Algorithms for Processing the Labeled Conley Index

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Joint with Sarah Day

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Setting

Goal: *automated* rigorous computations

to capture complex dynamics

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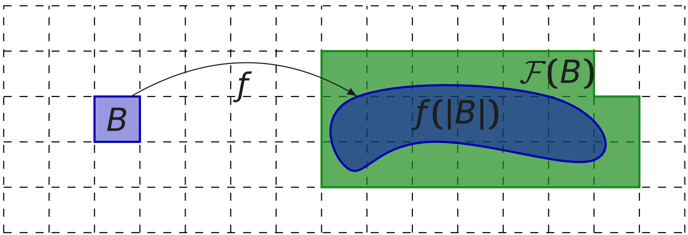
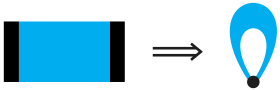
1. Discretize and compute combinatorial enclosure



2. Build Conley index pair
3. Compute index representative [CHomP]
4. Process the index information ⟵ *this talk*

Labeled Conley Index



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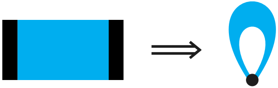
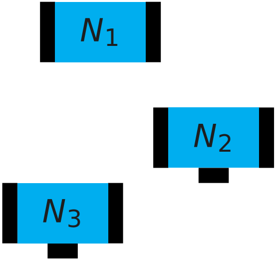


Ważewski Principle:

$$\mathrm{Con}(S, f) \neq [0]_s \implies S \neq \emptyset$$



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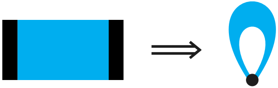

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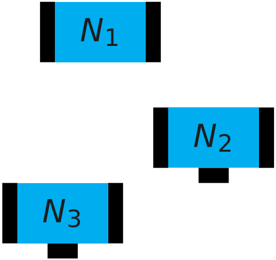
Corollary:
$\mathrm{Con}(S', f|_{N_3} \circ f|_{N_2} \circ f|_{N_1}) \neq [0]_s$



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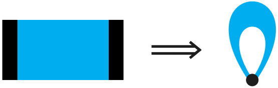

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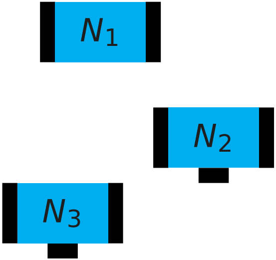
$\exists\, x \in S'$ with itinerary $\rho(x) = \overline{123}$



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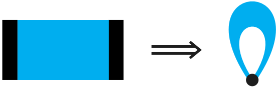

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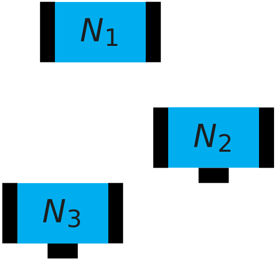

Labeled Conley index: set of all such cyclic Conley indices

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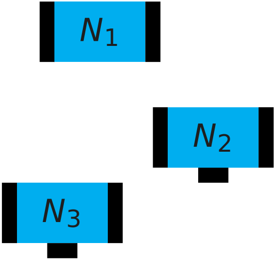

	N_1	N_2	N_2	N_3	N_3
N_1	0	−1	0	1	−1
N_2	1	0	0	0	0
N_2	1	0	0	0	0
N_3	0	1	0	0	1
N_3	0	0	−1	1	0

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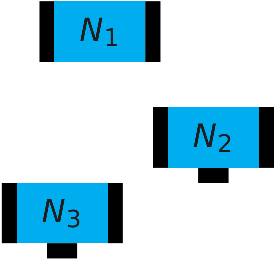

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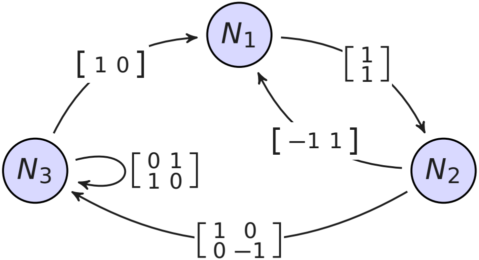

