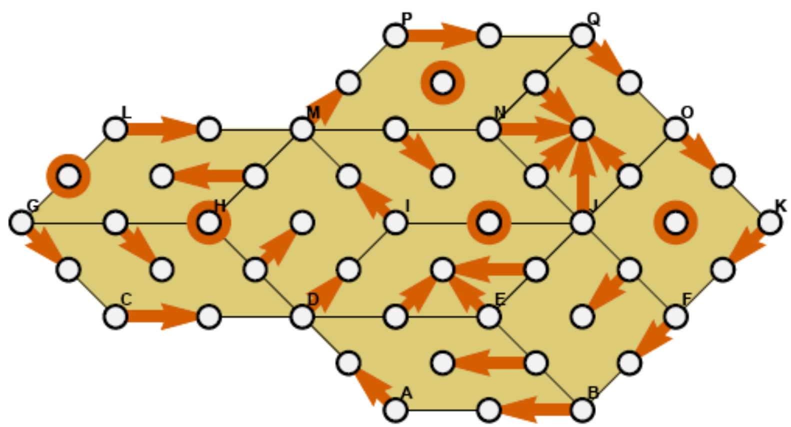
2017 SIAM Conference on Dynamical Systems
Topological Data Analysis of Time Series from Dynamical Systems
Snowbird, 23rd May 2017

The Conley Index for Sampled Dynamical Systems



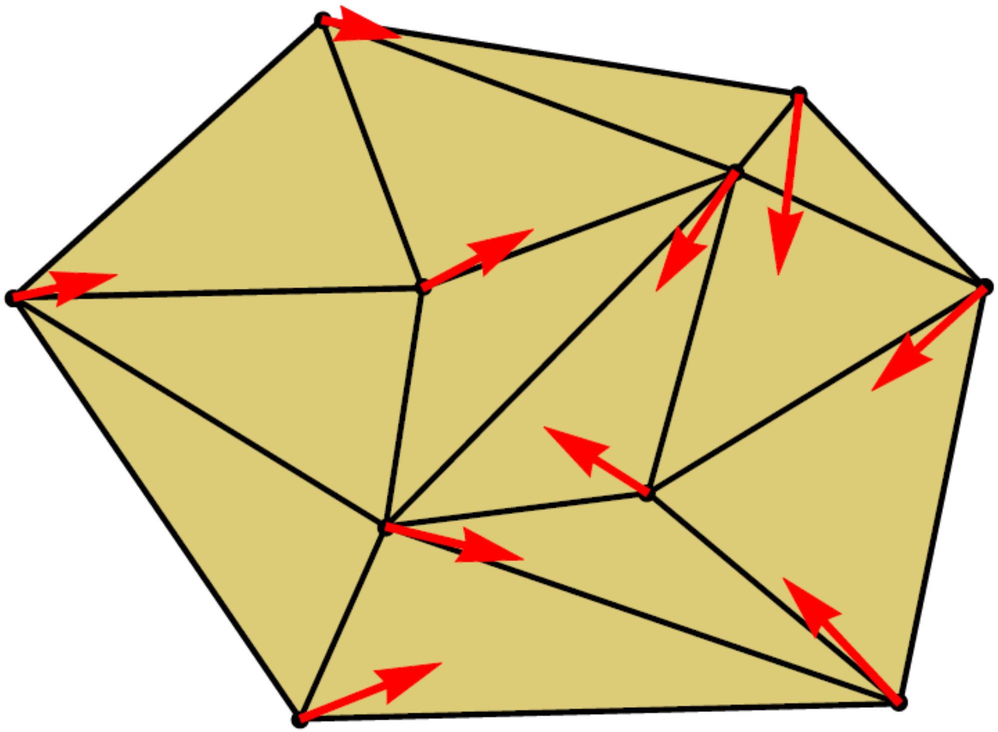
Marian Mrozek

Jagiellonian University, Kraków, Poland

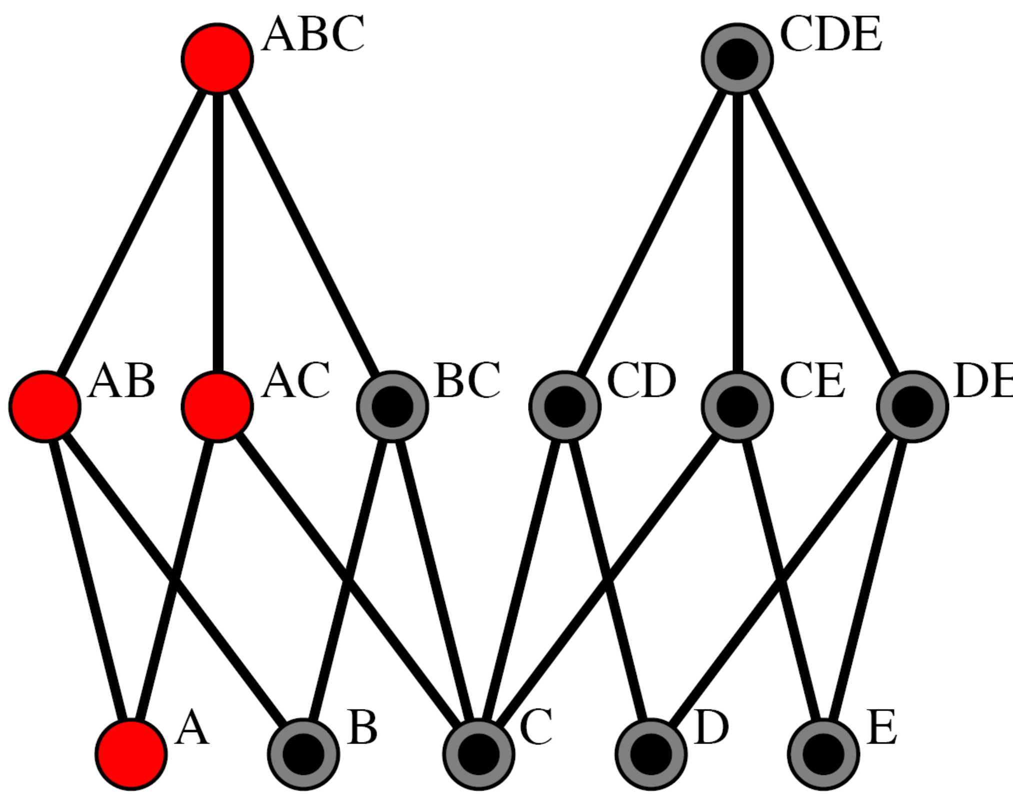
Based on research in collaboration with:

B. Batko, T. Dey, M. Juda, T. Kaczynski, T. Kapela, J. Kubica and Th. Wanner

Sampled Dynamics



Topology of finite sets.



