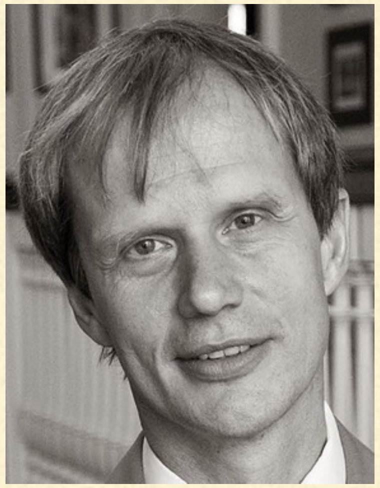
# Nonlinear Patterns and Waves: From Spectra to Stability and Dynamics


Margaret Beck


Stephanie Dodson

Björn Sandstede


Toan Nguyen


Arnd Scheel

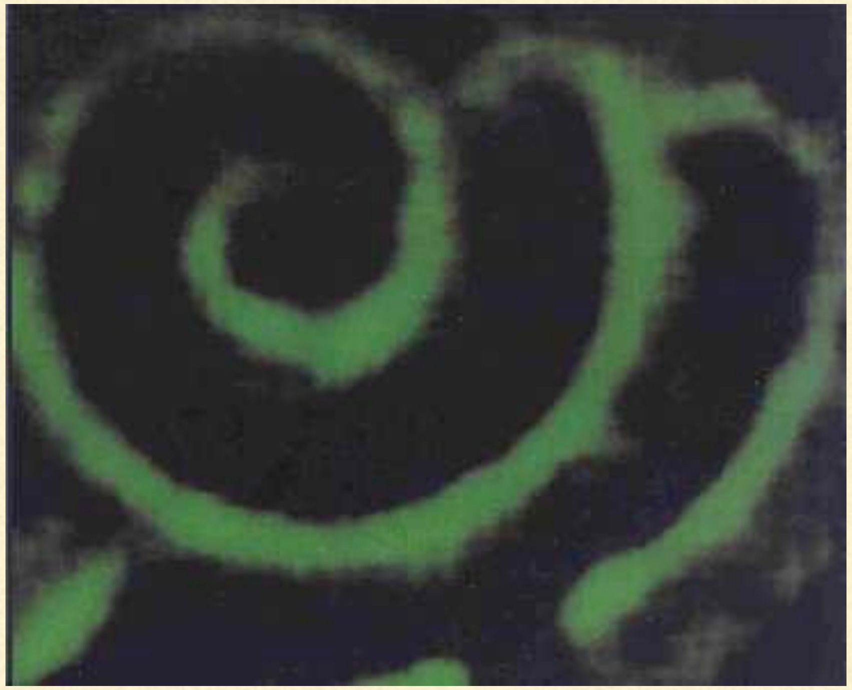

Kevin Zumbrun

# Patterns …

Zebrafish *Danio rerio*


Vertically vibrated copper beads [Umbanhowar et al.]

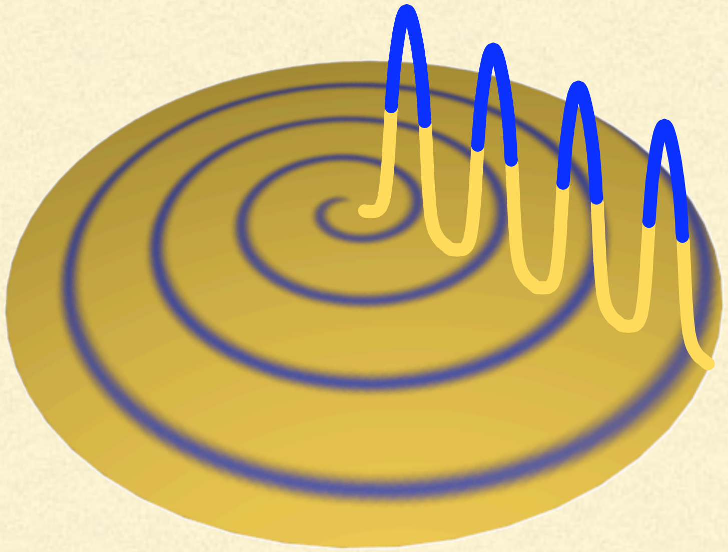
# … and waves



Calcium waves in
Xenopus oocytes
[Clapham et al.]

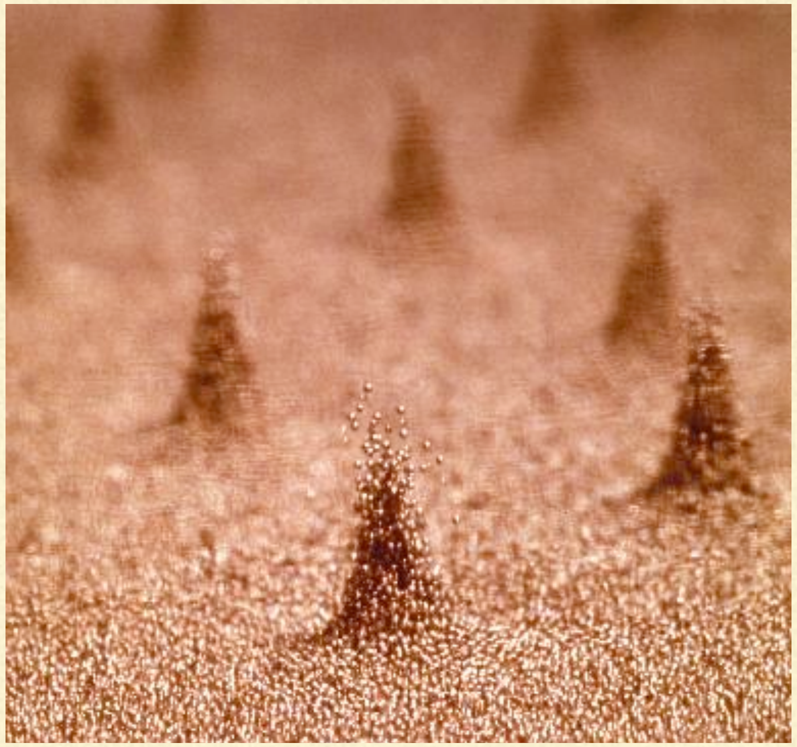
spiral waves

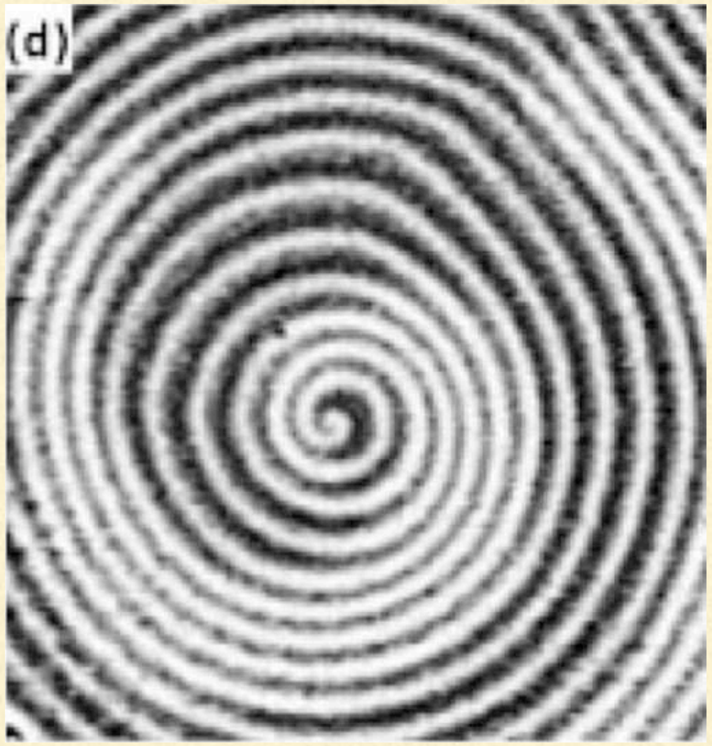
CO oxidation on Pt
[Nettesheim et al.]



