Rare event extinction on stochastic networks

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Outline

- Analytic approach for optimal (most probable) path to extinction
- Disease extinction in a static social network
  - Lindley, Shaw, and Schwartz, *Europhysics Letters* 108: 58008 (2014)
- Extinction in an adaptive social network

Optimal path to extinction

Extinction analysis

- Stochastic effects are described by the master equation for probability density $\rho(\mathbf{x},t)$ to find system in state $\mathbf{x}$ at time t

$$\frac{d\rho(\mathbf{x},t)}{dt} = \sum_{\mathbf{r}} \left[w(\mathbf{x}-\mathbf{r};\mathbf{r})\rho(\mathbf{x}-\mathbf{r},t) - w(\mathbf{x};\mathbf{r})\rho(\mathbf{x},t) \right]$$

  where r are transition vectors (change in x), and w are transition rates

- Use WKB/eikonal approximation:
  Assume probability distribution $\rho(\mathbf{x}) \sim e^{-N\,S(\mathbf{x})}$, where $S(\mathbf{x})$ is called the action

- Maximizing extinction probability requires minimizing action over all possible paths to extinction (calculus of variations problem)

(Kubo et al *J Stat Phys* 1973 , Gang *PRA* 36: 5782, 1987; Dykman et al *J Chem Phys* 100: 5735, 1994, Forgoston et al *Bull Math Biol* 2011)

Extinction analysis

- The position and momentum variables on the optimal path are described by the following equations of motion:

$$\dot{x} = \frac{\partial H}{\partial p}; \quad \dot{p} = -\frac{\partial H}{\partial x}$$

$$H(\mathrm{x}, \mathrm{p}; t) = \sum_{\mathrm{r}} w(\mathrm{x}; \mathrm{r})\left(e^{\mathrm{p}\cdot\mathrm{r}} - 1\right)$$

where H is the Hamiltonian and p is momentum (force due to the noise)

- The optimal path is the heteroclinic orbit from endemic to extinct state
- Task: Find this heteroclinic orbit; use action $S(\mathbf{x})$ on path to predict extinction time
- Challenge: This system is high dimensional

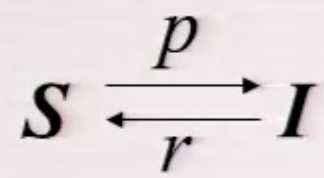
Disease extinction in a static social network

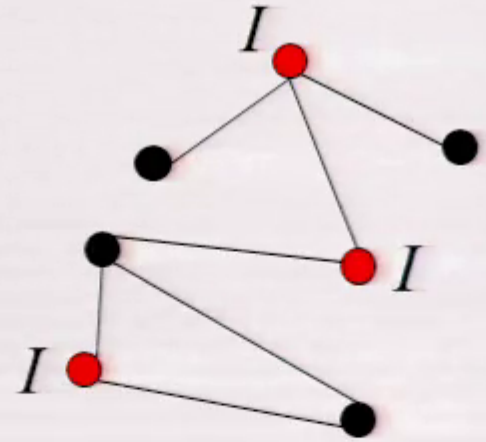
(Lindley, Shaw, and Schwartz, *EPL* 2014)

SIS model on static network

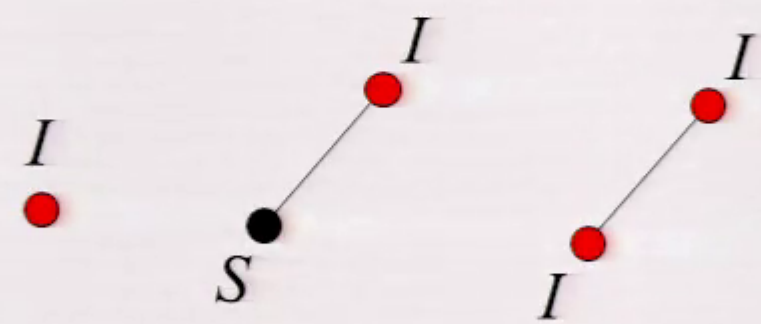
- Infection rate p, recovery rate r, conserved population (N nodes, K links)
- Consider transitions in number of nodes and number of links of each type, ignoring higher order structure (similar to pair-based proxy model, Rogers et al *JSTAT* 2012)
- Variables $P_I = \frac{N_I}{N}$, $L_{SI} = \frac{N_{SI}}{N}$, $L_{II} = \frac{N_{II}}{N}$ – 3 position variables, plus 3 momentum variables