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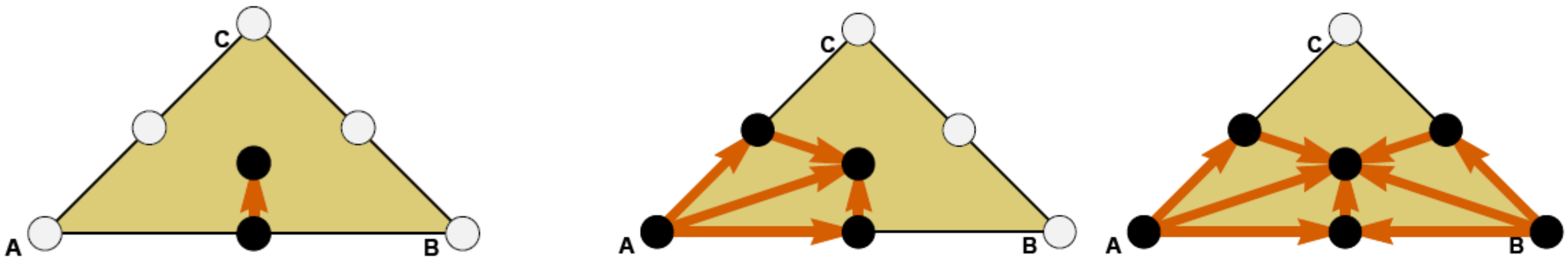
$\mathrm{mo}\, A := \mathrm{cl}\, A \setminus A$ is closed.

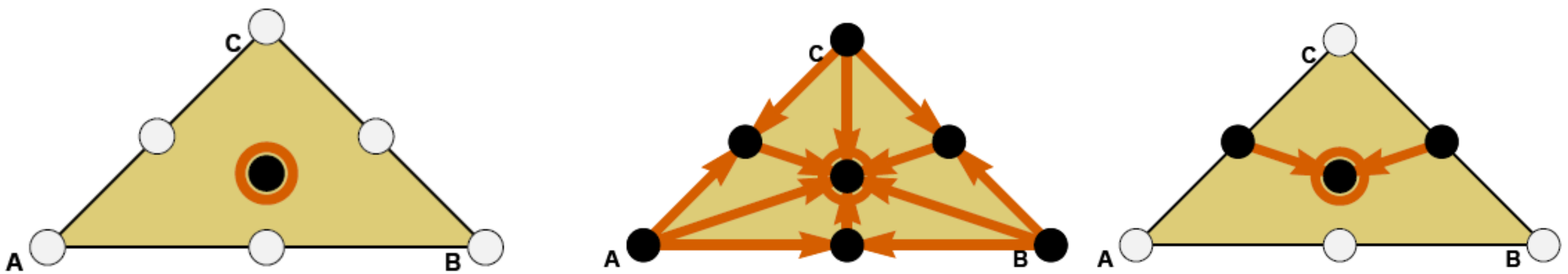
$$H^{\kappa}(A) := H(\mathrm{cl}\, A, \mathrm{mo}\, A)$$

Combinatorial Vectors and Multivectors

A multivector is a convex $V \subset X$.



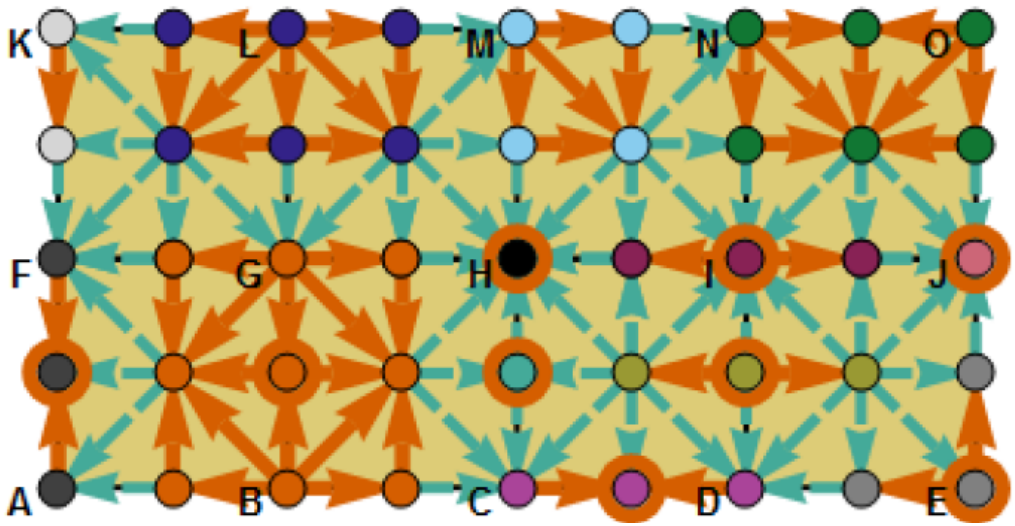
V is regular if $H(\operatorname{cl} V, \operatorname{mo} V) = 0$.



V is critical if $H(\operatorname{cl} V, \operatorname{mo} V) \neq 0$.

Combinatorial multivector fields

A multivector field is a partition $\mathcal{V}$ of X into multivectors.