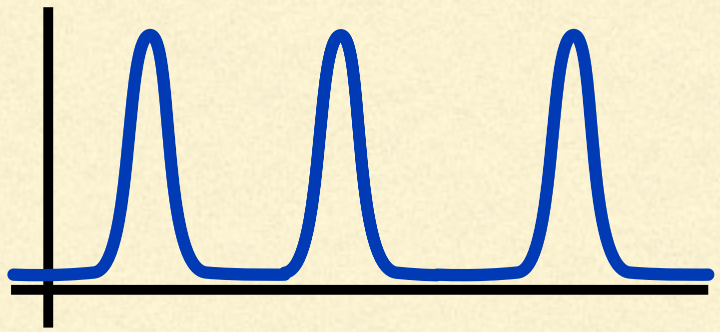
# Outline

- Existence: How many degrees of freedom do these structures have?
- Stability: What happens when spots or spiral waves are perturbed?
- Interaction: How do these structures interact with each other?

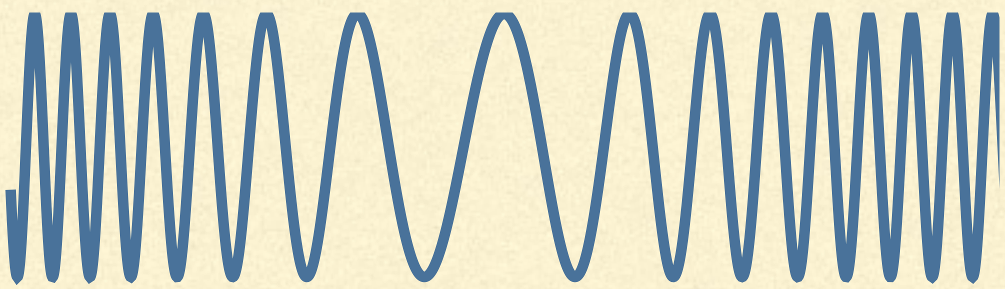




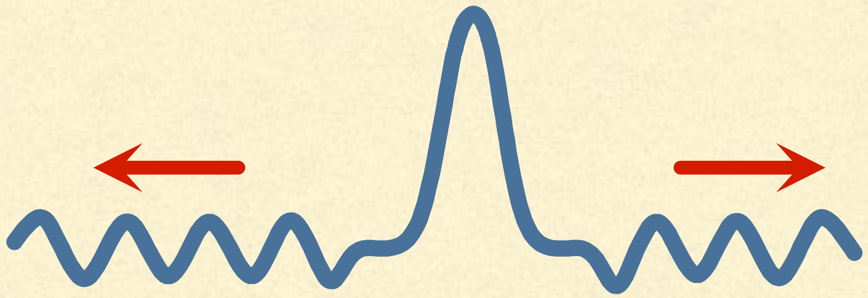

# Outline



Interactions of localized structures



Modulations of spatially-periodic structures

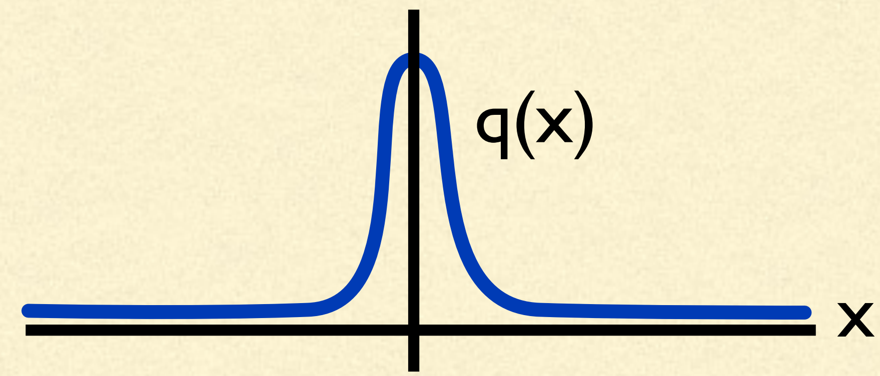


Dynamics of one-dimensional spiral waves

# Localized structures

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R}^n$$

u(x,t) represents vector of concentrations or displacements at position x and time t




localized steady state
u(x,t) = q(x)

Assess stability under small perturbations:

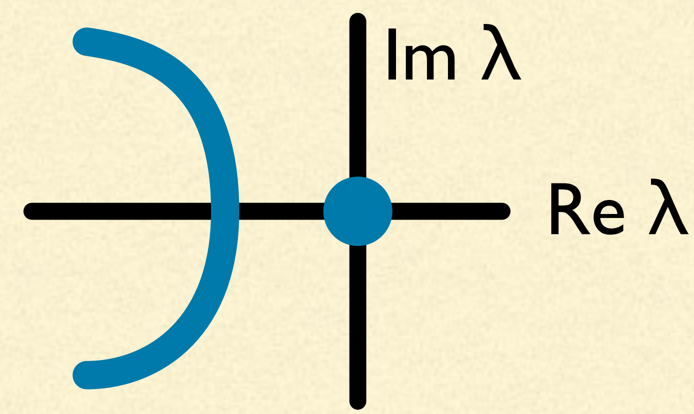
Setting u=q+v with |v| small gives

Perturbed solution converges exponentially in time to an appropriate translate of q(x)

$$\frac{\partial v}{\partial t} = D\frac{\partial^2 v}{\partial x^2} + f_u(q(x))v$$

has solutions of the form $v(x,t) = e^{\lambda t} v_0(x)$, where λ is in the spectrum of the operator

$$\mathcal{L} := D\frac{\partial^2}{\partial x^2} + f_u(q(x))$$

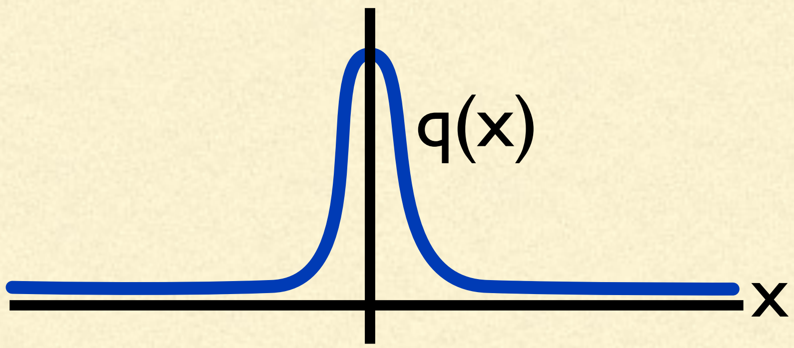



spectrum of the linearization about q(x)

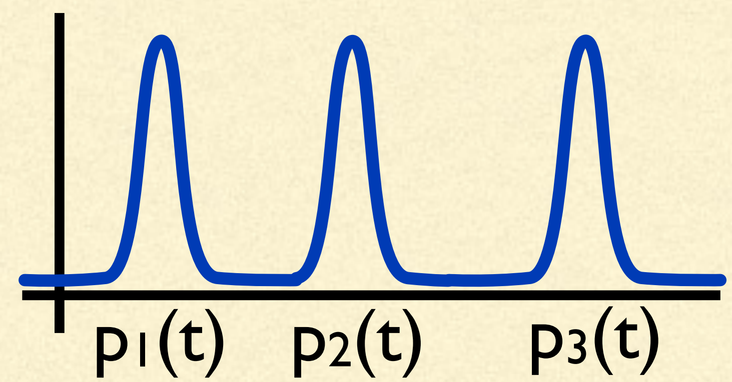
# Interaction of localized structures

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R}^n$$

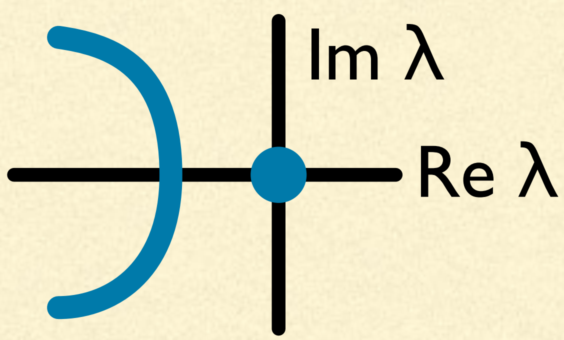
Interacting structures: $$u_0(x) = \sum_{j=1}^{N} q(x - p_j), \quad |p_i - p_j| \gg 1$$



