Dynamics, Mixing, and Coherence

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SIAM Annual Meeting
Boston, July 15, 2016

Outline

1. Mathematicians and scientists often look for **structure in complex processes**;
2. It is possible to find **particular observations that remain correlated for long times**.
3. These observations **directly provide important spatial information about dynamic structures that decay or mix slowly**, but are otherwise very difficult to identify (e.g. oceanic eddies and atmospheric vortices).

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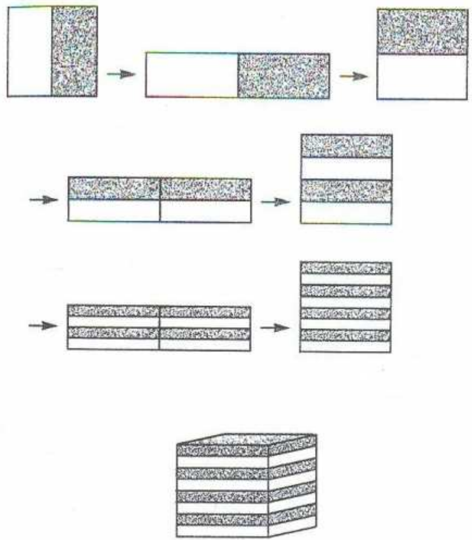
A prototype dynamical system



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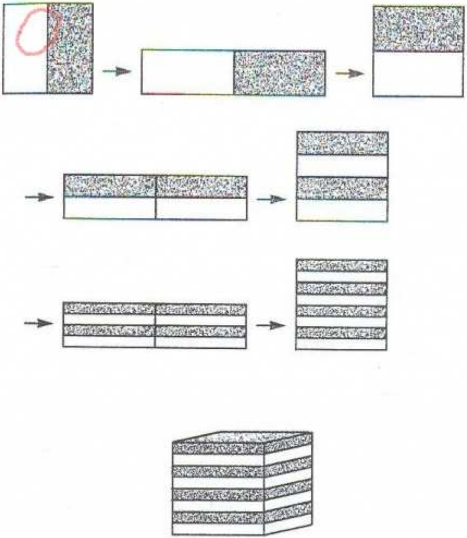

Poincaré Recurrence for the lamington map: Let A be a fixed region in the 2D lamington. Then under the action of the lamington map, almost all crumbs in A return infinitely often to A.

Theorem (Poincaré Recurrence (1890))

If a probability measure μ on X is preserved by the action of $T : X \to X$, then for any $A \subset X$ with positive μ-measure, μ-almost all points return infinitely often to A.

Basic ergodic theorems: recurrence

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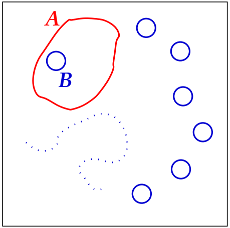
Proof sketch: Suppose there is a "bad set" $B \subset A$, with positive μ-measure, which does not recur to A infinitely often.



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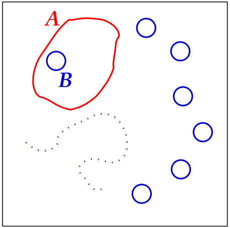


What about the **frequency** of returns?

Frequency of returns for lamingtons: For any fixed region A in the 2D lamington, the time-asymptotic frequency with which crumbs return to A is exactly the area of A.

Proof: Put f=$\mathbf{1}_A$ in the theorem below.

Theorem (Birkhoff's Ergodic Theorem (1931))

Let $f : X \to \mathbb{R}$ be an observable and define the n-step average

$$A_n[f](x) := \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x), \;\; x \in X.$$

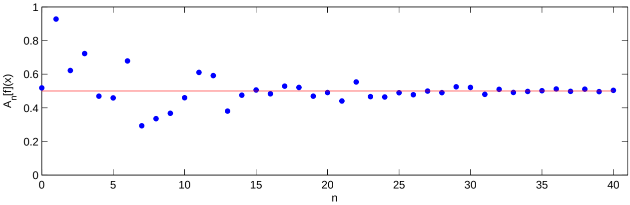
If μ is ergodic, then as $n \to \infty$,

$$A_n[f](x) \to \int_X f \; d\mu =: \mathbb{E}(f), \qquad \text{for } \mu \text{ almost all } x \in X.$$

Fluctuations in finite-time averages

- Birkhoff's Theorem says

$$A_n[f](x) := \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k(x) \to \mathbb{E}_\mu(f), \text{ for } \mu \text{ a.e. } x \in X.$$



- What about the error $|A_n[f](x) - \mathbb{E}(f)|$? This is more subtle and depends on how dependent or **correlated** the observables $f \circ T^k$ are.