$\mathcal{V}$-digraph:

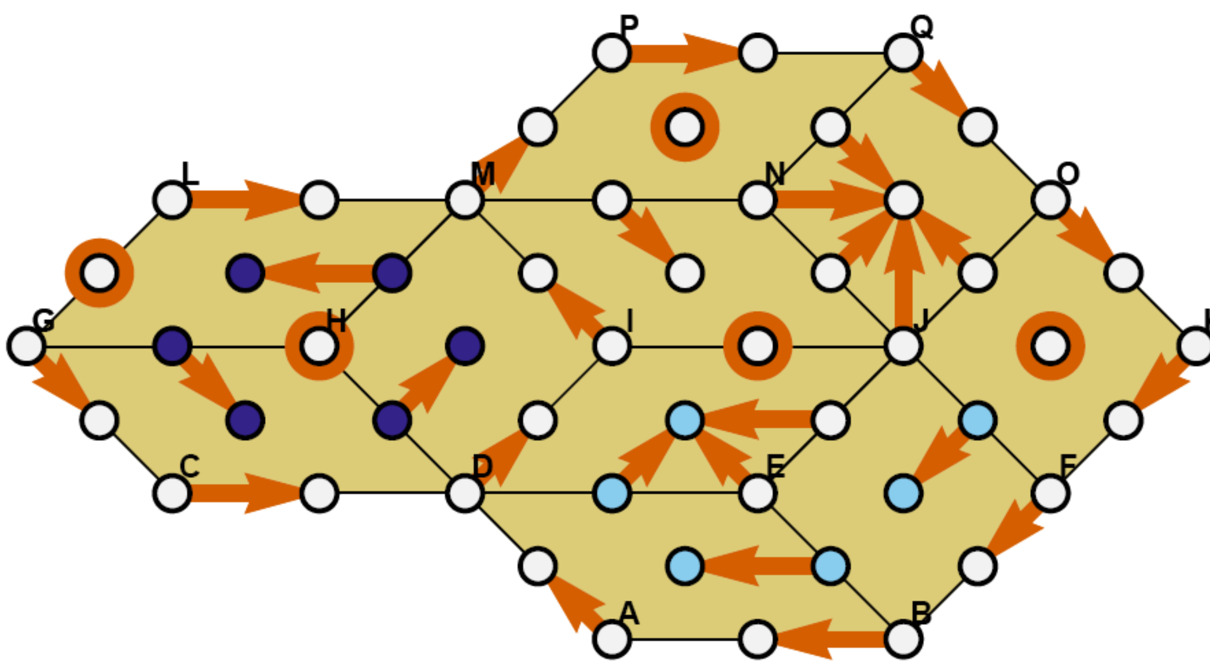
- Vertices: elements of X
- An edge from x to y:
  - explicit: if $\dim x < \dim y$ and $y \in [x]$
  - implicit: if $\dim x > \dim y$ and $y \in \operatorname{mo}[x]$
  - implicit critical: if $\dim x \geqslant \dim y$, $y \in [x]$ and $[x]$ is critical

The multivalued map $\Pi_{\mathcal{V}} : X \rightrightarrows X$ assigns to x all endpoints of edges originating from x.

Solutions and paths

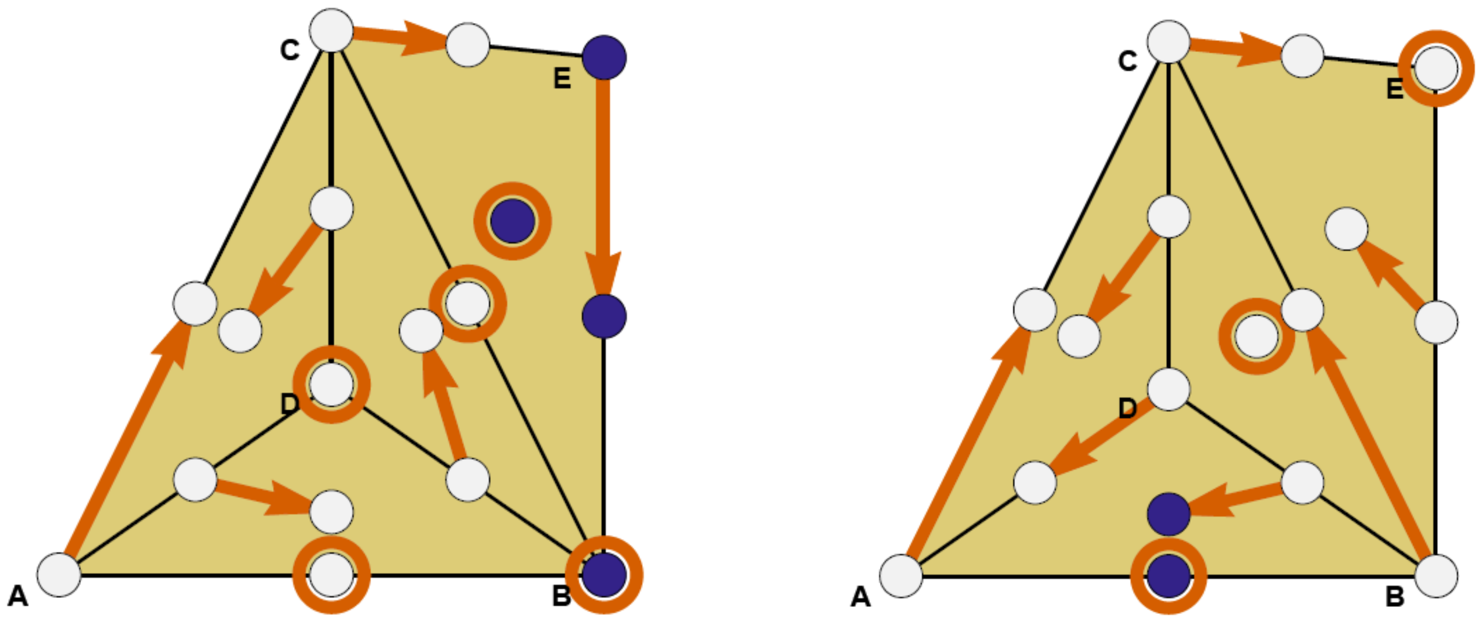
A partial map $\gamma : \mathbb{Z} \nrightarrow X$ is a solution of $\mathcal{V}$ if it is a walk in the $\mathcal{V}$-digraph, that is:

$$\gamma(i+1) \in \Pi_{\mathcal{V}}(\gamma(i)) \text{ for } i, i+1 \in \operatorname{dom} \gamma.$$



Isolating blocks

A solution $\gamma : \mathbb{Z} \to \operatorname{cl} N$ is an internal tangency to N if for some $n_1 < n_2 < n_3$ we have $\gamma(n_1), \gamma(n_3) \in N$ but $\gamma(n_2) \notin N$.