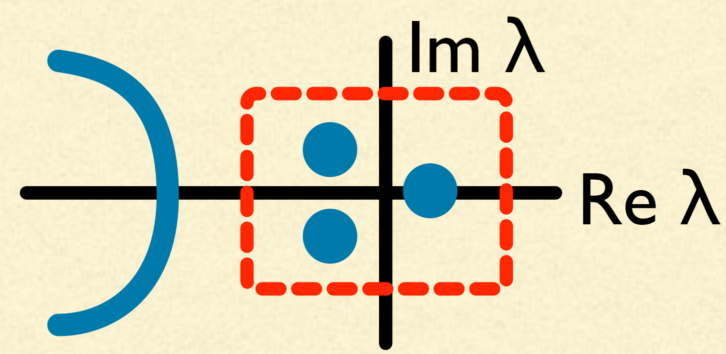

localized stable steady state




spectrum of linearization about q(x)

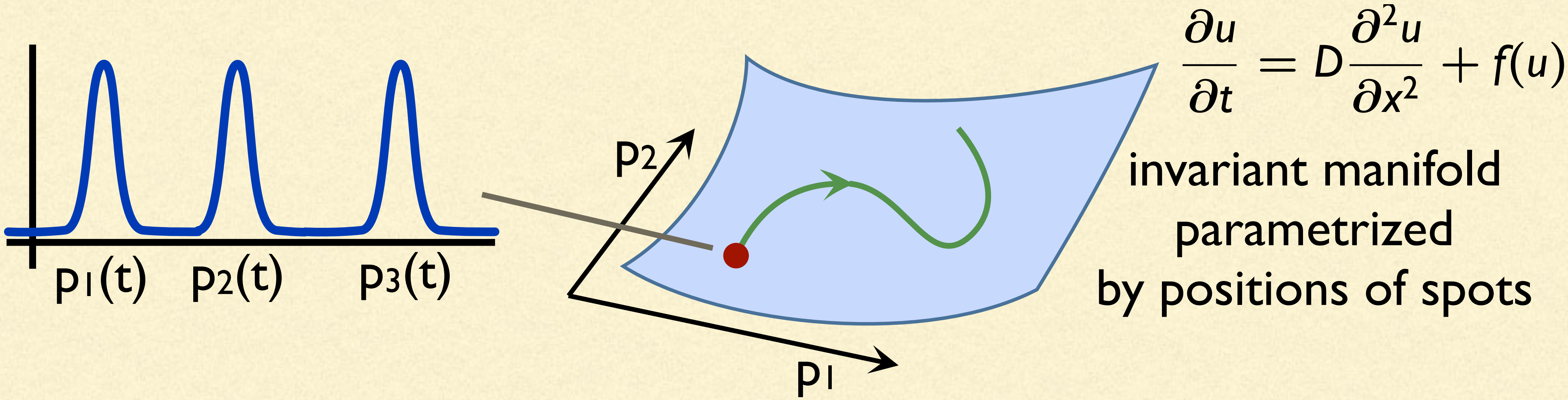


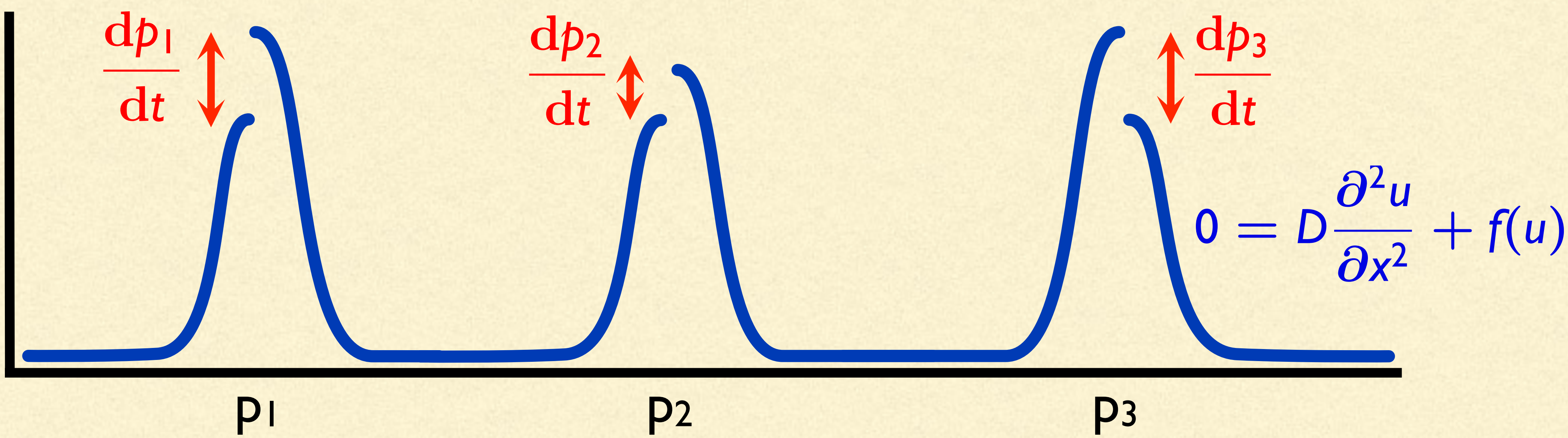

spectrum of linearization about $u_0(x)$




N eigenvalues near zero

# Interaction of localized structures

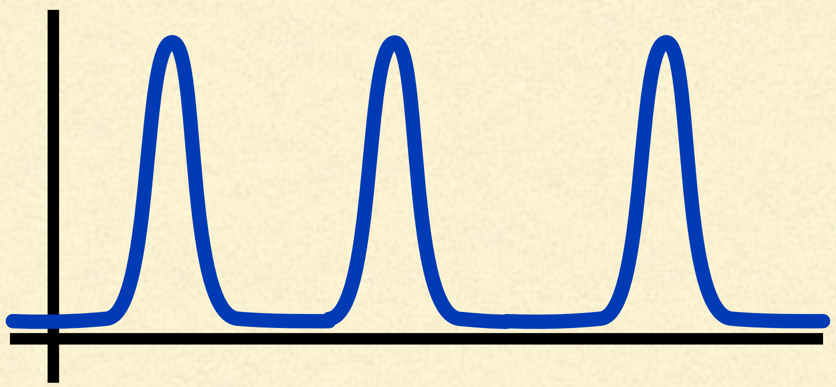


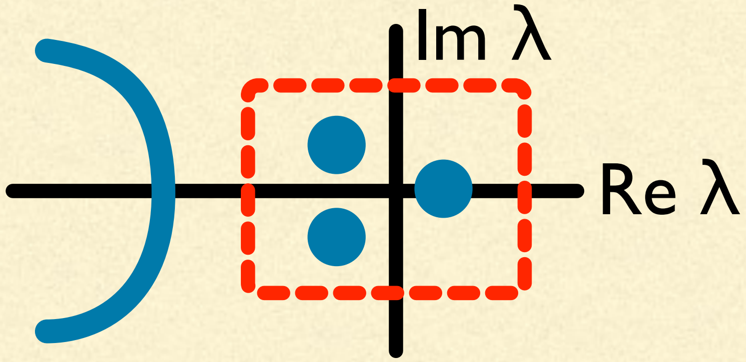




- Nearest-neighbor coupling (to leading order)
- Exponential small in distances

# Summary: interaction of localized structures

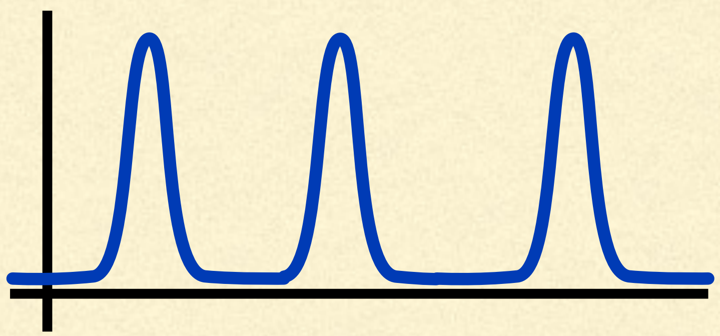
- PDE dynamics of spots described by system of ODEs for positions
- To leading order, nearest-neighbor coupling
- Interaction through tails, hence exponentially slow dynamics
- Spectrum: finitely many eigenvalues