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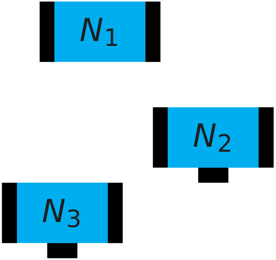

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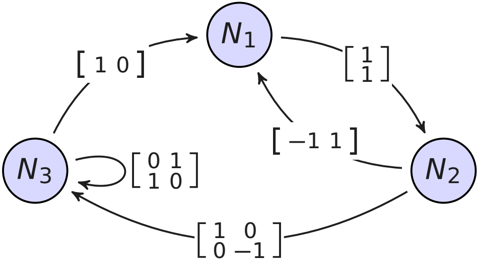
$$\mathrm{Con}(S', f|_{N_3} \circ f|_{N_2} \circ f|_{N_1})$$



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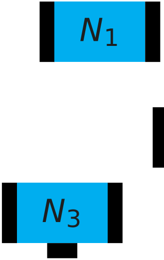
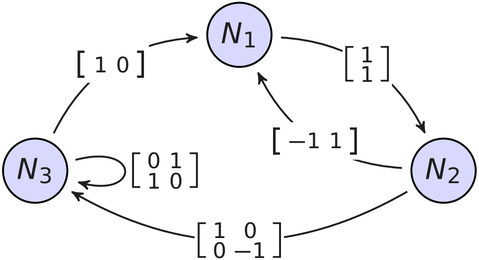


$$
\begin{array}{c} \\ N_1 \\ N_2 \\ N_2 \\ N_3 \\ N_3 \end{array}
\begin{array}{c|cc|cc}
N_1 & N_2 & N_2 & N_3 & N_3 \\
0 & -1 & 0 & 1 & -1 \\
\hline
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 & 0
\end{array}
$$

$$
\mathrm{Con}(S', f|_{N_3} \circ f|_{N_2} \circ f|_{N_1}) = \left(M^{31} M^{23} M^{12}\right)_s
$$



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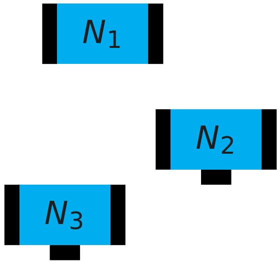

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1 & 0 & 0 & 0 & 0 \\ \hline
0 & 1 & 0 & 0 & 1 \\
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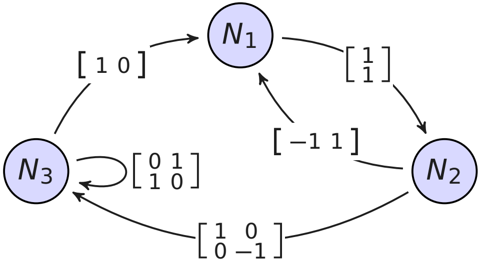
\end{array}
$$

$$
\begin{aligned}
&\mathrm{Con}(S', f|_{N_3} \circ f|_{N_2} \circ f|_{N_1}) \\
&= \left(M^{31} M^{23} M^{12}\right)_s \\
&= \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)_s \\
&= [1]_s \neq [0]_s \\
&\implies \exists x \; \rho(x) = \overline{123}
\end{aligned}
$$



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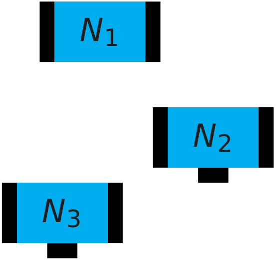

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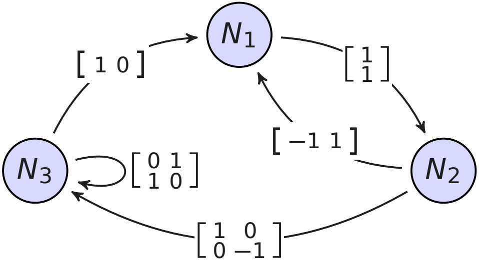