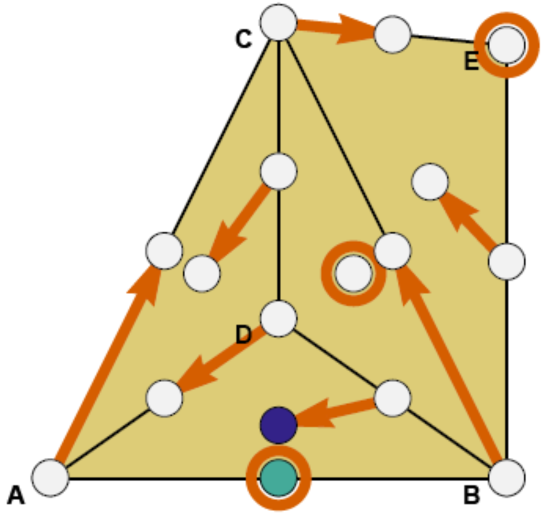
A convex set N is an isolating block if it admits no internal tangencies.

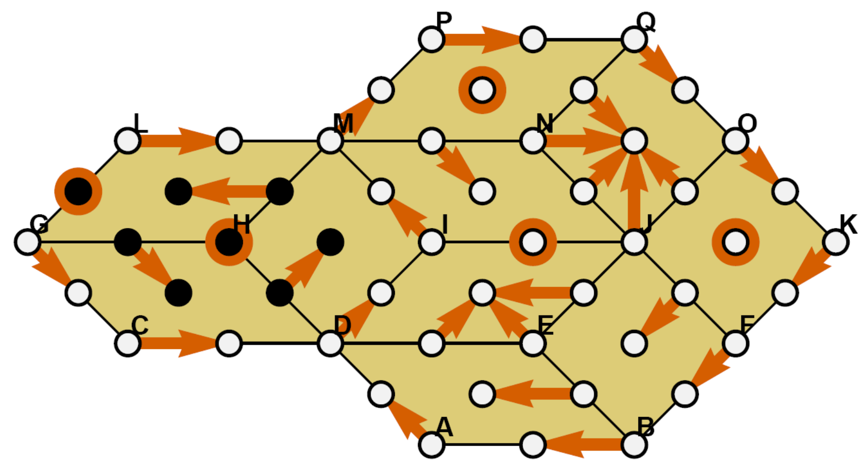
Isolated invariant sets

$$\operatorname{Sol}(x, A) := \{\, \varrho : \mathbb{Z} \to A \text{ a solution s.t. } \varrho(0) = x \,\}.$$

$$\operatorname{Inv} A := \{\, x \in X \mid [x] \subset A \text{ and } \exists y \in [x] :\ \operatorname{Sol}(y, A) \neq \varnothing \,\}$$

A set $S \subset X$ is $\mathcal{V}$-invariant if $\operatorname{Inv} S = S$.



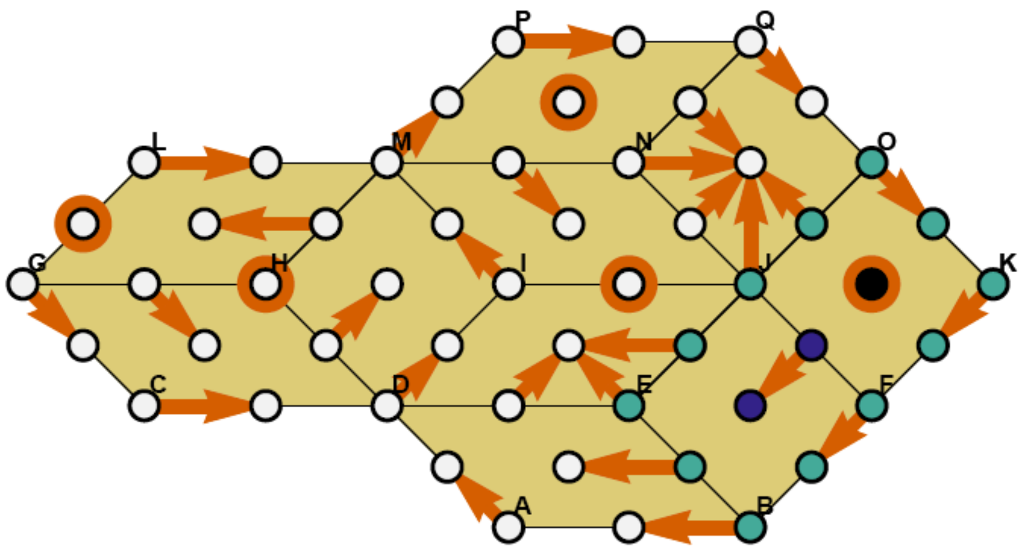


A set $S \subset X$ is an isolated invariant set if $S = \operatorname{Inv} N$ for an isolating block N.

Theorem. Let $S \subset X$ be invariant. Then, S is an isolated invariant set if and only if S is convex.

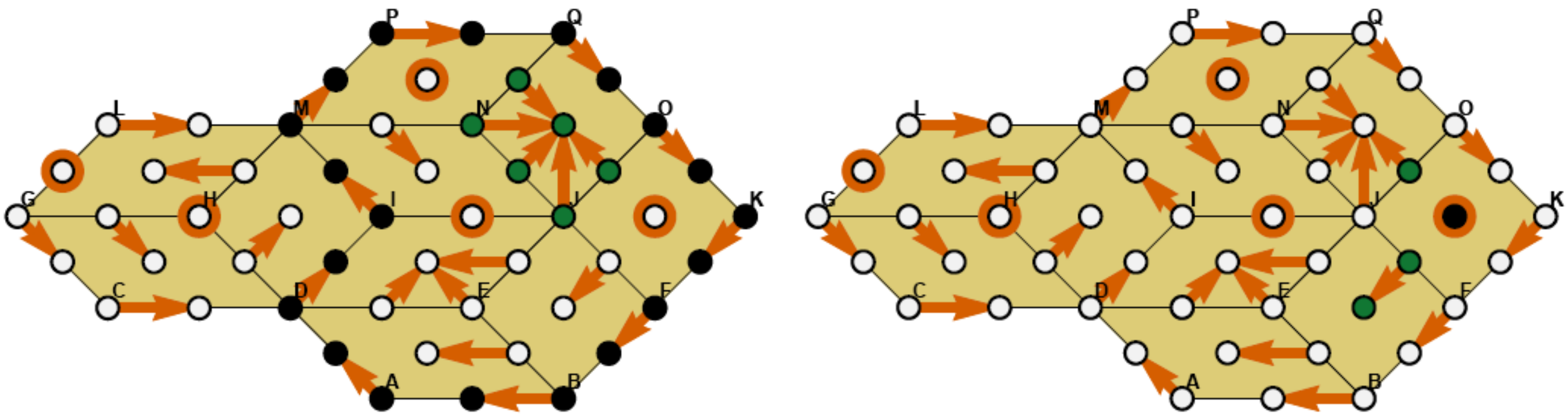
Theorem.