Epidemic model:



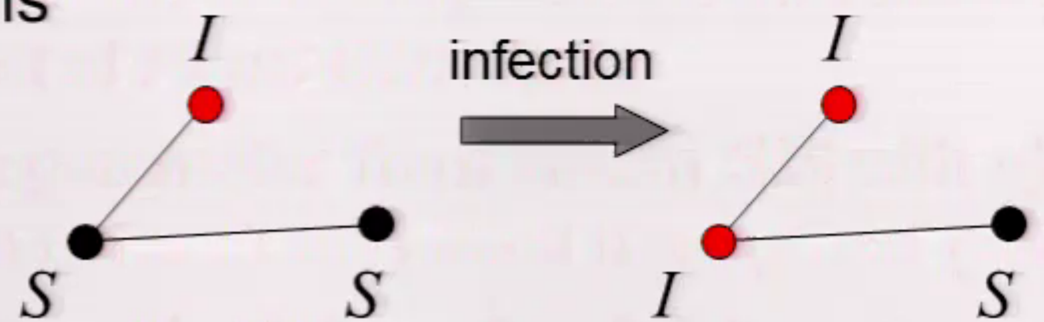


Variables:



State transitions

- Assume each node has neighbors given by network averages
- Enumerate transitions



- Transition 1: Infection along network

$$w_1 = \varepsilon p N_{SI} \qquad \mathrm{r}_1 = \left[1, \frac{2N_{SS}}{N_S} - \left(1 + \frac{N_{SI}}{N_S}\right), \left(1 + \frac{N_{SI}}{N_S}\right)\right]$$

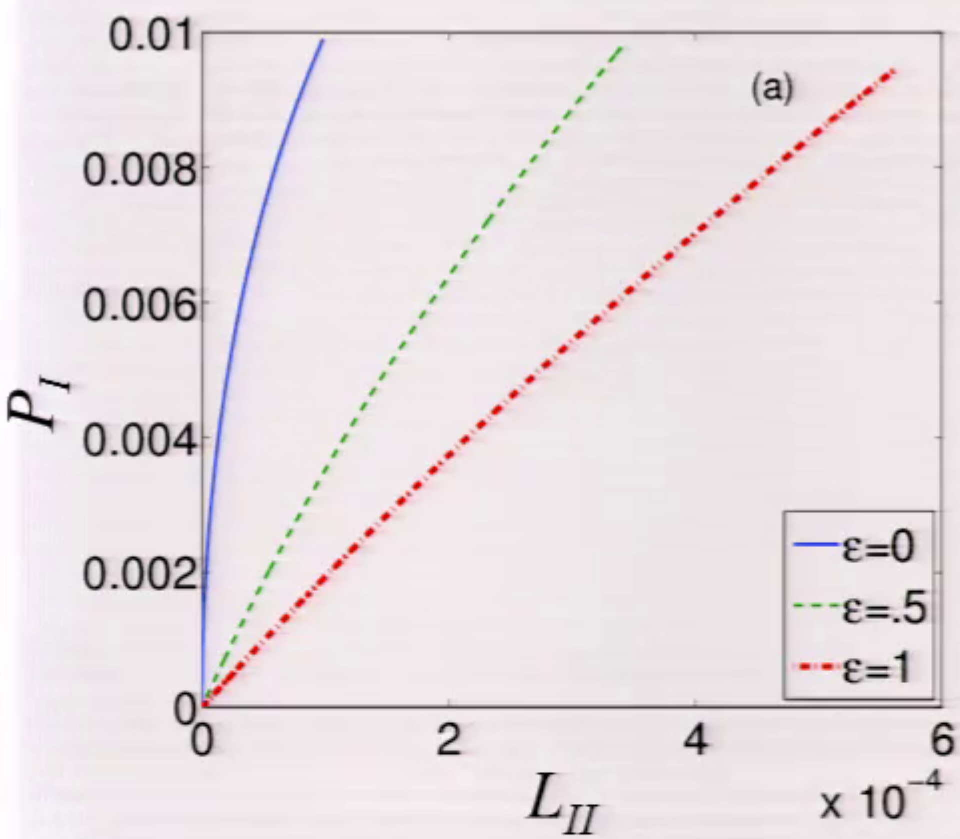
- Transition 2: Infection through global coupling (mass action)