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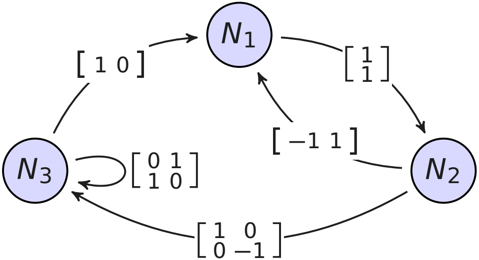
\end{array}\right]
\end{array}
$$

$$\mathrm{Con}(S', f|_{N_2} \circ f|_{N_1})$$
$$= \left(M^{21}M^{12}\right)_s$$
$$= \left(\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)_s = [0]_s$$
$$\not\Longrightarrow \; \exists x \; \rho(x) = \overline{12}$$



Symbolic dynamics

Goal: prove semi-conjugacies to symbolic dynamics

Why: entropy, periodic orbits, etc.

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- Finite alphabet $\mathcal{A}$ *(here regions of phase space)*
- Bi-infinite seq $x = \dots x_{-2}\ x_{-1}\ x_0\ x_1\ x_2 \dots$ with $x_i \in \mathcal{A}$
- Full shift $X_{\text{full}} = \mathcal{A}^{\mathbb{Z}}$
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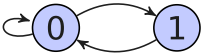
SFT (vertex shift)



$\mathcal{F} = \{11\}$

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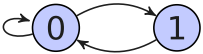

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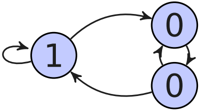

$\mathcal{F} = \{101, 10001, 1000001, \ldots\}$

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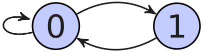

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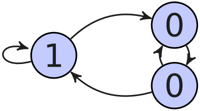

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In both cases, topological entropy $h(X_G) = \log \lambda_{\max}(G)$

Constructing Semi-Conjugacies

Idea: find lots of cycles yielding nontrivial Conley index

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$$X^P := \mathrm{cl}\left(\bigcup \left\{ \overline{x_1 \cdots x_k} : \mathrm{Con}(S', f|_{N_{x_k}} \circ \cdots \circ f|_{N_{x_1}}) \neq [0]_s \right\} \right)$$

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$$= \mathrm{cl}\Big(\bigcup \big\{ \overline{x_1 \cdots x_k} : [M^{x_1 x_k} M^{x_k x_{k-1}} \cdots M^{x_2 x_1}]_s \neq [0]_s \big\} \Big)$$

Aside: cocyclic subshifts [Kwapisz 2000]

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Aside: cocyclic subshifts [Kwapisz 2000]

Proposition: f is semi-conjugate to (any subshift of) X^P

But *computing* X^P seems daunting...

Previous Work: SFTs

[with Sarah Day and Rodrigo Treviño]

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- Remove edges, until you can prove...
- All cycles in the graph have nontrivial index

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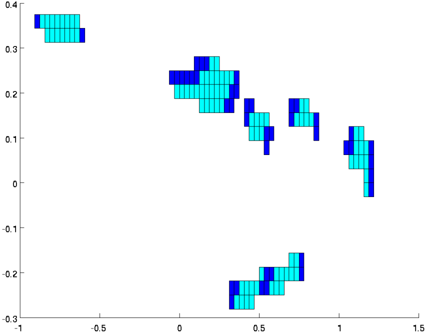
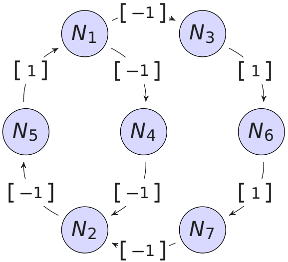
Could be infinitely many cycles!
Technique: look for scalar multiples of matrix products seen before, and proceed by induction