- Every isolated invariant set S is its own isolating block.
- If N and M are isolating blocks for S, then $H^{\kappa}(N)$ and $H^{\kappa}(M)$ are isomorphic.



The Conley index of S is the homology $H^{\kappa}(N)$ for any isolating block N of S.

Attractors and repellers

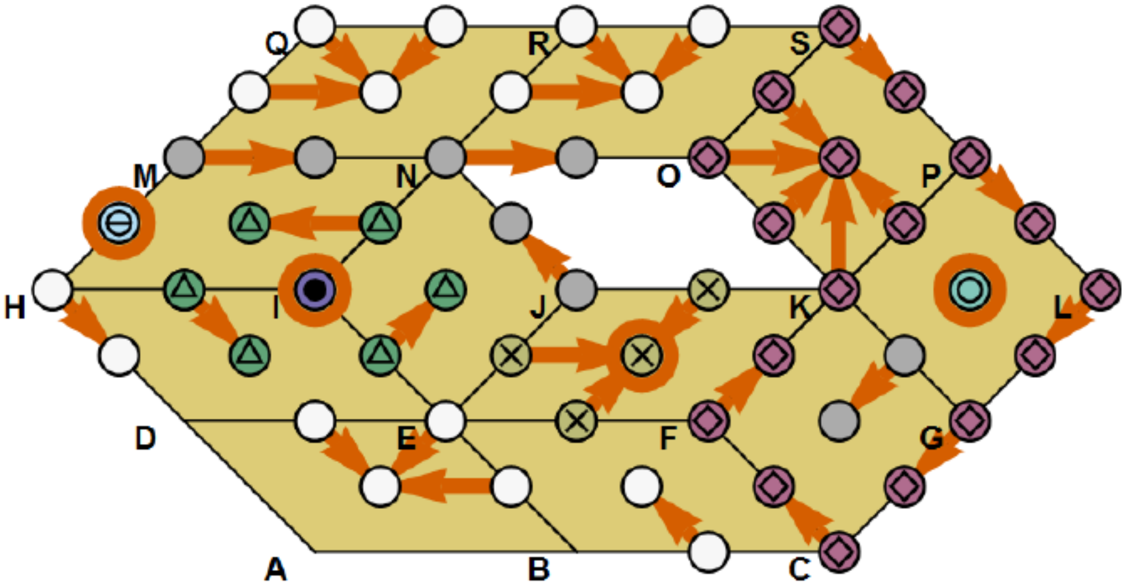


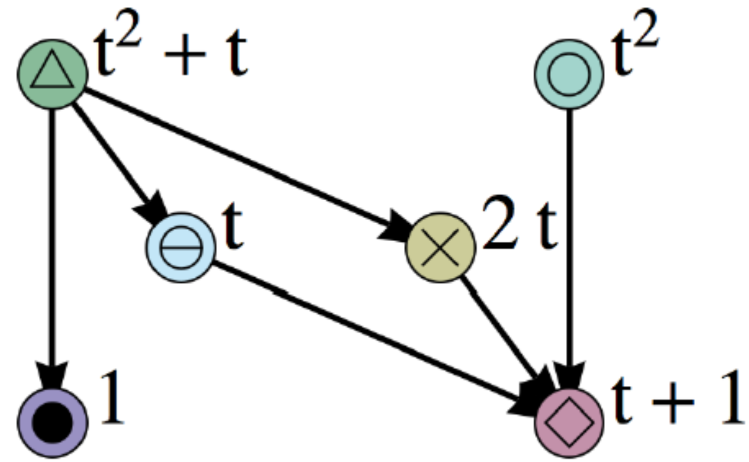
Theorem. The following conditions are equivalent:
(i) A is an attractor,
(ii) A is isolated invariant and closed in S.

Theorem. The following conditions are equivalent:
(i) R is a repeller,
(ii) R is isolated invariant and open in S.

Morse decompositions

Morse-Conley graph and Morse inequalities





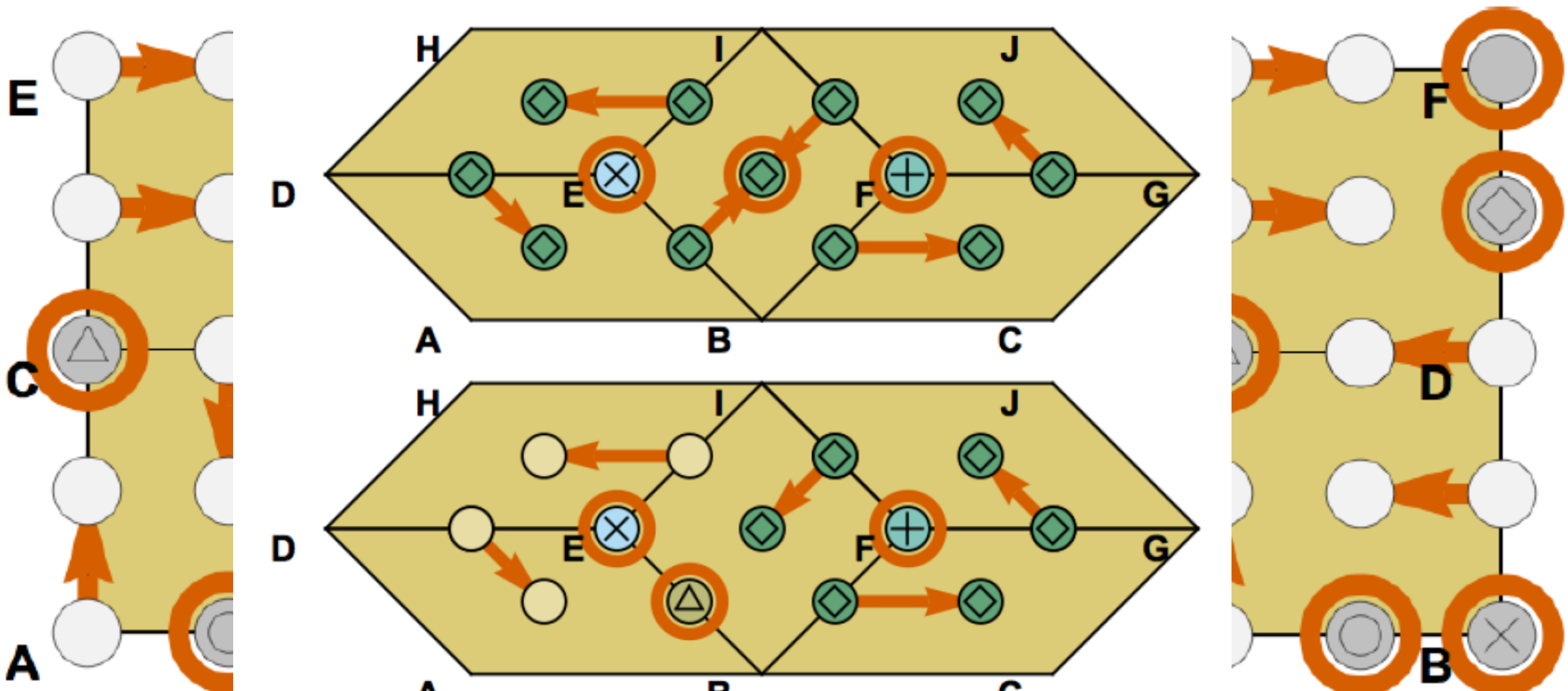
Theorem. Given a Morse decomposition $M = \{ M_\iota \mid \iota \in P \}$ of an isolated invariant set S we have

