Nonlinear Patterns and Waves: From Spectra to Stability and Dynamics

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Zebrafish *Danio rerio*

Vertically vibrated copper beads
[Ummanhowar et al.]
Calcium waves in Xenopus oocytes [Clapham et al.]

CO oxidation on Pt [Nettesheim et al.]

s... and waves
Outline

- **Existence:** How many degrees of freedom do these structures have?
- **Stability:** What happens when spots or spiral waves are perturbed?
- **Interaction:** How do these structures interact with each other?
Outline

Interactions of localized structures

Modulations of spatially-periodic structures

Dynamics of one-dimensional spiral waves
Localized structures

\[ \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R}^n \]

\( u(x,t) \) represents vector of concentrations or displacements at position \( x \) and time \( t \)

Localized steady state

\[ u(x,t) = q(x) \]

Assess stability under small perturbations:

Setting \( u = q + v \) with \( |v| \) small gives

\[ \frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} + f_u(q(x))v \]

Perturbed solution converges exponentially in time to \( q(x) \), an appropriate translate of \( q(x) \), has solutions of the form \( v(x,t) = e^{\lambda t} v_0(x) \), where \( \lambda \) is in the spectrum of the operator

\[ \mathcal{L} := D \frac{\partial^2}{\partial x^2} + f_u(q(x)) \]

Spectrum of the linearization about \( q(x) \)
Interaction of localized structures

\[
\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u), \quad x \in \mathbb{R}, \quad u \in \mathbb{R}^n
\]

Interacting structures: \( u_0(x) = \sum_{j=1}^{N} q(x - p_j), \quad |p_i - p_j| \gg 1 \)

![Diagram showing localized stable steady state, spectrum of linearization about \( q(x) \), and spectrum of linearization about \( u_0(x) \) with \( N \) eigenvalues near zero.\]
Interaction of localized structures

\[ \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u) \]

Invariant manifold parametrized by positions of spots

- Nearest-neighbor coupling (to leading order)
- Exponential small in distances
Summary: interaction of localized structures

- PDE dynamics of spots described by system of ODEs for positions
- To leading order, nearest-neighbor coupling
- Interaction through tails, hence exponentially slow dynamics
- Spectrum: finitely many eigenvalues

[Neu], [Ei], [S.], [Mielke & Zelik]
Outline

Interactions of localized structures

Modulations of spatially-periodic structures

Dynamics of one-dimensional spiral waves
Infinitely many interacting spots

finitely many spots

infinitely many spots

infinitely many interaction eigenvalues form curve
Slowly varying modulations

\[ \frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2} + f(u) \]

\[ u(x, t) = u_*(kx - \omega_*(k)t; k) \]

\begin{align*}
    k & \quad \text{wavenumber} \\
    \omega = \omega_*(k) & \quad \text{temporal frequency} \\
    c_p = \omega_*/k & \quad \text{phase velocity} \\
    c_g = d\omega_*/dk & \quad \text{group velocity} \\
\end{align*}

\[ \frac{\partial q}{\partial t} + c_g \frac{\partial q}{\partial x} = 0 \]
Summary: slowly-varying modulations

- Modulations are transported with uniform speed given by the group velocity.
- Spectrum consists of a curve touching the imaginary axis.

[Howard & Kopell], [Kuramoto], [Doelman et al.], [Johnson et al.]
Outline

Interactions of localized structures

Modulations of spatially-periodic structures

Dynamics of one-dimensional spiral waves
Modulations of spiral waves
Dynamics of spiral waves

Perturbing a spiral wave has two effects:
- Position of the spiral may change
- Far-field wave trains may be modulated

source

group velocities point away from core

c_g

c_g

time

space
Spectra of sources

\[ \frac{\partial v}{\partial t} = D \frac{\partial^2 v}{\partial x^2} + f_u(u_*(x, t)) v \]

Spectrum of wave trains

\[ L^2 \text{ space} \]

\[ L^2_{\exp} := \{ v : \|v(x)e^{-|x|}\|_2 < \infty \} \]
Defect core converges exponentially to new position.

Modulation interface is localized and travels with speed given by the group velocity.
Caveats

Long-time dynamics for small localized initial data

Heat equation
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \]

Reaction diffusion
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u^3 \]

Burgers equation
\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \frac{\partial u}{\partial x} \]

Spectrum of \( \mathcal{L} = \frac{\partial^2}{\partial x^2} \)
Summary and outlook

- Spectra generally predict nonlinear dynamics
- Proofs utilize dynamical-systems and PDE techniques to account for nonlinear terms
- Open problems: planar spiral waves are not nearly as well understood
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- Spectra generally predict nonlinear dynamics
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- **Open problems:** planar spiral waves are not nearly as well understood

Alternans / period-doubled spiral waves