- Tricky argument
- Fairly slow, but gets the job done

Small Hénon Example





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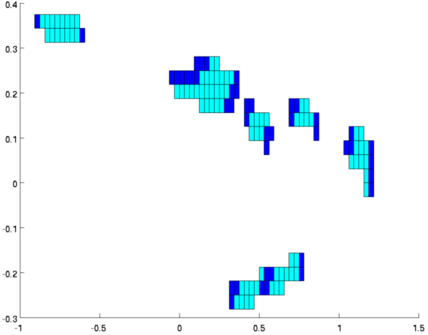
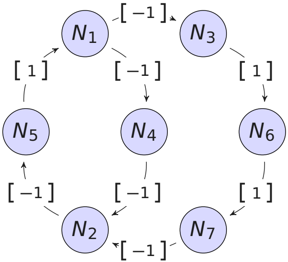
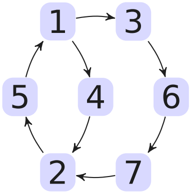
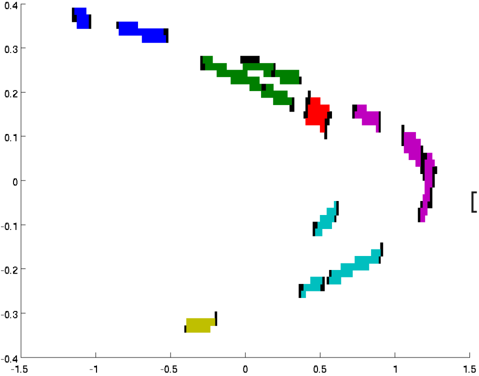
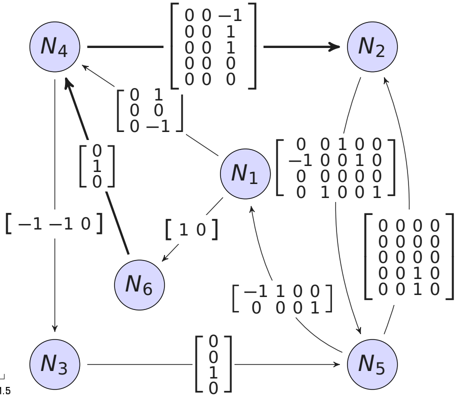






Medium Hénon Example





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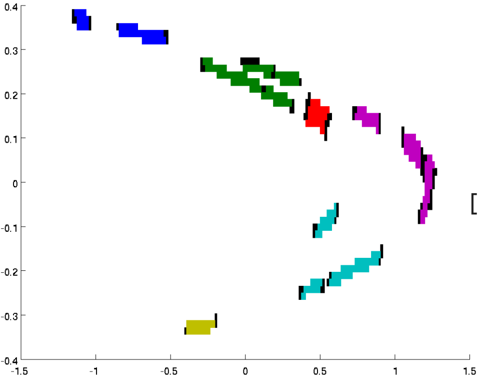
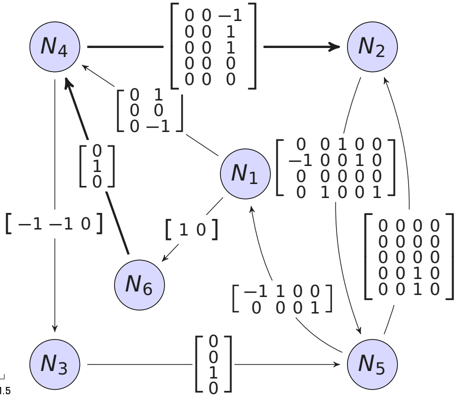
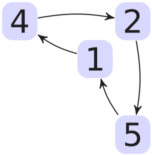
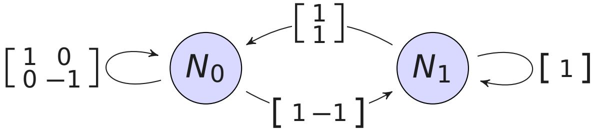






Smaller Nontrivial Example



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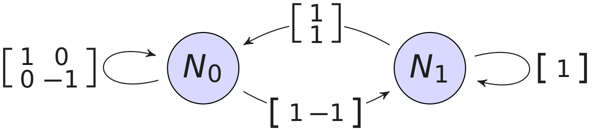

SFT approach: have to cut edge $(0, 1)$ or $(1, 0)$ because $M^{01}M^{10} = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

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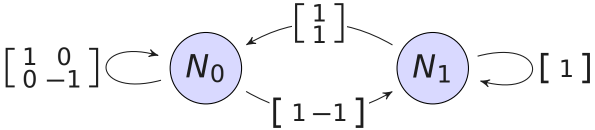

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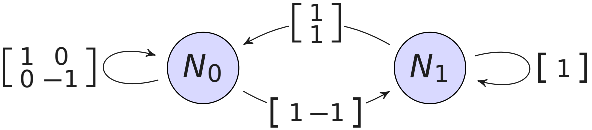
Let's dream big. What is X^P?