$$\sum_{\iota \in P} p_{M_\iota}(t) = p_S(t) + (1+t)q(t)$$

for some non-negative polynomial q. In particular,

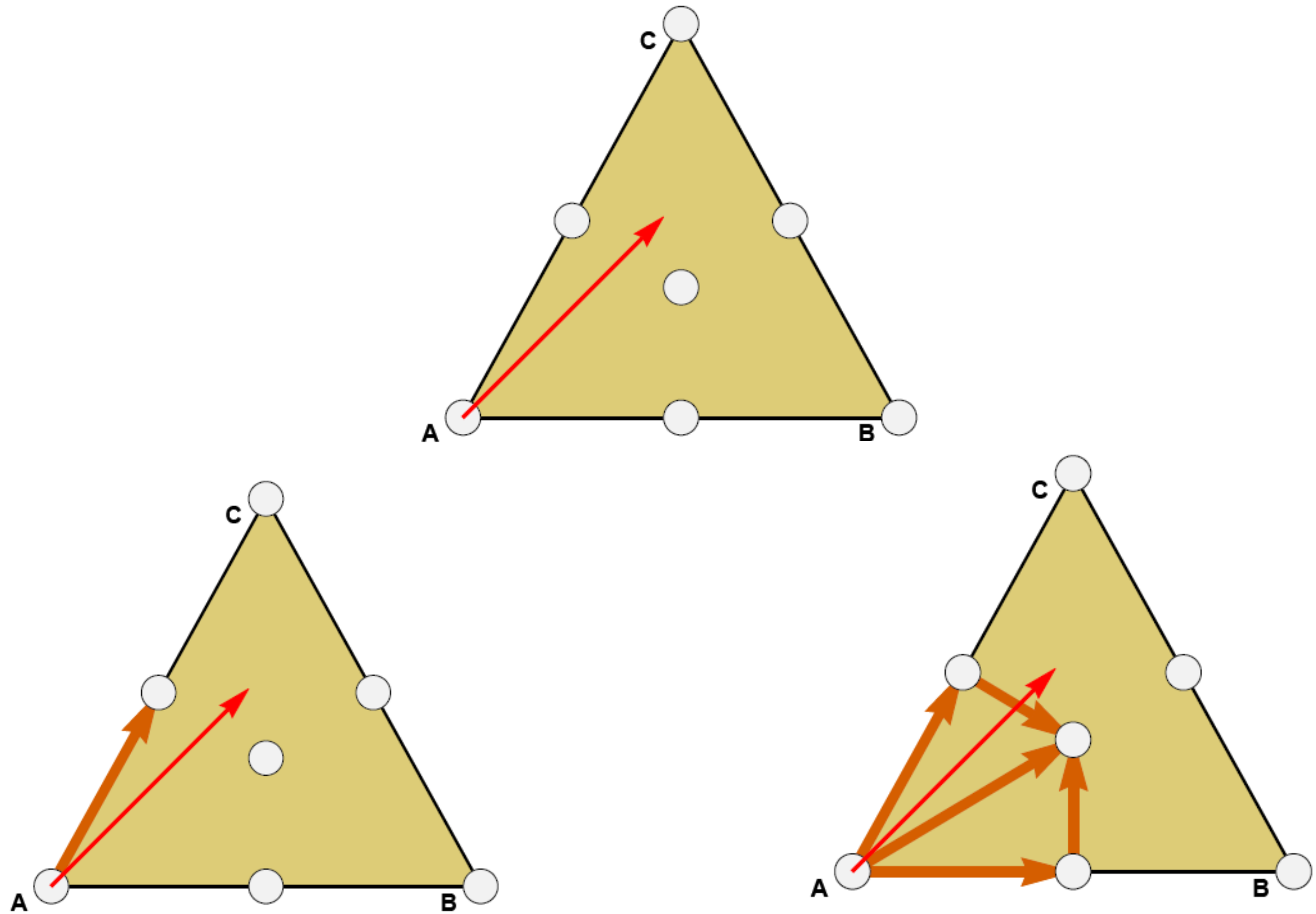
$$\sum_{r \in P} \operatorname{rank} H_k^\kappa(M_r) \geqslant \operatorname{rank} H_k^\kappa(X).$$

Refinements.