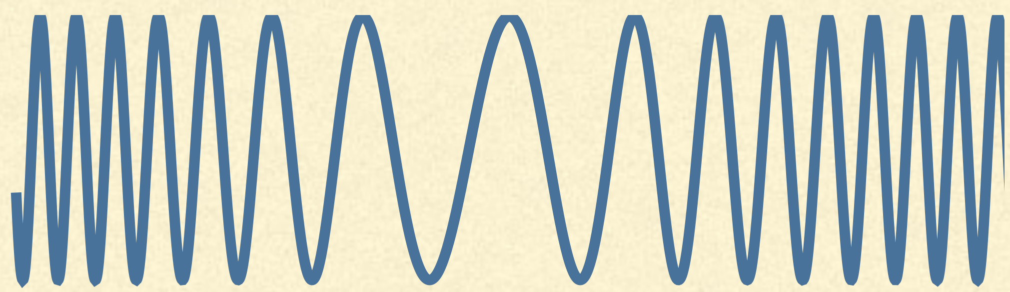



[Neu], [Ei], [S.], [Mielke & Zelik]

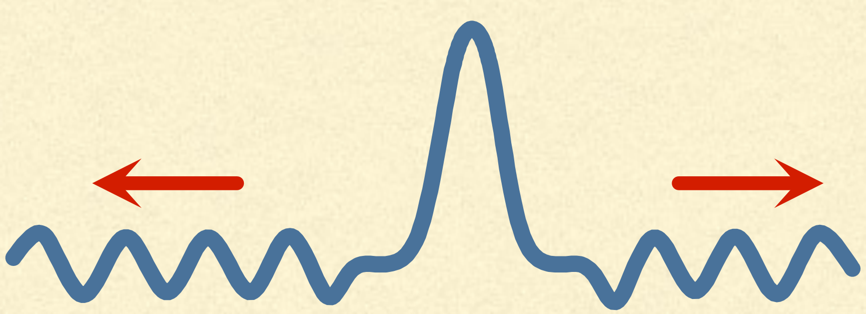
# Outline



Interactions of localized structures

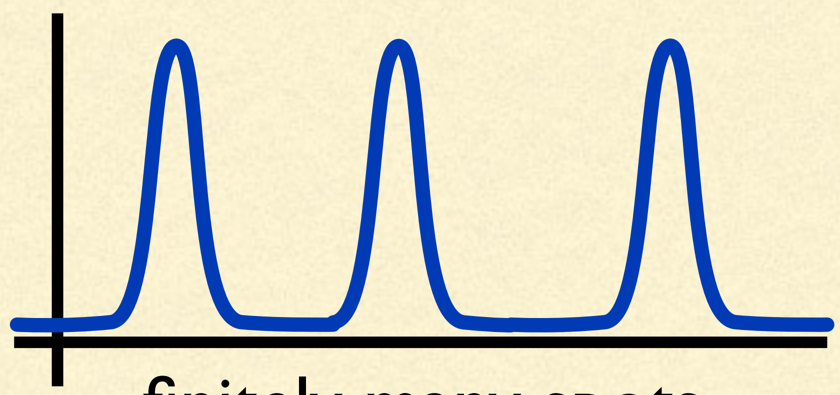


Modulations of spatially-periodic structures

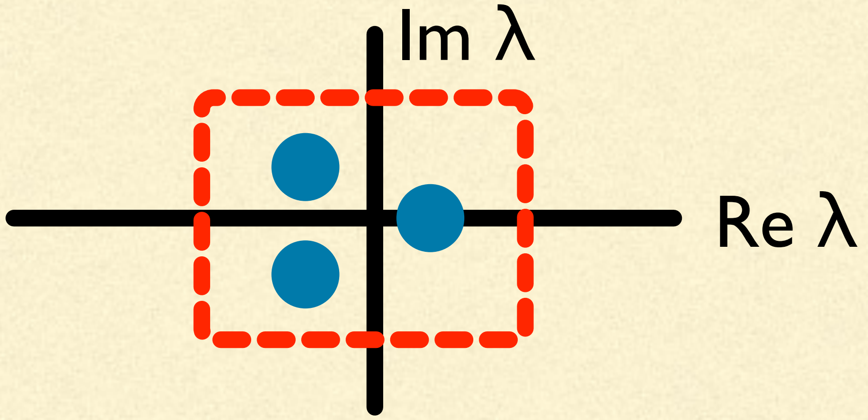


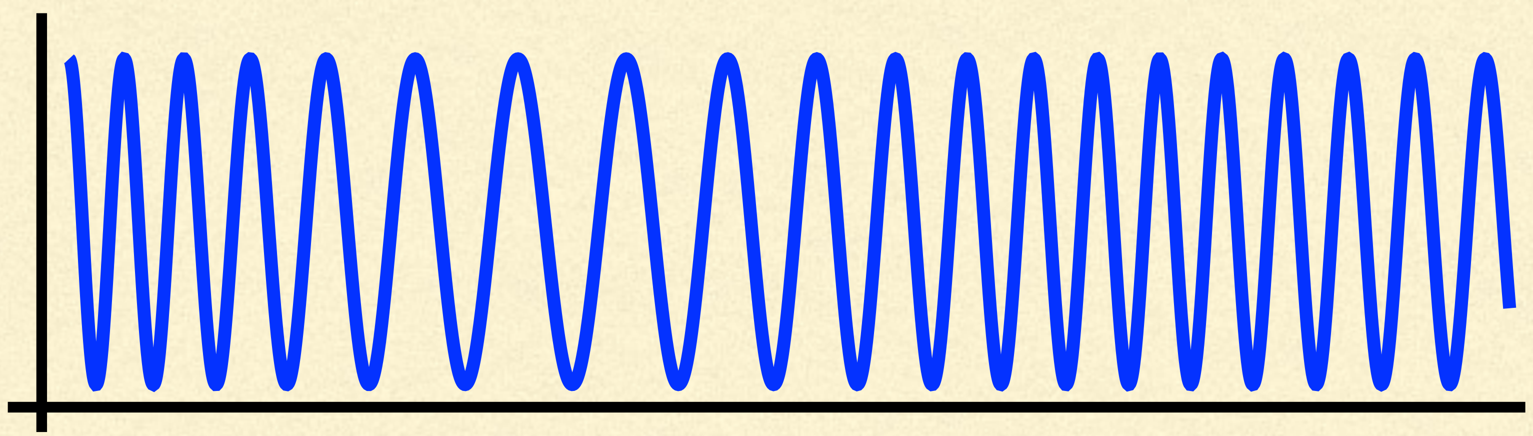
Dynamics of one-dimensional spiral waves