Smaller Nontrivial Example



SFT approach: have to cut edge $(0,1)$ or $(1,0)$ because $M^{01}M^{10} = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

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sofic *shift, but not SFT*

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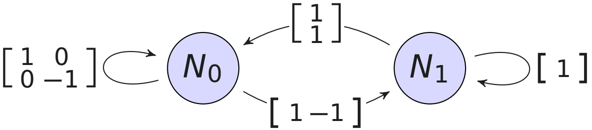

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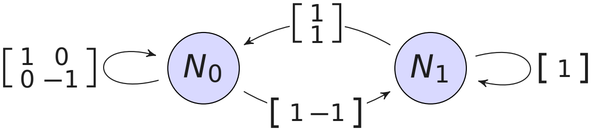

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Algorithm: Sofic Processor

Require: Labeled Conley index $M \in \mathbb{Z}^{n \times n}$, # iter τ
Ensure: Vertex presentation G of a sofic subshift of X^P

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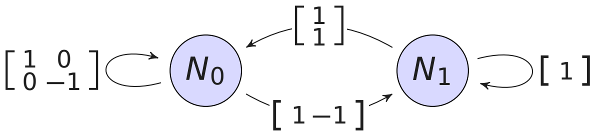
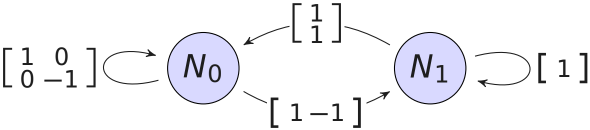
procedure SoficProcessor(M, τ)
    $G \leftarrow$ empty graph with nodes $\{a, I_a\}$ for all $a \in \mathcal{A}$
    mark all nodes of G
    while marked nodes left, or until τ iterations, **do**
        unmark some marked node $\{a, A\}$ of G
        for $b \in \mathcal{A}$ **do**
            $B \leftarrow \mathrm{ech}(M^{ab} A)$
            if $B = 0$, **continue**
            if $\{b, B\} \notin G$ **then**
                add node $\{b, B\}$ to G and mark it
            add edge $(\{a, A\}, \{b, B\})$ to G
    return sink strongly-connected component of G

Example Run



$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
N_0
$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\begin{bmatrix} 1 & -1 \end{bmatrix}$
N_1
$\begin{bmatrix} 1 \end{bmatrix}$

Example Run



$\left\{0, \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\}$ $\{1, \begin{bmatrix}1\end{bmatrix}\}$

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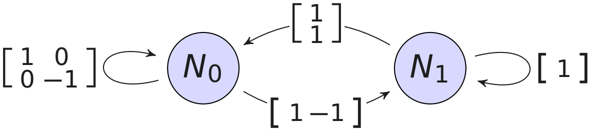
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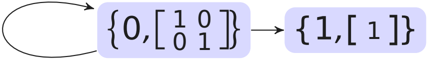
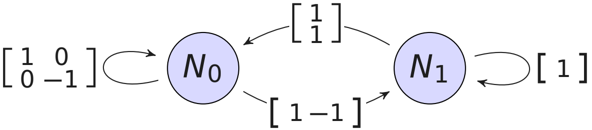

$\left\{0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$
$\{1, [\,1\,]\}$

Example Run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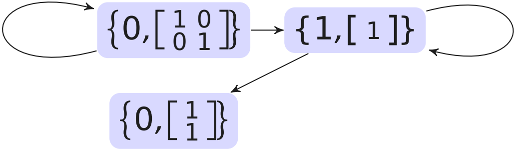
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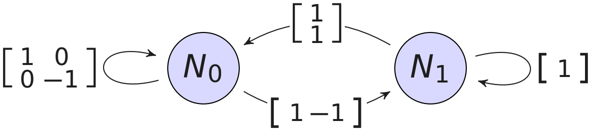