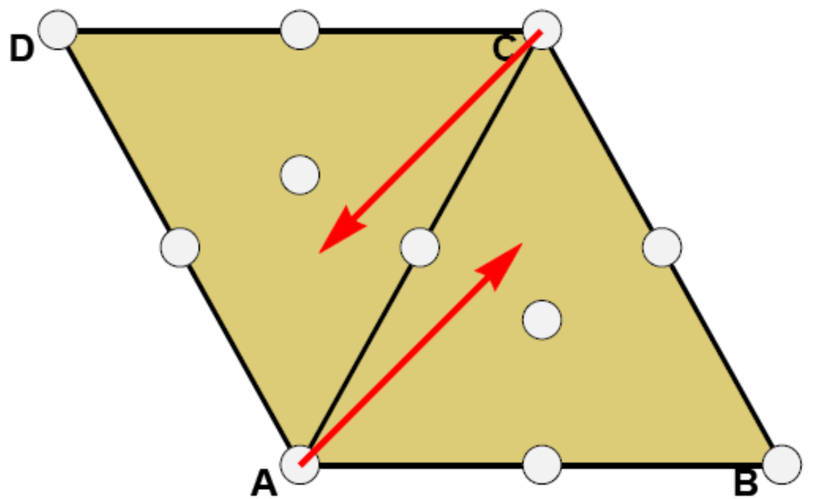


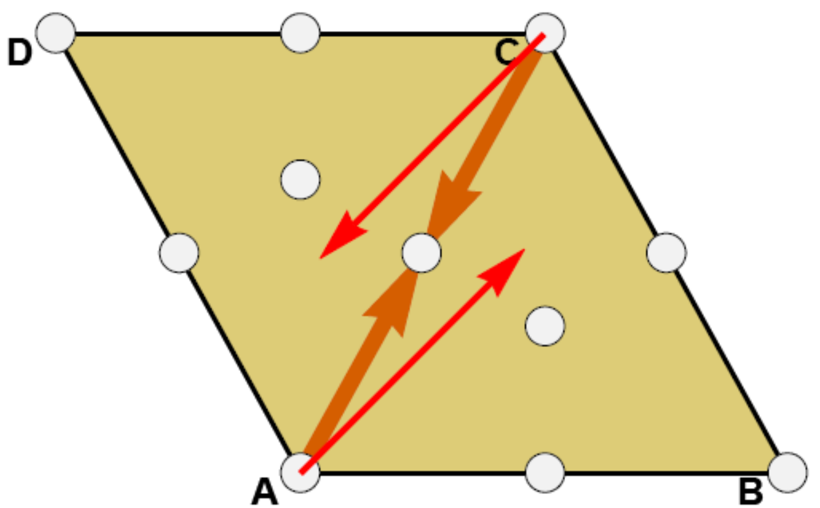
$$\mathcal{V} \sqsubset \mathcal{W} :\Leftrightarrow \forall V \in \mathcal{V} \exists W \in \mathcal{W} : V \subset W$$

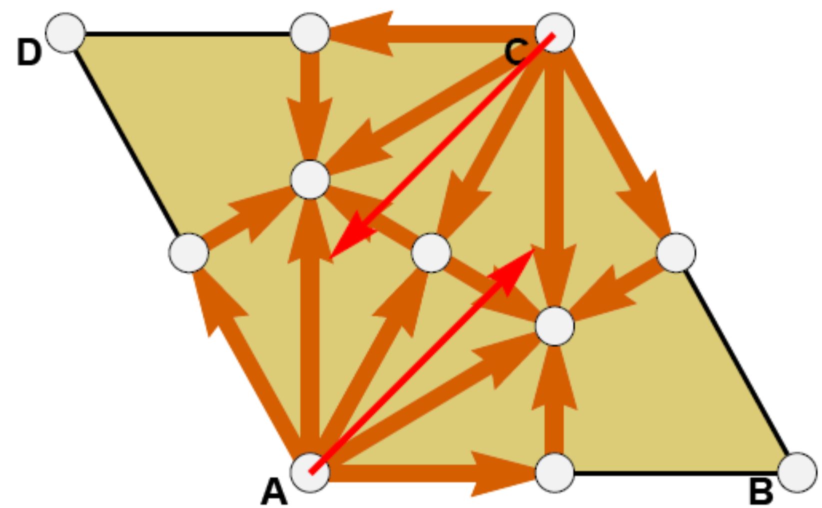
Multivector field construction.



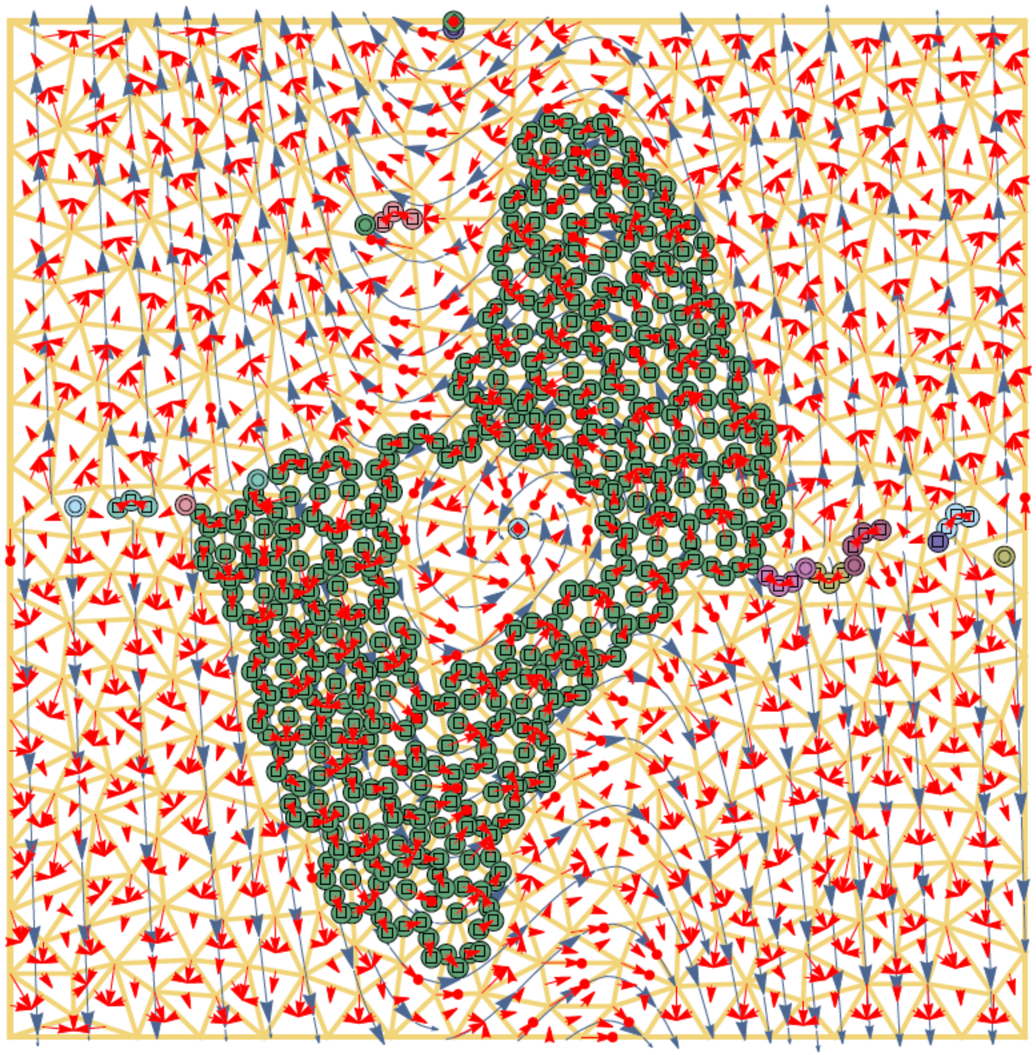
Conflict resolution.