Temporal correlations of observables

- Suppose I have two observables $f, g : X \to \mathbb{R}$ and I observe f now, but wait k units of time before observing g. How are the observables f and $g \circ T^k$ correlated?
- Thinking of f, g as random variables (e.g. concentration of chocolate sauce on the lamington or CO_2 in the atmosphere):

$$\begin{aligned} cov(f, g \circ T^k) &= \mathbb{E}_\mu \left[(f - \mathbb{E}_\mu(f)) \cdot (g \circ T^k - \mathbb{E}_\mu(g \circ T^k)) \right] \\ &= \mathbb{E}_\mu(f \cdot g \circ T^k) - \mathbb{E}_\mu(f)\mathbb{E}_\mu(g \circ T^k) \\ &= \mathbb{E}_\mu(f \cdot g \circ T^k) - \mathbb{E}_\mu(f)\mathbb{E}_\mu(g) \end{aligned}$$

- Let's suppose that $cov(f, g \circ T^k) \to 0$ as $k \to \infty$. What is the **rate** at which $cov(f, g \circ T^k) \to 0$? This subtle question requires **smoothness**.

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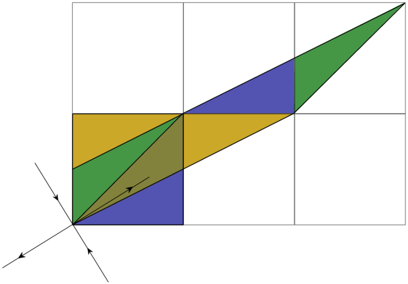
- Many wonderful things happen when you combine differential structure with probability $\rightsquigarrow$ *smooth ergodic theory*.
- Let X be a manifold and $T : X \circlearrowleft$ a differentiable bijection (a *diffeomorphism*).
- The map T is called **uniformly hyperbolic** (Anosov, Sinai, Smale) if one can identify local directions in the tangent space of X at every point $x \in X$ in which there is either strict local expansion or strict local contraction.



Smooth ergodic theory and temporal correlations

Theorem (Sinai'72, Bowen'75, Ruelle'76)

If T is C^2 and uniformly hyperbolic, f is C^1, and g is bounded, then there is a $0 < \lambda < 1$ such that

$$cov(f, g \circ T^k) \leq C(f,g)\lambda^k \text{ for all } k \geq 0.$$

*That is, T has "***exponential decay of correlations***".*

- **Q:** What is driving this decay of correlation?
- **A:** The exponential separation of nearby trajectories caused by the strict local expansion of T.
- Local expansion is a common feature in many dynamical systems. *This is why the weather is hard to predict one week in advance using observations from the present.*

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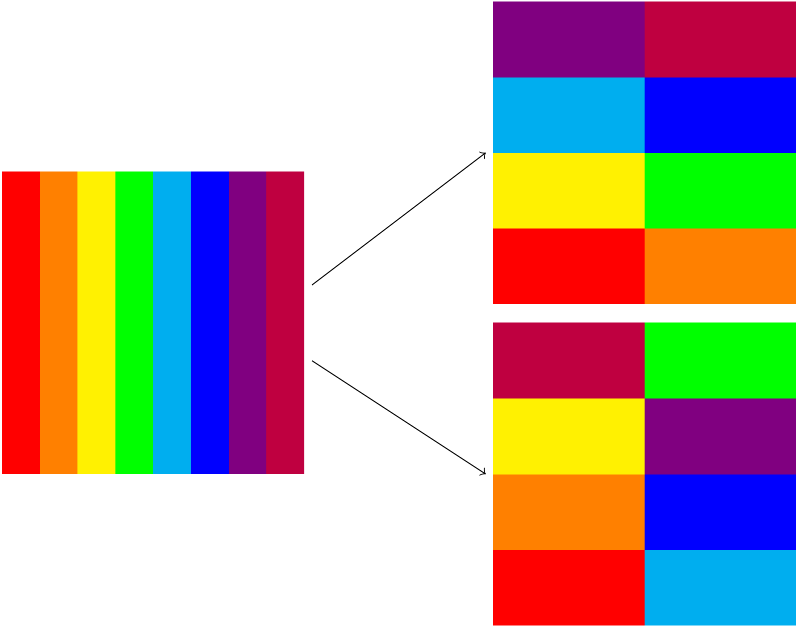
So what is this rate of decay?