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$\begin{bmatrix}1\\ 1\end{bmatrix}$
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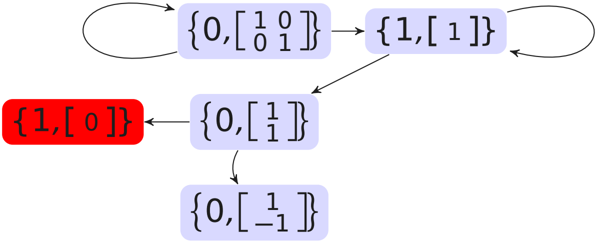

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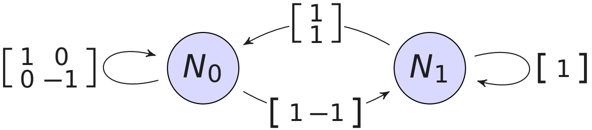
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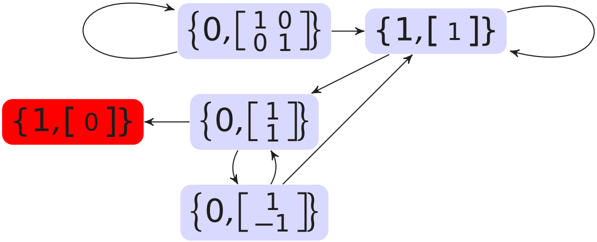
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$\{1, [\,0\,]\}$
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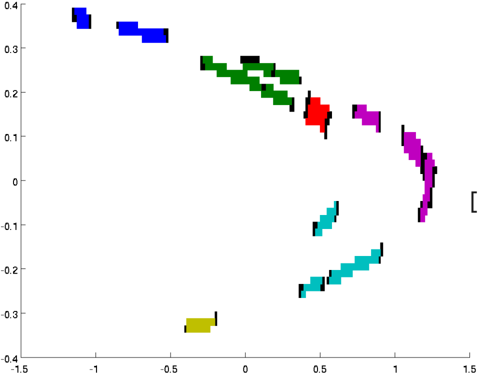
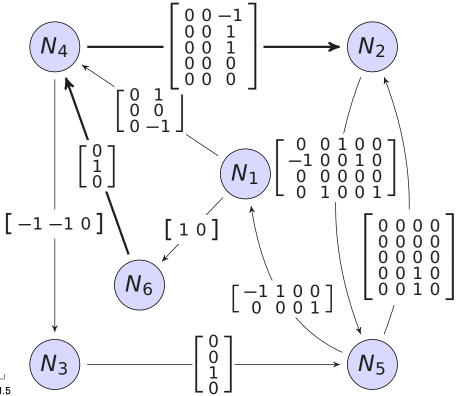
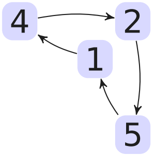
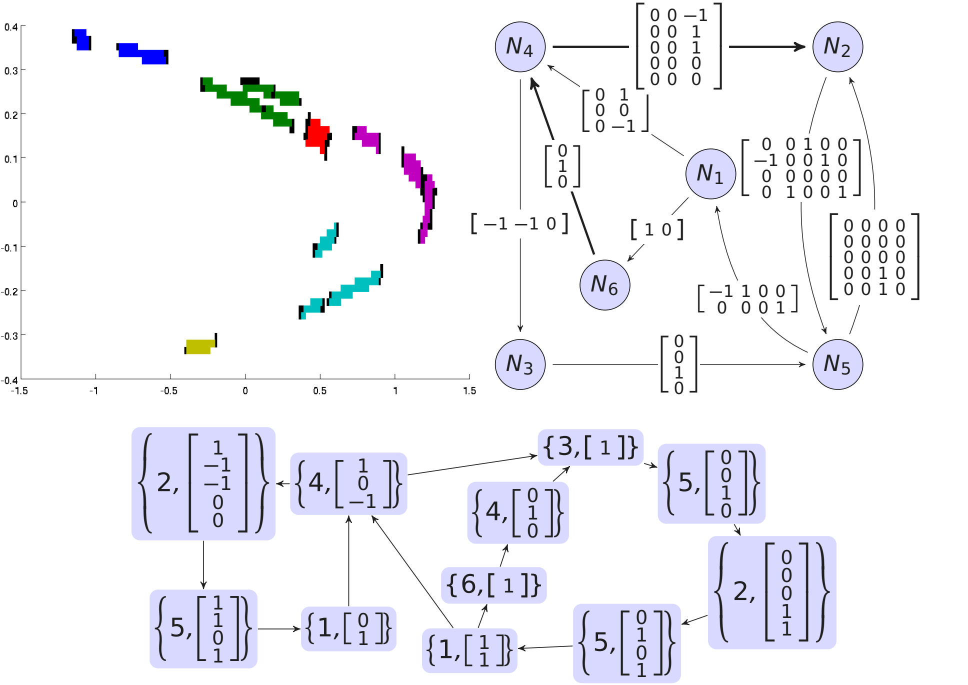
Medium Hénon Example