$$w_2 = (1-\varepsilon)\, p \frac{2K}{N^2} N_S N_I \qquad \mathrm{r}_2 = \left[1, \frac{2N_{SS}}{N_S} - \frac{N_{SI}}{N_S}, \frac{N_{SI}}{N_S}\right]$$

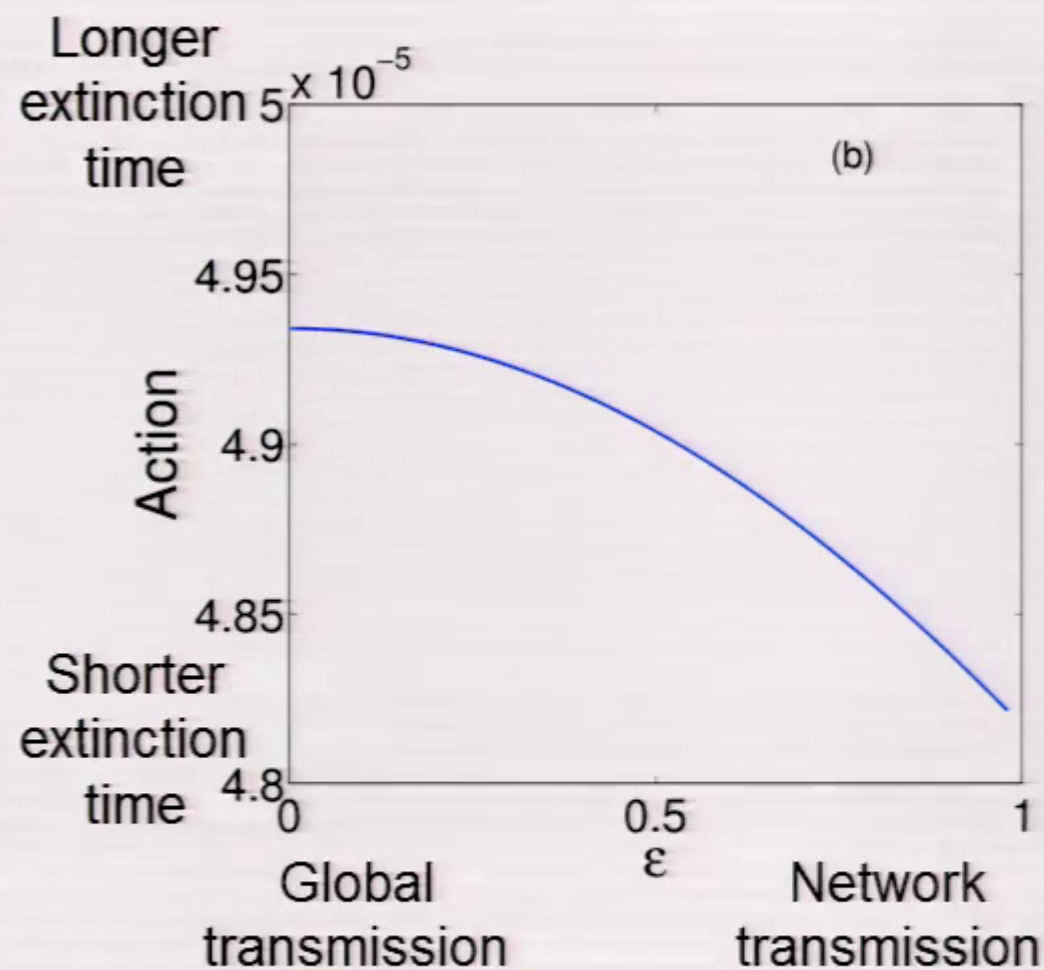
- Transition 3: Recovery

Global vs network transmission

Path to extinction:



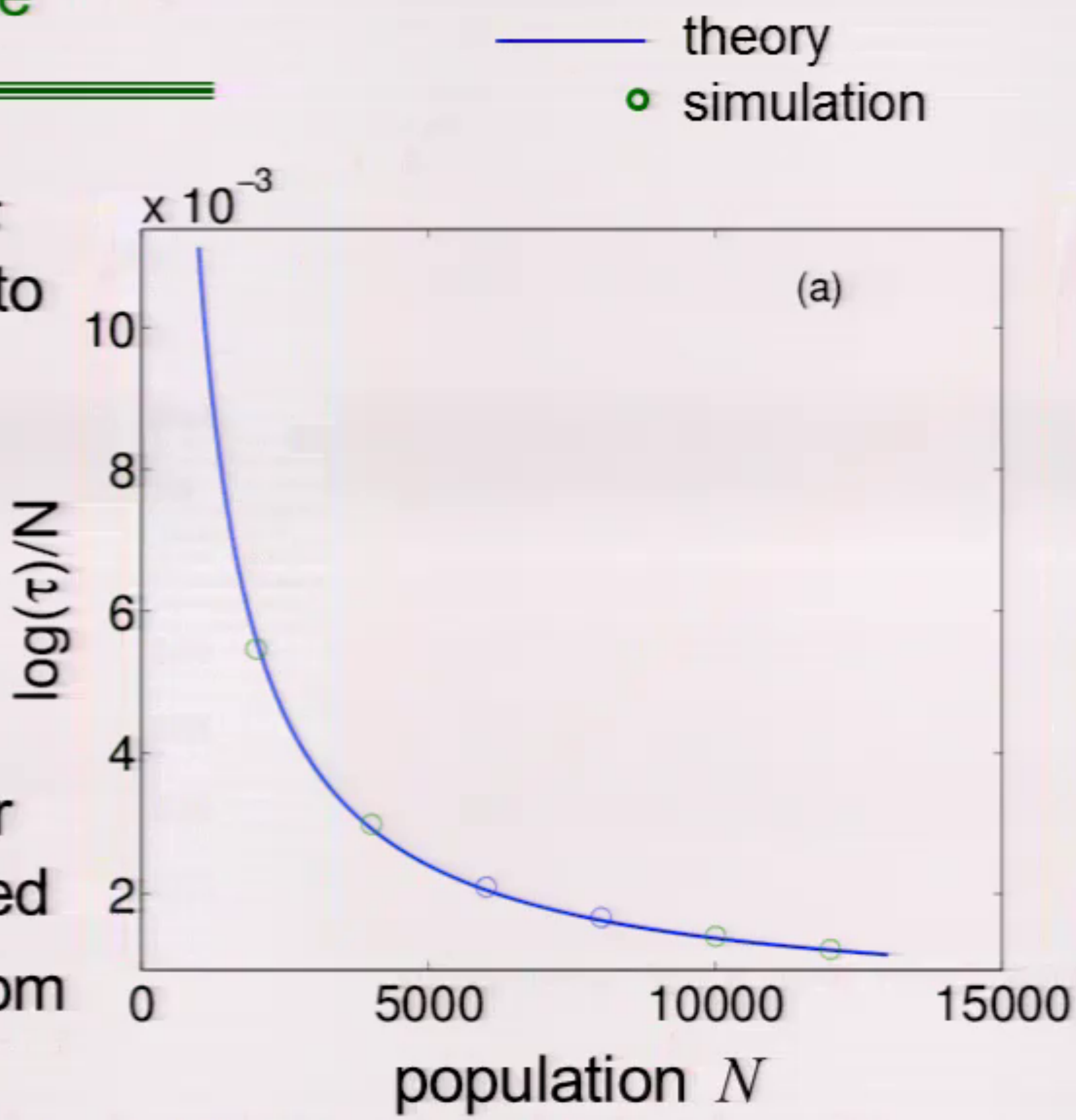
Effect on lifetime:



p=1.03×10^{-4}, r=0.002, K/N=10, predictions from IAMM

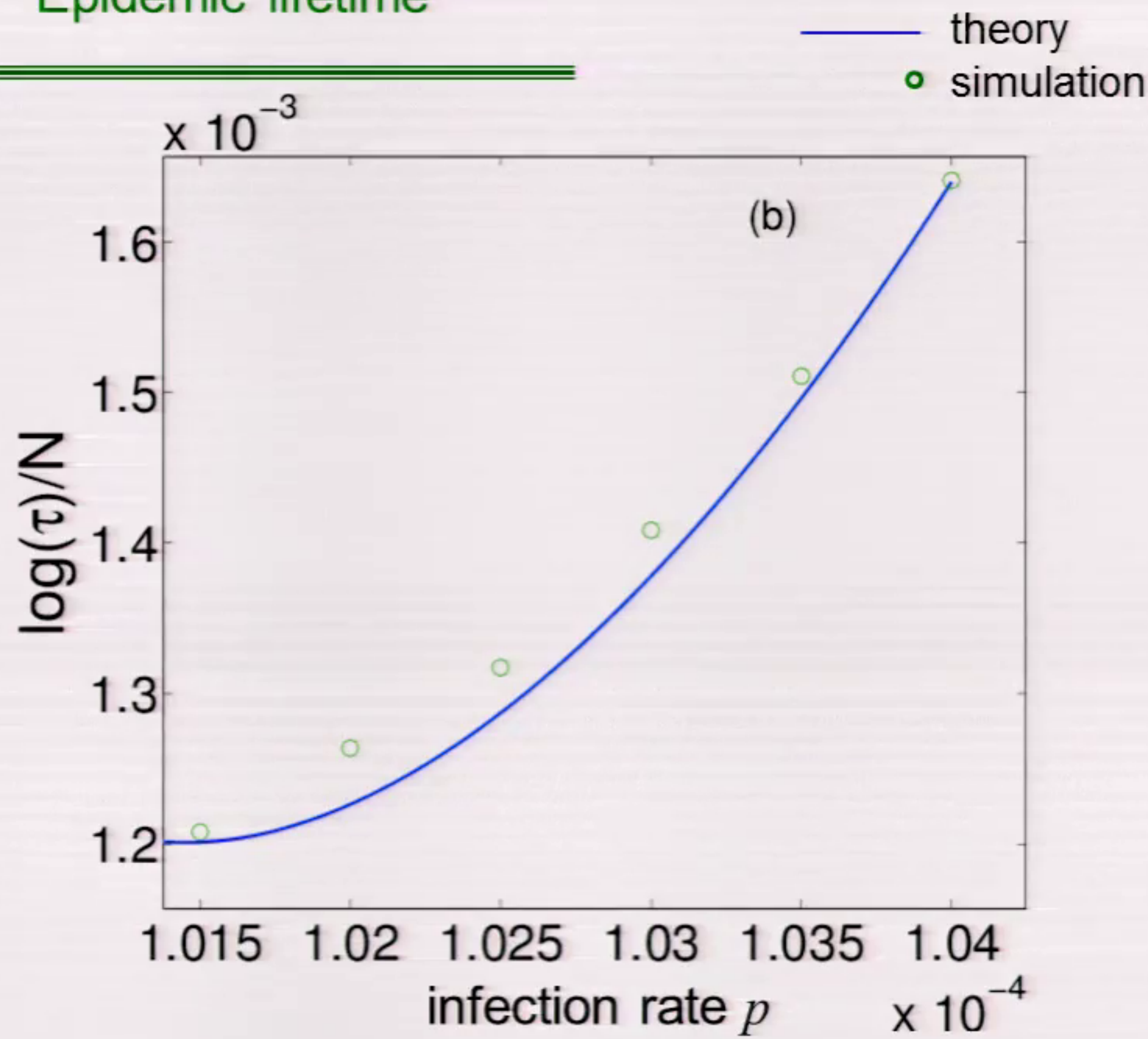
Epidemic lifetime

- Expect mean lifetime τ inversely proportional to probability
- Recall assumption $\rho(\mathbf{x}) \sim e^{-N\,S(\mathbf{x})}$
- This implies $(\log \tau)/N$ independent of N
- Not so here – prefactor (order of $N^{-1/2}$) is needed
- Use $\tau = B\,e^{N\,S}$ with B from globally coupled SIS model (Billings et al *PLoS ONE* 2013)