So what is this rate of decay?

- The figure on the left shows the evolution of a small square of points under the “standard” 4-fold lamington map.
- The figure on the right is the tweaked lamington map.
- **Both lamington maps have expansion factors of 4**, meaning nearby trajectories separate by a factor 4 at each iteration.
- However, the “standard” version (on the left) **appears to mix faster**. What’s going on?

A dual point of view

- We now write

$$\mathbb{E}_\mu(f \cdot g \circ T^k) = \int_X f \cdot g \circ T^k \; d\mu =: \int_X \underbrace{\mathcal{P}^k f}_{\sim f \circ T^{-k}} \cdot g \; d\mu,$$

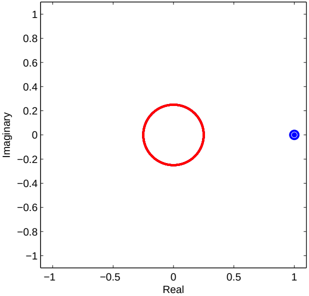
  where the **Perron-Frobenius operator** or **transfer operator** $\mathcal{P}$ is defined via a change of variables using T^k.
- If $f \in L^1(X), g \in L^\infty(X)$,

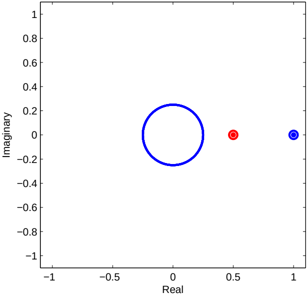
$$\left| \int_X \mathcal{P}^k f \cdot g \; d\mu \right| \leq \|\mathcal{P}^k f\|_{L^1} \cdot \|g\|_{L^\infty}, \quad k \geq 0.$$

- Thus, the **spectrum of** $\mathcal{P}$ is important for controlling covariances and upper bounds of rates of decay of correlations.
- Typically, one considers $\mathcal{P} : \mathcal{B} \circlearrowleft$, where $\mathcal{B}$ is a Banach space of suitably regular functions, strictly contained in L^1.

Decay rates from the spectrum of the transfer operator

Left: Spectrum of $\mathcal{P}$ for the "standard" lamington map;
Right: Spectrum of $\mathcal{P}$ for the tweaked lamington map.





Thus, the rate of decay of correlation is **not** a function of expansion rates only, or "more chaotic" does not necessarily equal "faster decay of correlations" or "faster mixing" (Dellnitz/F/Sertl'00, Collet/Eckmann'04, F'07)

Visualising the eigenfunction of $\mathcal{P}$ corresponding to λ_2

- Experiments of dye-mixing in periodically forced fluids (eg. [Voth *et al.* '02]) have shown that intricate, persistent patterns can develop from an initial dye distribution.

- You are watching convergence to f_2, where $\mathcal{P}f_2 = \lambda_2 f_2$ and λ_2 is the second largest eigenvalue of $\mathcal{P}$.

Garbage patches in the ocean

[van Sebille/England/F, '12]; see also [Maximenko '11, Khatiwala/Visbeck/Cane '05]

Time-dependent dynamics

- In applications, many systems are time-dependent, meaning that the **underlying dynamical rules change over time**.
- For example, the three-dimensional velocities of ocean currents are governed by changing internal variations in density controlled by salinity and heat, which in turn are affected by changing external inputs.
- In the atmosphere, similar variations occur on much faster timescales.
- Dynamical systems models of time-dependent evolution take the forms:
    - **Continuous time:** A time-dependent ODE $\dot{x} = f(x, t)$ rather than $\dot{x} = f(x)$.
    - **Discrete time:** A concatenation $\cdots T_k \circ T_{k-1} \circ \cdots \circ T_2 \circ T_1$, where $T_i, i = 1 \ldots, k$ are **different maps**, rather than T^k, iteration of a single map T.

Slow mixing structures in time-dependent systems

- There is no reason to expect the slowly-mixing structures to be **fixed in space** (like almost-invariant sets) in time-dependent systems.
- In fact, they can be **highly mobile**, making their detection considerably more difficult.