# Infinitely many interacting spots

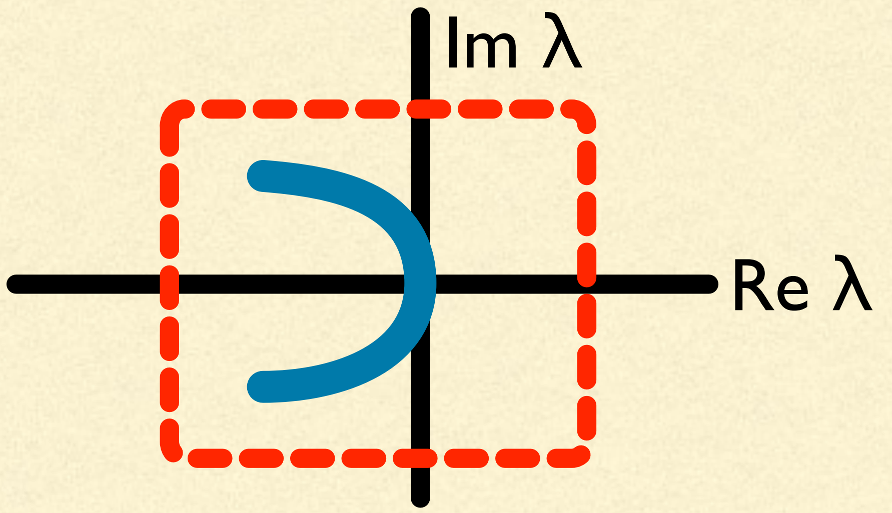


finitely many spots



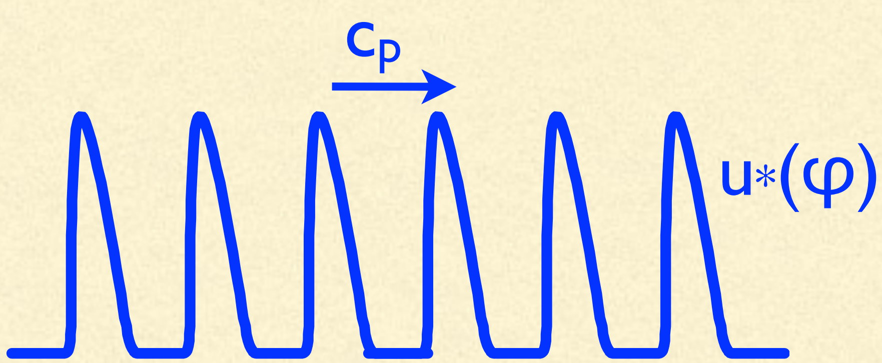
infinitely many spots




infinitely many interaction
eigenvalues form curve

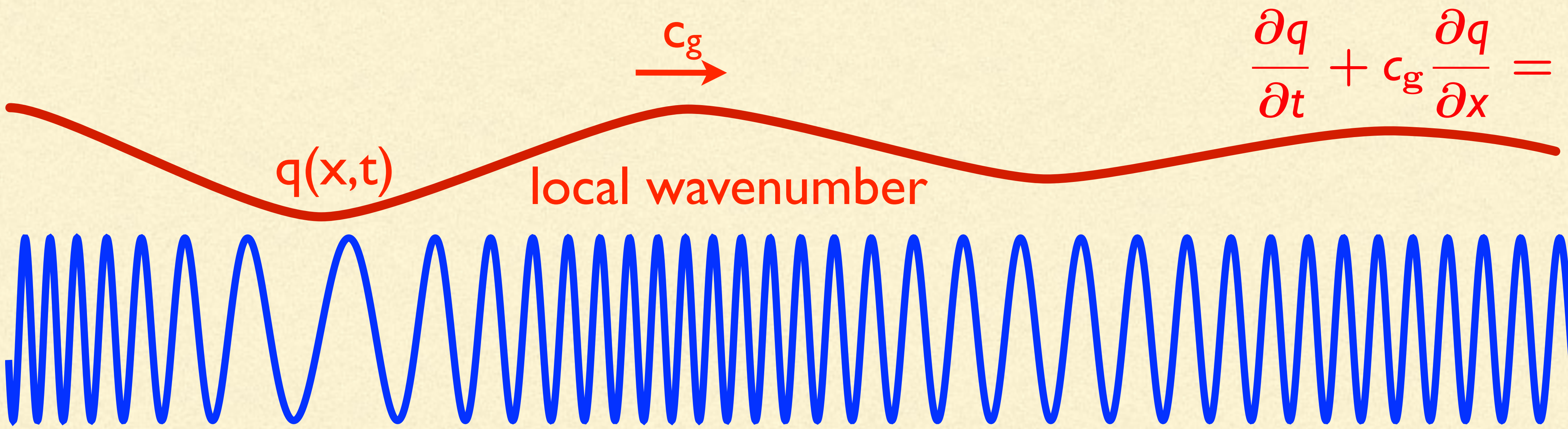
# Slowly varying modulations

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + f(u) \qquad u(x,t) = u_*(kx - \omega_*(k)t; k)$$




| | |
|---|---|
| k | wavenumber |
| $\omega=\omega_*(k)$ | temporal frequency |
| $c_p=\omega_*/k$ | phase velocity |
| $c_g=d\omega_*/dk$ | group velocity |

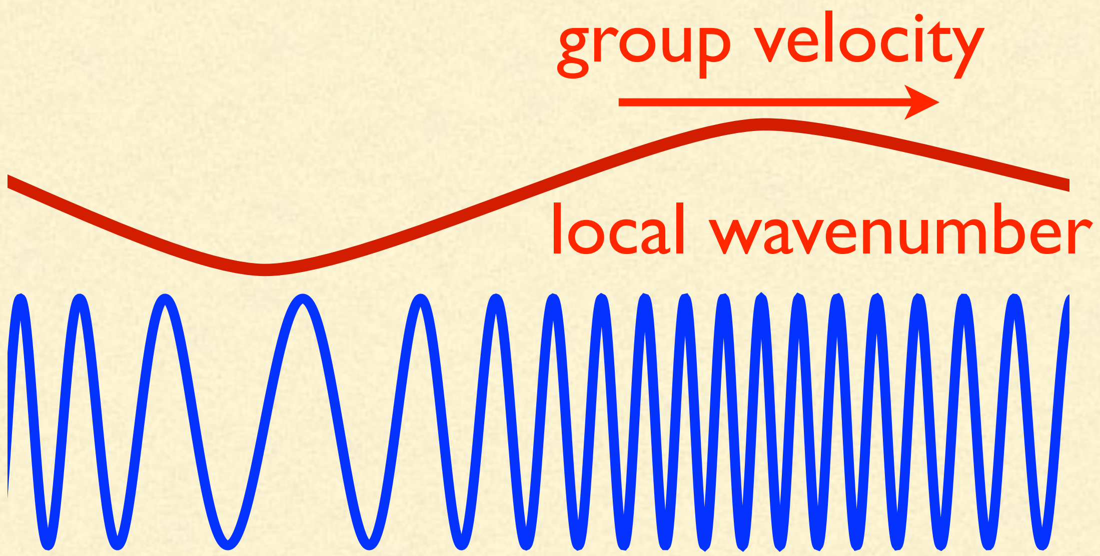
$$\frac{\partial q}{\partial t} + c_g\frac{\partial q}{\partial x} = 0$$

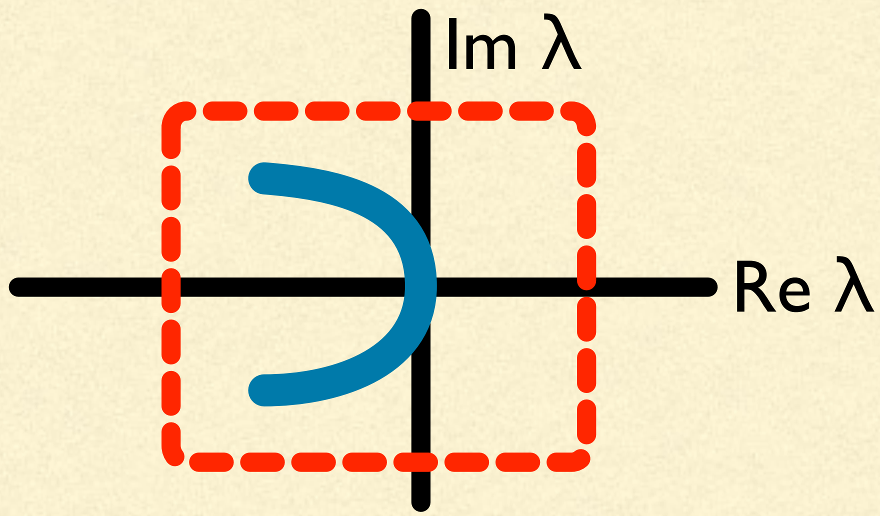



slowly varying modulations of wavenumber

# Summary: slowly-varying modulations

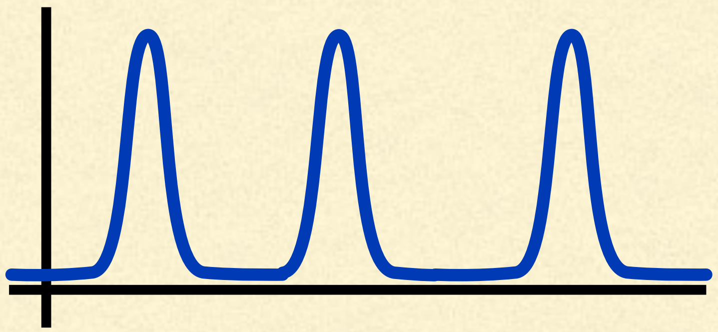
- Modulations are transported with uniform speed given by the group velocity
- Spectrum consists of a curve touching the imaginary axis



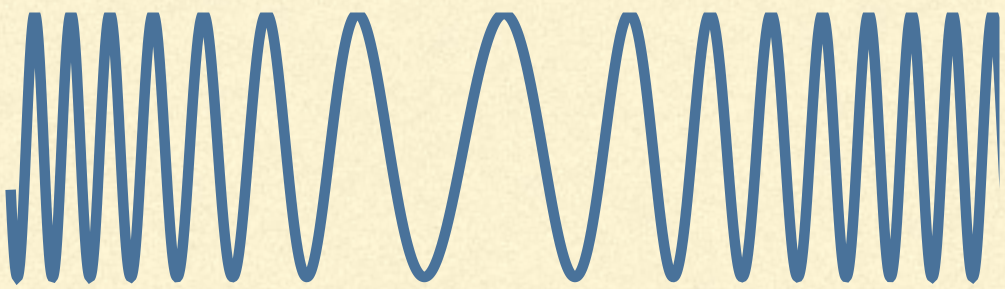




[Howard & Kopell], [Kuramoto], [Doelman et al.], [Johnson et al.]