0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-1.5
-1
-0.5
0
0.5
1
1.5
N_4
N_2
N_1
N_6
N_3
N_5

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Theorem (Day, F.)

If Algorithm 1 (sofic processor) terminates before the cutoff τ, and returns G, then $\overline{X_G} = X^P$, and X^P is sofic.

Optimality

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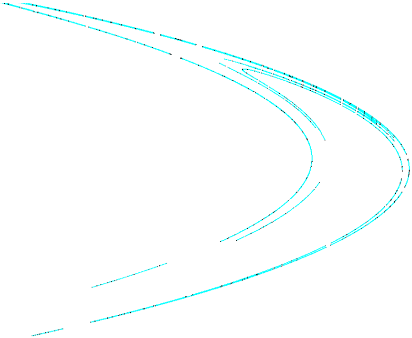
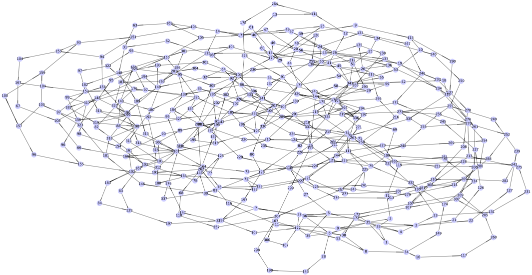
If Algorithm 1 (sofic processor) terminates before the cutoff τ, and returns G, then $\overline{X_G} = X^P$, and X^P is sofic.

Always happens in practice!

100s of examples and counting...

Massive Hénon Example





Index pair: built from 350 low-period orbits
185030 boxes (+ 3193 exit set), 342 regions $\leftarrow \mathcal{A}$

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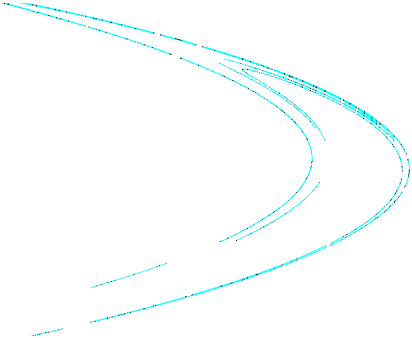

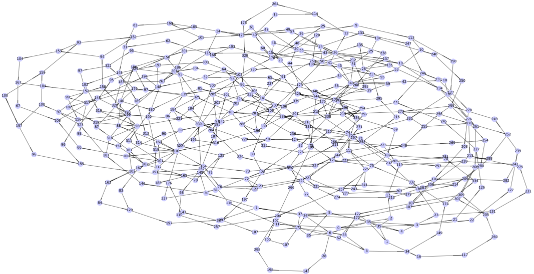

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X^P is sofic! G has 388 vertices and 586 edges.
$h(\text{Hénon}) \geq h(X^P) \geq 0.4555$ *up from* 0.4320

Summary

Automated rigorous tools to capture complex dynamics

Optimal conversion: Conley index → symbolic dynamics

Code available soon...
email `raf@colorado.edu` if interested

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