Decay in time-dependent systems

- **Time-independent case**
  - We found the eigenfunction f_2 corresponding to the second largest eigenvalue λ_2. Thus,

$$\|\mathcal{P}^k f_2\| \leq C(f_2)\lambda_2^k, \quad \text{for all } k \geq 0.$$

  - *But what are "eigenvalues" and "eigenfunctions" in the time-dependent setting?*
- **Time-dependent case**
  - The analogous growth rate expression is

$$\|\mathcal{P}_{T_k} \circ \cdots \circ \mathcal{P}_{T_2} \circ \mathcal{P}_{T_1} f\| \leq C(f)\lambda_2^k.$$

  - Or:

$$\lim_{k\to\infty} \frac{1}{k} \log \|\mathcal{P}_{T_k} \circ \cdots \circ \mathcal{P}_{T_2} \circ \mathcal{P}_{T_1} f\| \leq \log \lambda_2.$$

  - Note that the $\mathcal{P}_{T_i}$ are *linear* operators (or in numerical experiments, matrices), so $\log \lambda_2$ is a **Lyapunov exponent**.
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- The Oseledets Multiplicative Ergodic Theorem (MET), proven in Oseledets' thesis in 1965, **creates time-dependent generalisations of eigenvalues and eigenvectors for concatenations of matrices**.
- Building on the work of Ruelle, Mañé, Thieullen, extensions of Oseledets' MET have been developed [F, González-Tokman, Lloyd, Quas,...] to **enable application to time-dependent dynamical systems**.
- The Oseledets vectors corresponding to the second Lyapunov exponent λ_2 are the **unique collection of fs that decays as slowly as possible and evolve consistently with the time-dependent dynamics**:

$$\lim_{k\to\infty} \frac{1}{k} \log \|\mathcal{P}_{T_k} \circ \cdots \circ \mathcal{P}_{T_2} \circ \mathcal{P}_{T_1} f\|$$

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- When studying systems over **finite time durations**, one uses **singular vectors** of $\mathcal{P}_{T_k} \circ \cdots \circ \mathcal{P}_{T_2} \circ \mathcal{P}_{T_1}$, which approximate Oseledets vectors.

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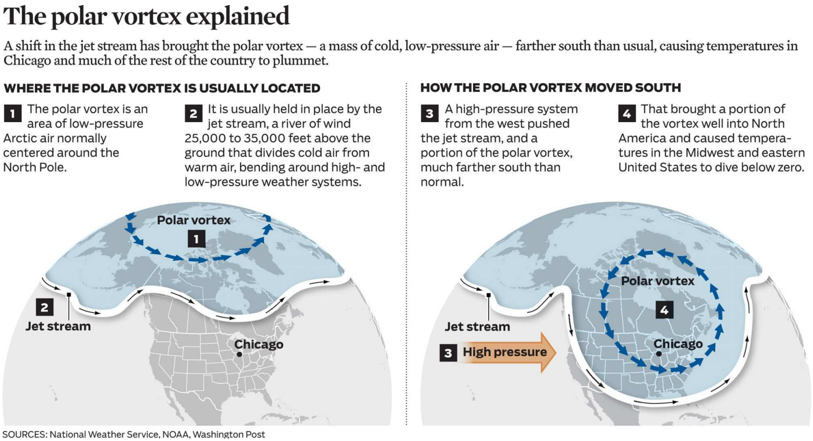
The North American 2013-14 “Polar Vortex Winter”.



Source: National Weather Service, NOAA, Washington Post.

- In the stratosphere over the south pole, there are **strong persistent transport barriers** that give rise to the **Antarctic polar vortex**.
- Previous studies include Boffetta *et al.* '01, Koh/Legras '02, Rypina *et al.* '07, Lekien/Ross '10, de la Cámara *et al.* '12.
- We numerically approximate transfer operators $\mathcal{P}$ using ECMWF vector fields, compute singular vectors, and **resolve the polar vortex as the slowest decaying object**.
- We initialise the flow at September 1, 2008 on a 475K isentropic surface and follow the flow for two weeks until September 14.