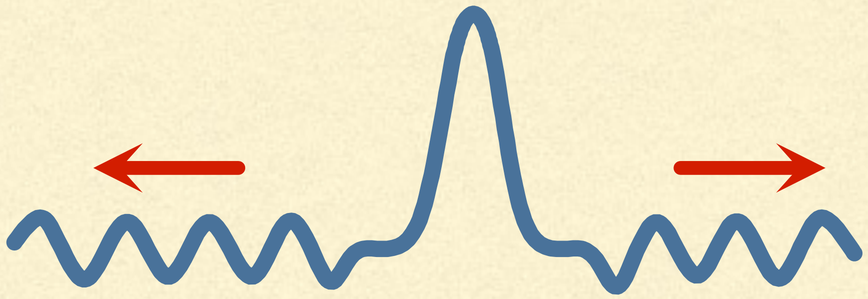
# Outline



Interactions of localized structures

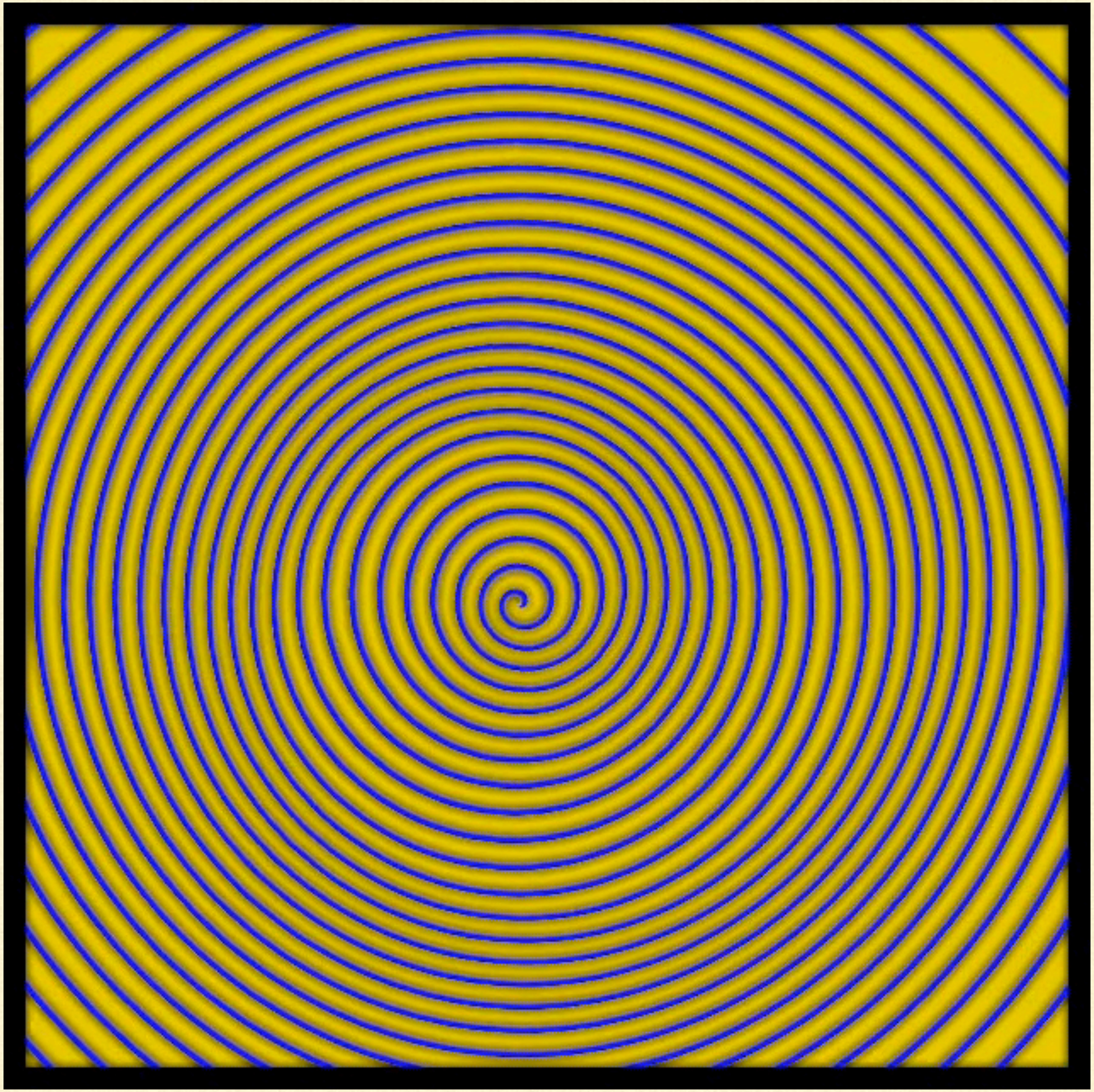


Modulations of spatially-periodic structures



Dynamics of one-dimensional spiral waves

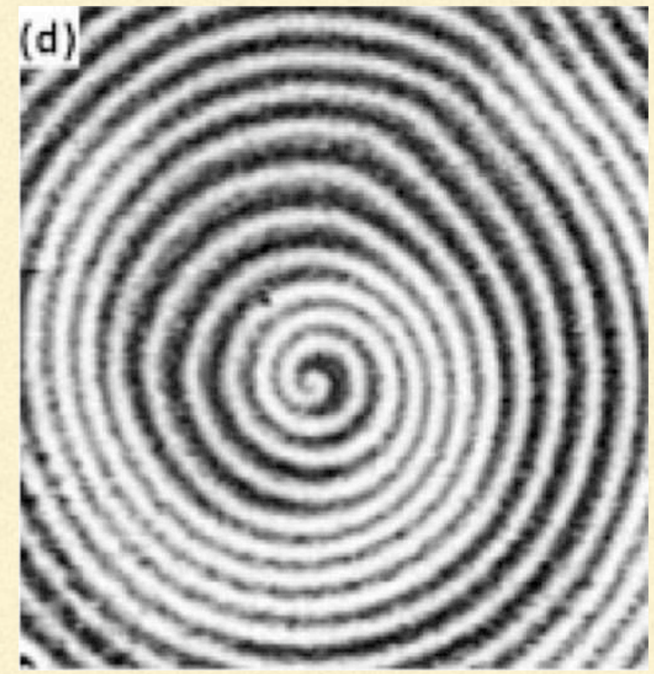
# Modulations of spiral waves

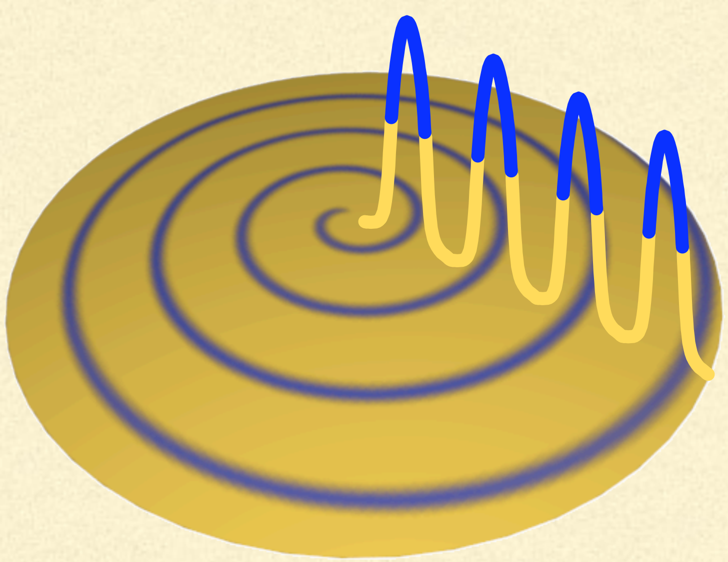


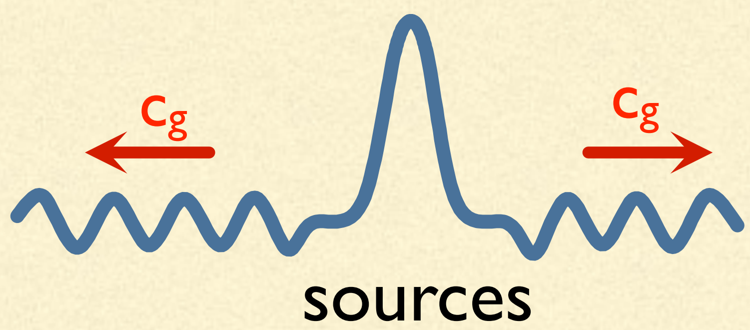
# Dynamics of spiral waves

Perturbing a spiral wave has two effects:

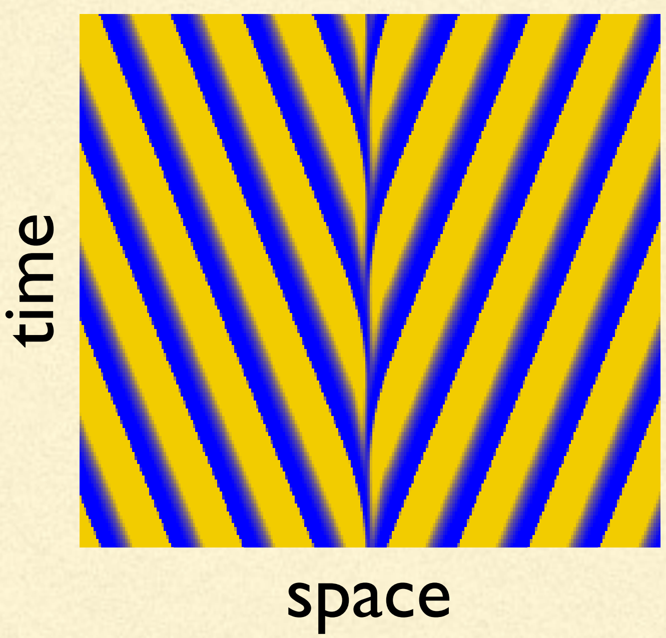
- Position of the spiral may change
- Far-field wave trains may be modulated