$p=1.03\times10^{-4}, r=0.002, \varepsilon=1, N=10^4, K=10^5$

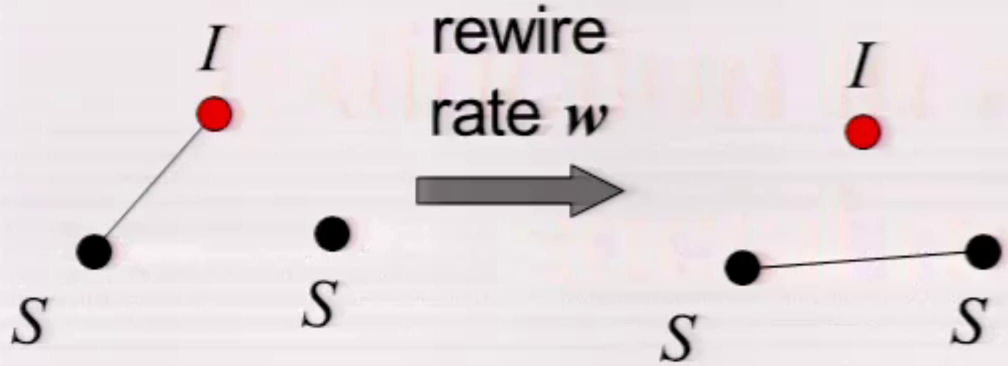
Epidemic lifetime



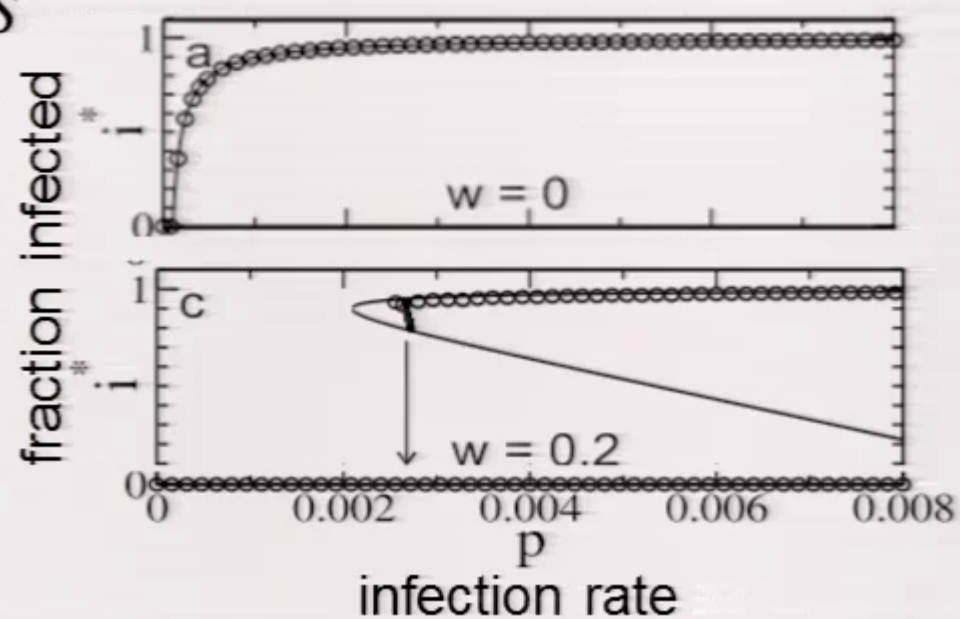
Add avoidance rewiring

Epidemic dynamics: $S \underset{r}{\overset{p}{\rightleftarrows}} I$

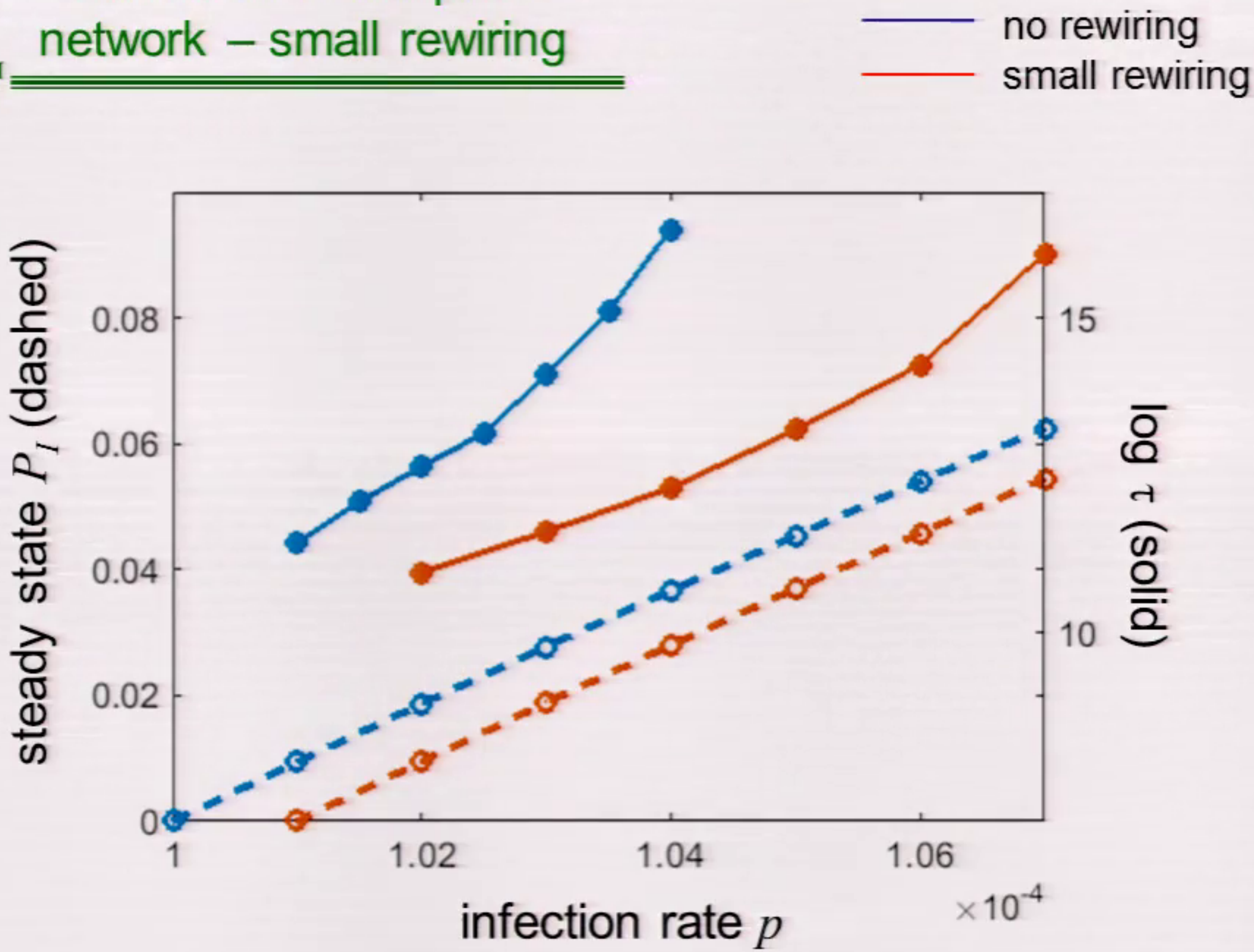
Network dynamics—avoidance:



Gross et al, *PRL* 96: 208701, 2006



Extinction in adaptive network – small rewiring



w=0 or w=2×10^{-5}, r=0.002, N=10^4, K=10^5

Conclusions

- Apply optimal path approach to extinction on a network
  - Node and link transitions from pair-based proxy model
  - Perturb from known solution for globally coupled system
  - Use IAMM to track optimal path
- Predicted path to extinction matches stochastic network simulations
- Extinction time matches simulations when appropriate prefactor is used
- Add network adaptation (small avoidance rewiring)
  - Extinction times substantially reduced