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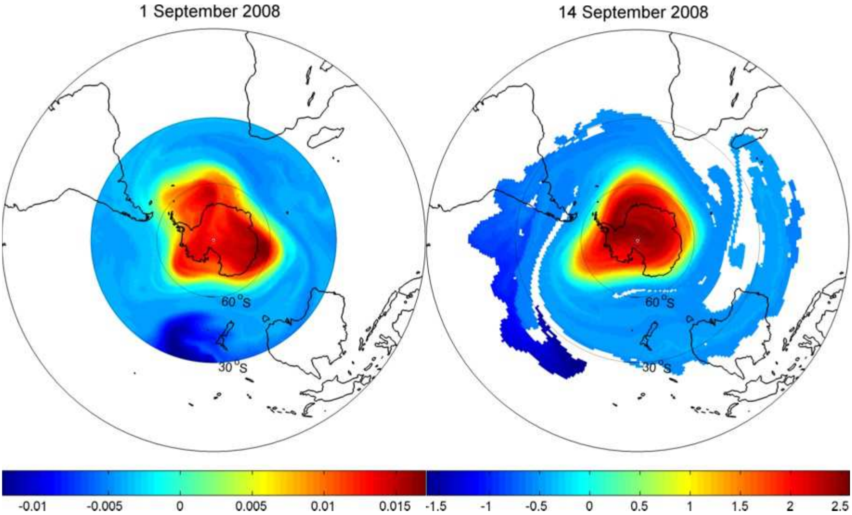
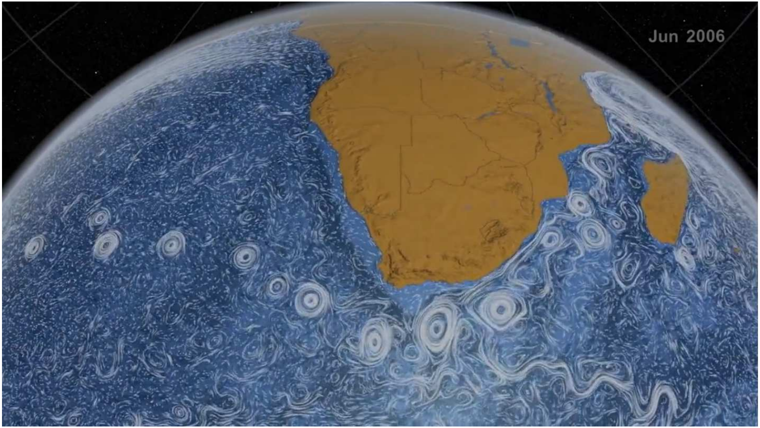
The second singular vectors



See [F/Santitissadeekorn/Monahan, *Chaos*, 2010.]

Particle simulation demonstrating the identified vortex inhibits global mixing

Application 2: Tracking Agulhas Rings

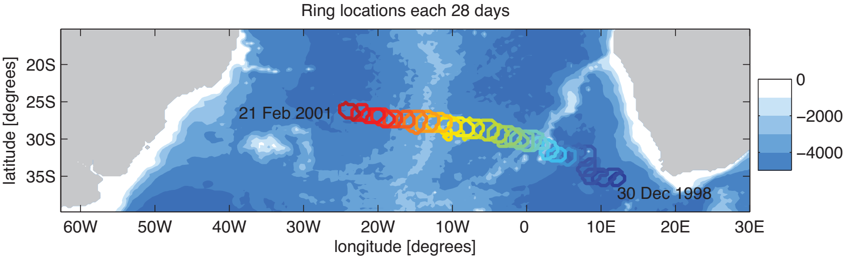


NASA combined a general ocean circulation model with observations (eg. sea surface height from satellites) to create a (somewhat smoothed) visualisation of surface ocean currents.

Application 2: Tracking Agulhas Rings

- The transport of **warm saline waters** from the Indian Ocean into the upper Atlantic Ocean is substantially affected by the advection of large anticyclonic eddies or Agulhas Rings that detach periodically at the Agulhas current retroflection eg. [De Ruijter *et al.* 1999; Lutjeharms, 2006; Doglioli *et al.*, 2006].
- How much heat and salt an Agulhas Ring transports, and how far into the North Atlantic the Ring transports these tracers, is sensitive to **how long the water remains within a Ring** as well as **its path** [Treguier *et al.* 2003].
- Previous LCS-based studies include Poje/Haller '99, Beron-Vera *et al.* '08, Bettencourt *et al.* '11, Beron-Vera *et al.* '13, Karrasch *et al.* 15, Wang *et al.* '16.