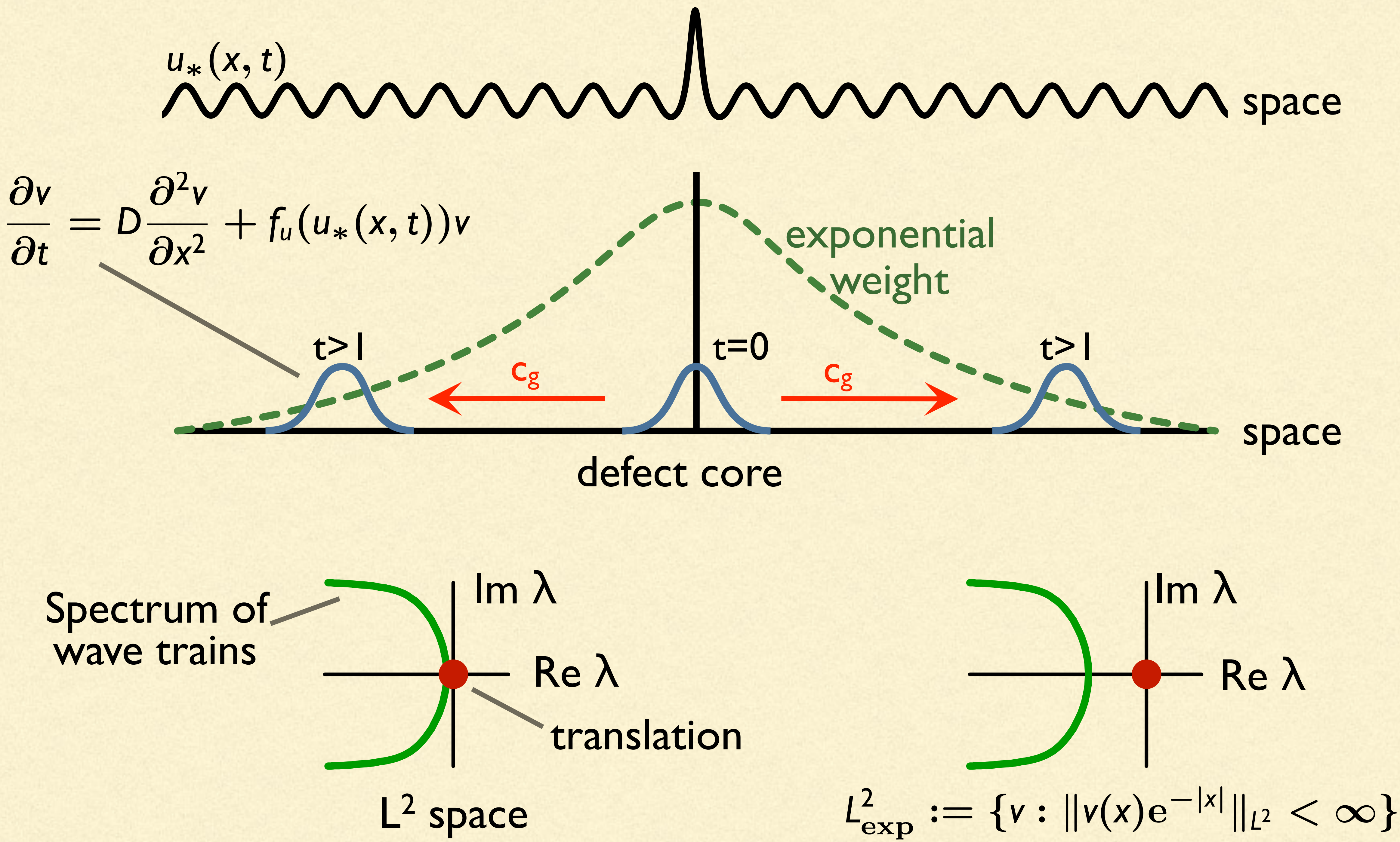





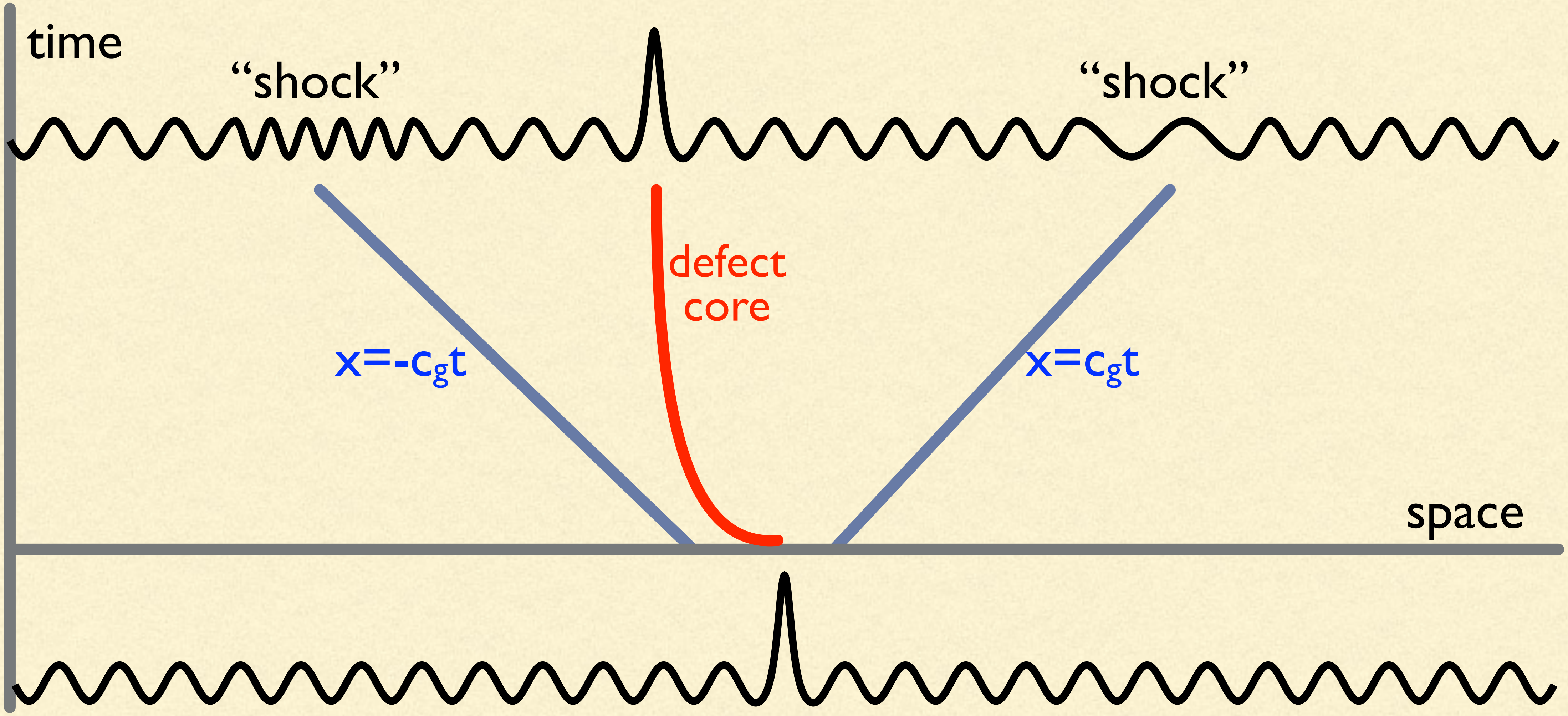

group velocities point away from core



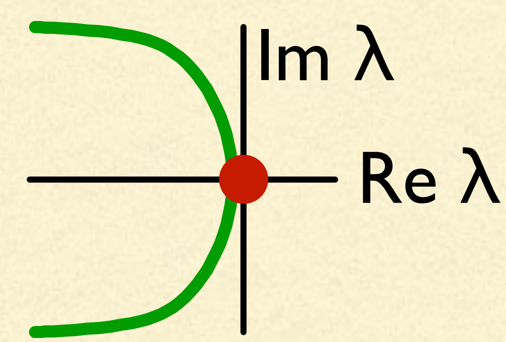
# Spectra of sources



# Dynamics of sources

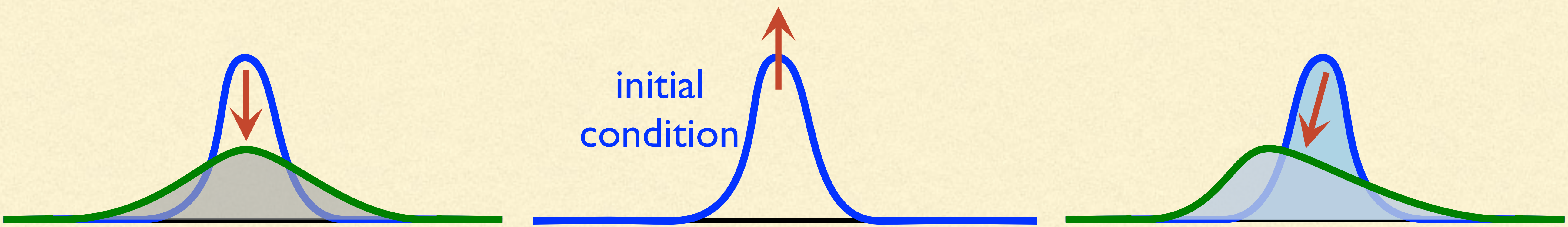



- Defect core converges exponentially to new position
- Modulation interface is localized and travels with speed given by the group velocity

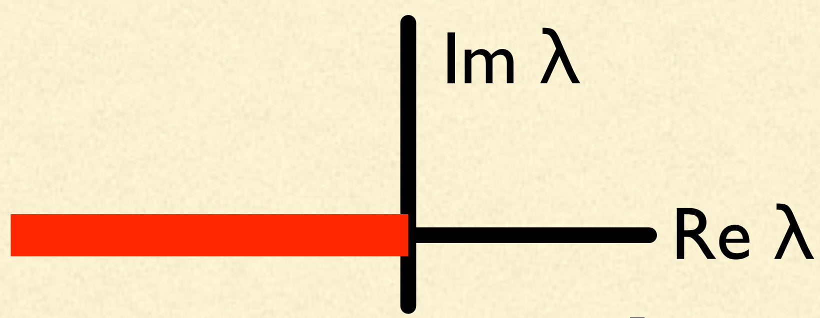


# Caveats

Long-time dynamics for small localized initial data




| Heat equation | Reaction diffusion | Burgers equation |
|---|---|---|
| $\frac{\partial u}{\partial_t} = \frac{\partial^2 u}{\partial x^2}$ | $\frac{\partial u}{\partial_t} = \frac{\partial^2 u}{\partial x^2} + u^3$ | $\frac{\partial u}{\partial_t} = \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x}$ |




spectrum of $\mathcal{L} = \frac{\partial^2}{\partial x^2}$

# Summary and outlook

- Spectra generally predict nonlinear dynamics
- Proofs utilize dynamical-systems and PDE techniques to account for nonlinear terms
- Open problems: planar spiral waves are not nearly as well understood

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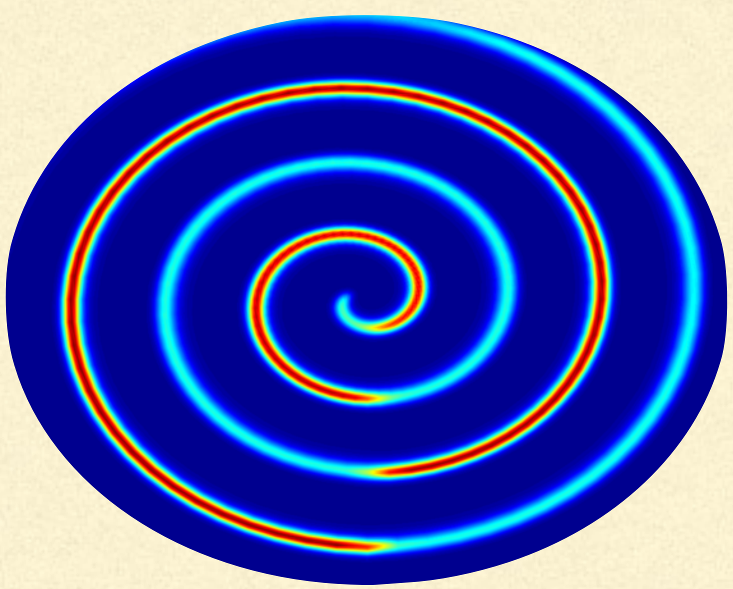

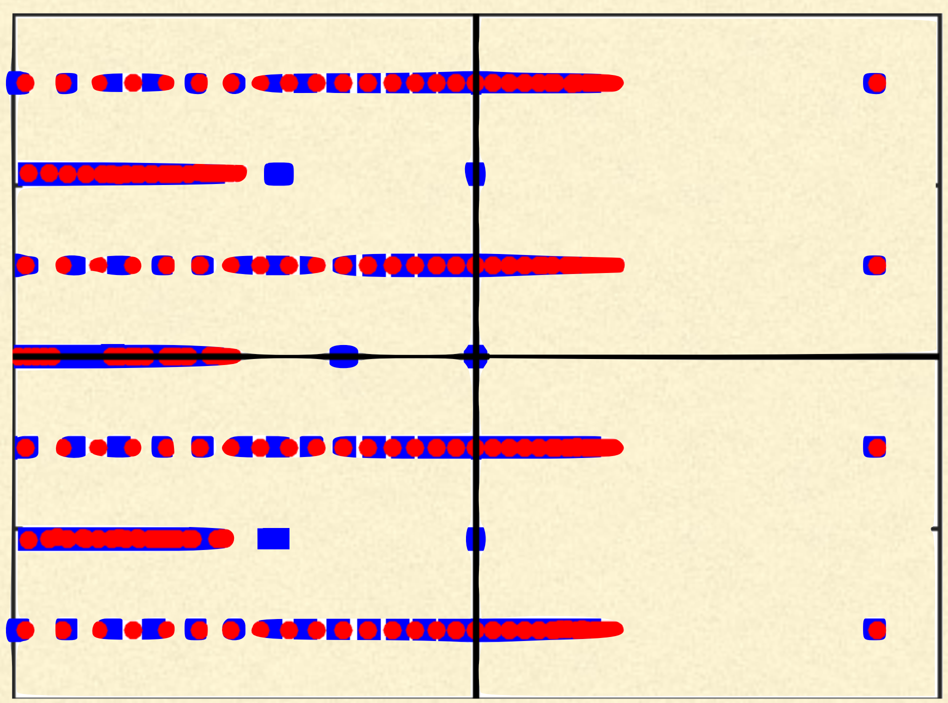

Alternans / period-doubled spiral waves