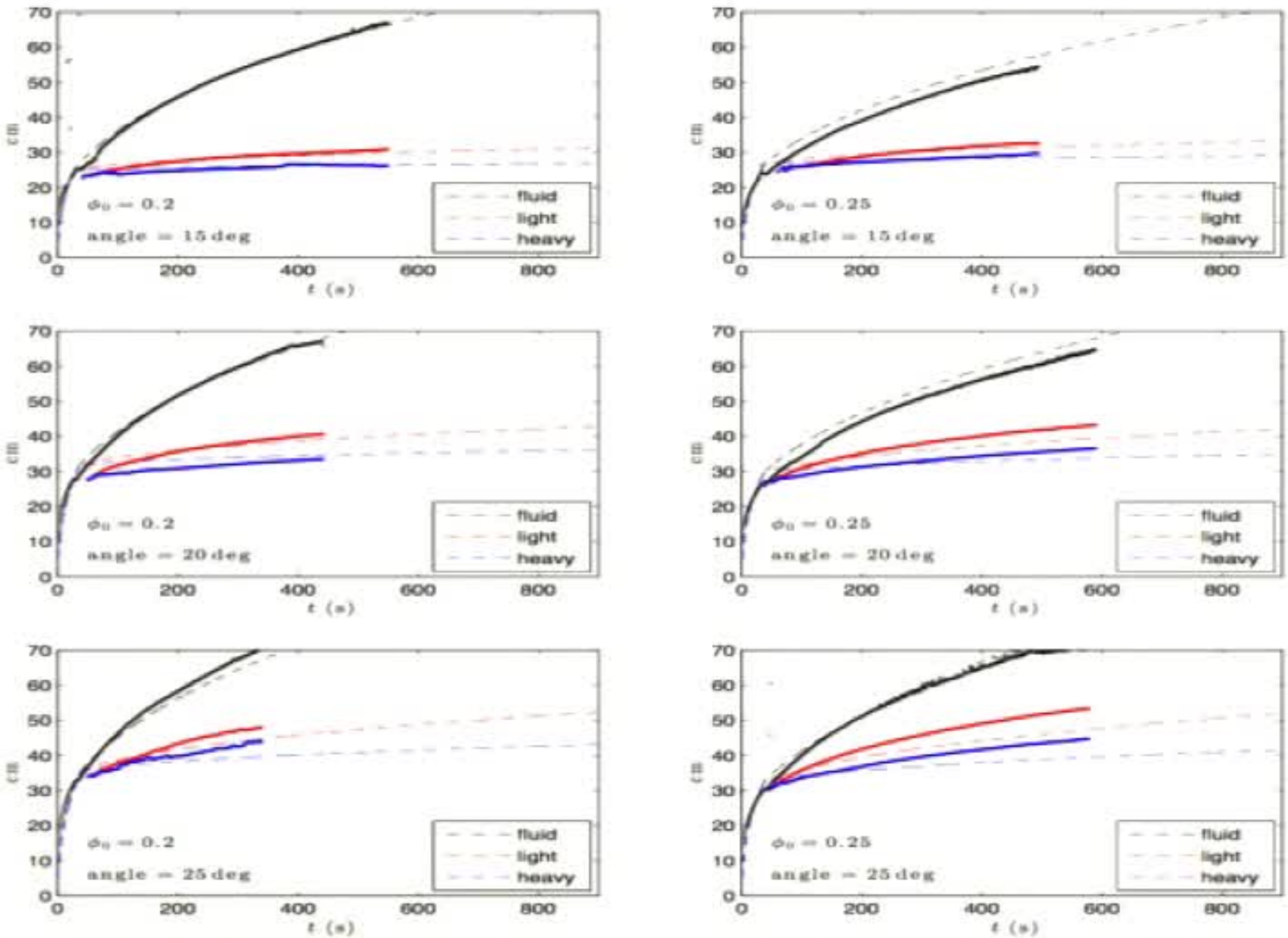
$$\partial_t \phi + \mathbf{u} \cdot \nabla \phi + \nabla \cdot \mathbf{J} = 0$$

where $\mathbf{J} = \sum_{i=1}^{m} \mathbf{J}_i = \mathbf{J}_{grav,i} + \mathbf{J}_{tracer,i} + \mathbf{J}_{drift}$

- $\phi_i = \frac{n_i}{h}$
- $\partial_t h + \nabla \cdot \left(h^3 F\left(\phi_1, \phi_2\right)\right) = 0$
- $\partial_t n_1 + \nabla \cdot \left(h^3 G_1\left(\phi_1, \phi_2\right)\right) = 0$
- $\partial_t n_2 + \nabla \cdot \left(h^3 G_2\left(\phi_1, \phi_2\right)\right) = 0$

Bidensity Slurries – theory vs. experiment

Theory by Jeffrey Wong; Exps by REU student team



Phase separation mechanism: helical separators

helical separator
(debris+mud)

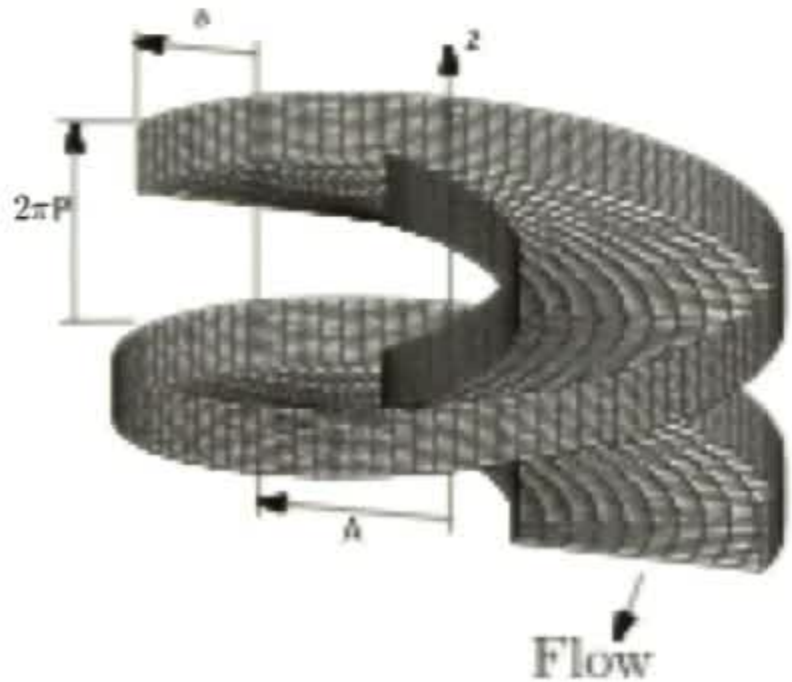
- used in mineral processing since 1940's to separate **solid-fluid** mixtures
- simultaneous application of **gravitational** and **centripetal** forces
- requires **no moving parts** (simple and robust)
- **no fundamental understanding** of the separation mechanism (mostly trial and error)

Phase separation mechanism: helical separators

helical separator
(debris+mud)

modeling	experiments
no 2-way coupling b/w fluids and particle dynamics	varying results depending on the separator used

Slurry flow in spiral separators (slides from Sungyon)



curvature

$$\epsilon = \frac{aA}{A^2 + P^2}$$

torsion

$$\tau = \frac{aP}{A^2 + P^2}$$

inclination angle

$$\alpha_0 = \arctan\left(\tau/\epsilon\right)$$

rectangular channel