Agulhas ring as a mass transporter and slowly decaying object



- We use velocity fields derived from satellite sea-surface height data to construct numerical transfer operators.
- We then compute the 2nd Oseledets vectors and identify an Agulhas ring as the **slowest decaying object**, and track its movement for 26 months. [F/Horenkamp/Rossi/SenGupta/vanSebille, 2015].

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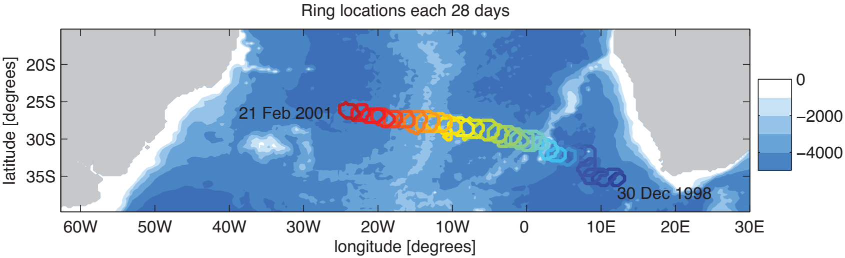

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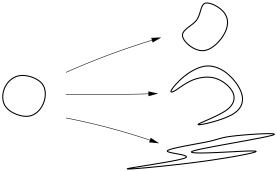
Particles initialised in the first ring at Dec 30, 1998

Mixing and Geometry

There is a beautiful relationship between coherence of a set in terms of **slow mixing** and coherence of a set in terms of its **boundary remaining small** [F'15].



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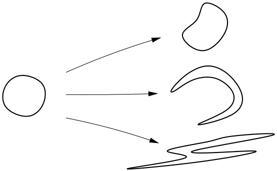

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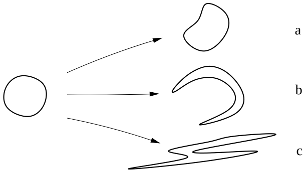

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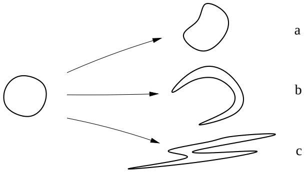

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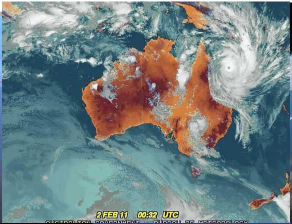
Summary

- Slowly decaying structures in dynamical systems are revealed by observations that remain temporally correlated for long times.
- These highly correlated observables are eigenvectors (resp. Oseledets/singular vectors) of transfer operators in time-independent (resp. time-dependent) dynamics.
- These ideas also apply to **time-dependent PDEs**.
- Accurately mapping and tracking slowly decaying structures is of great importance in models of geophysical flows because these structures are the **predictable components of often highly unpredictable dynamics**.
- Ultimate aim is to produce automated algorithms to process input and present results **in near-real time for predictive use**.

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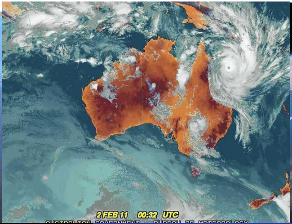

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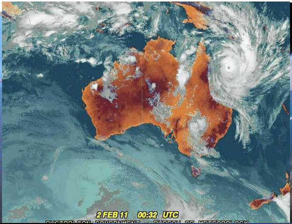

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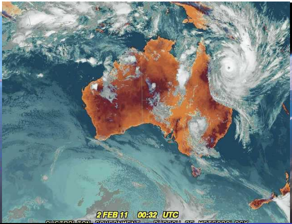

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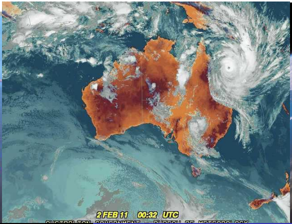

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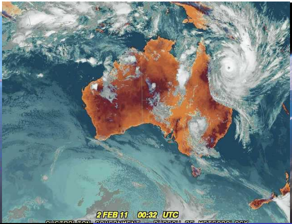

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Thanks go to:

Australian Government

Australian Research Council

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UNIVERSITIES

and my collaborators:

M. Dellnitz (Paderborn), M. England (UNSW),
S. Lloyd (X'ian-Liverpool), A. Monahan (UVic), A. Quas (UVic),
V. Rossi (Balearic Is.), N. Santitissadeekorn (Surrey),
A. SenGupta (UNSW), R. Stuart (Copenhagen),
E. van Sebille (Imperial).