Reverse van der Pol equations 20



Let $\mathcal{R}$ denote the family of strongly connected components of $G_\mathcal{V}$. The Morse

Consider a sequence $(\mathcal{V}_i)_{i=0}^n$ of multivector fields on X. We have the sequence

$$(X, \mathcal{T}_{\mathcal{V}_1}) \longleftarrow (X, \mathcal{T}_{\mathcal{V}_1 \bar{\cap} \mathcal{V}_2}) \longrightarrow (X, \mathcal{T}_{\mathcal{V}_2}) \longleftarrow \ldots$$

$$\ldots \longleftarrow (X, \mathcal{T}_{\mathcal{V}_{n-1} \bar{\cap} \mathcal{V}_n}) \longrightarrow (X, \mathcal{T}_{\mathcal{V}_n}).$$

Proposition. Each Morse set M is a connected component of

The persistence of $(\mathcal{V}_i)_{i=0}^n$ is the zig-zag persistence of

$$H(X, \mathcal{T}_{\mathcal{V}_1}) \longleftarrow H(X, \mathcal{T}_{\mathcal{V}_1 \bar{\cap} \mathcal{V}_2}) \longrightarrow H(X, \mathcal{T}_{\mathcal{V}_2}) \longleftarrow \ldots$$

$$\ldots \longleftarrow H(X, \mathcal{T}_{\mathcal{V}_{n-1} \bar{\cap} \mathcal{V}_n}) \longrightarrow H(X, \mathcal{T}_{\mathcal{V}_n}).$$

is co

$$\mathcal{K}(X, \mathcal{T}) := \{ \{x_0, x_1, \ldots x_k\} \mid x_i \leqslant x_{i+1} \}$$

Theorem. (M. C. McCord, 1966) For every finite T_0 topological space the map

$$\mu_{(X,\mathcal{T})} : |\mathcal{K}(X, \mathcal{T})| \ni x \mapsto \min |x| \in X,$$

is continuous and a weak homotopy equivalence.

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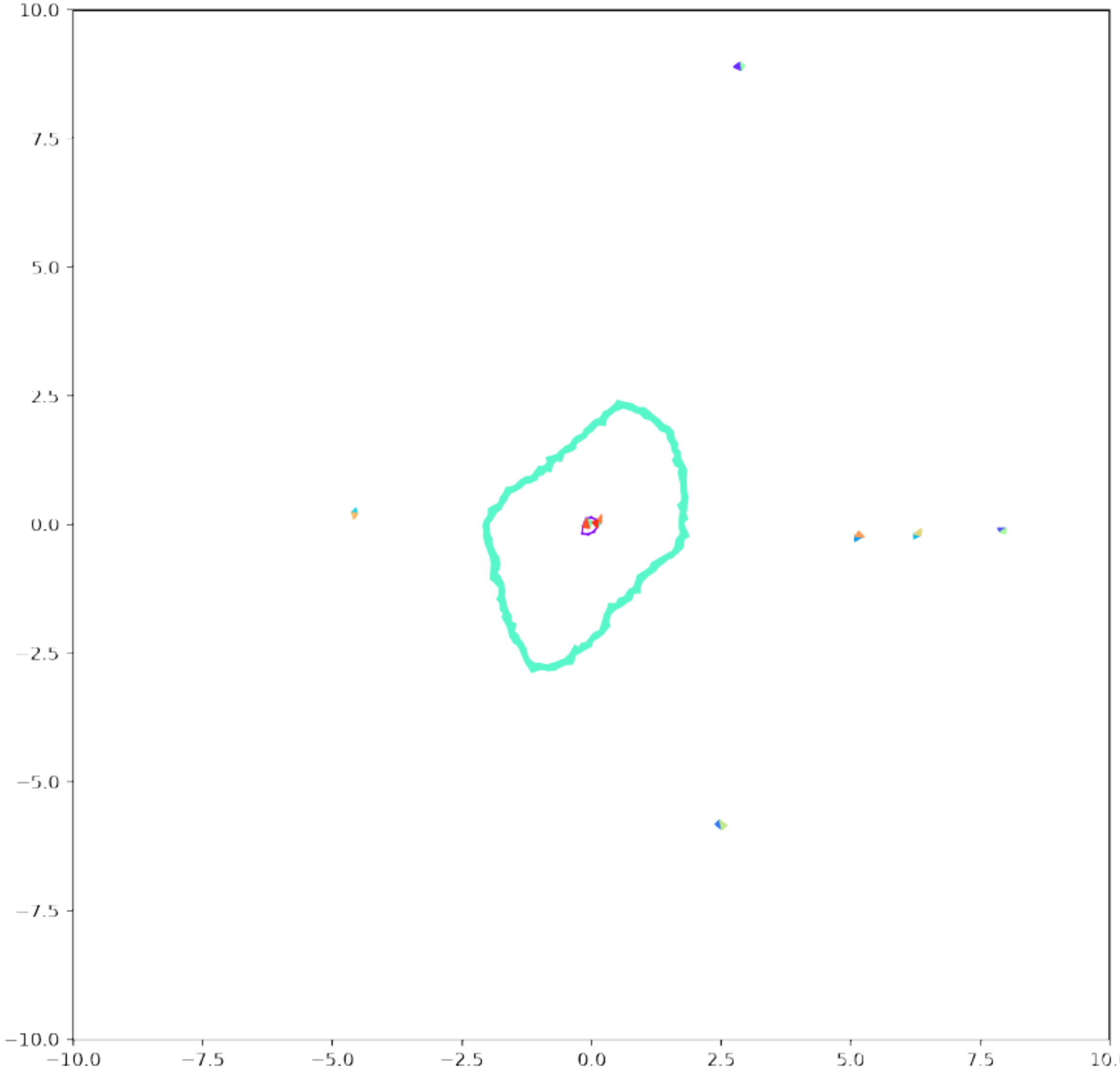
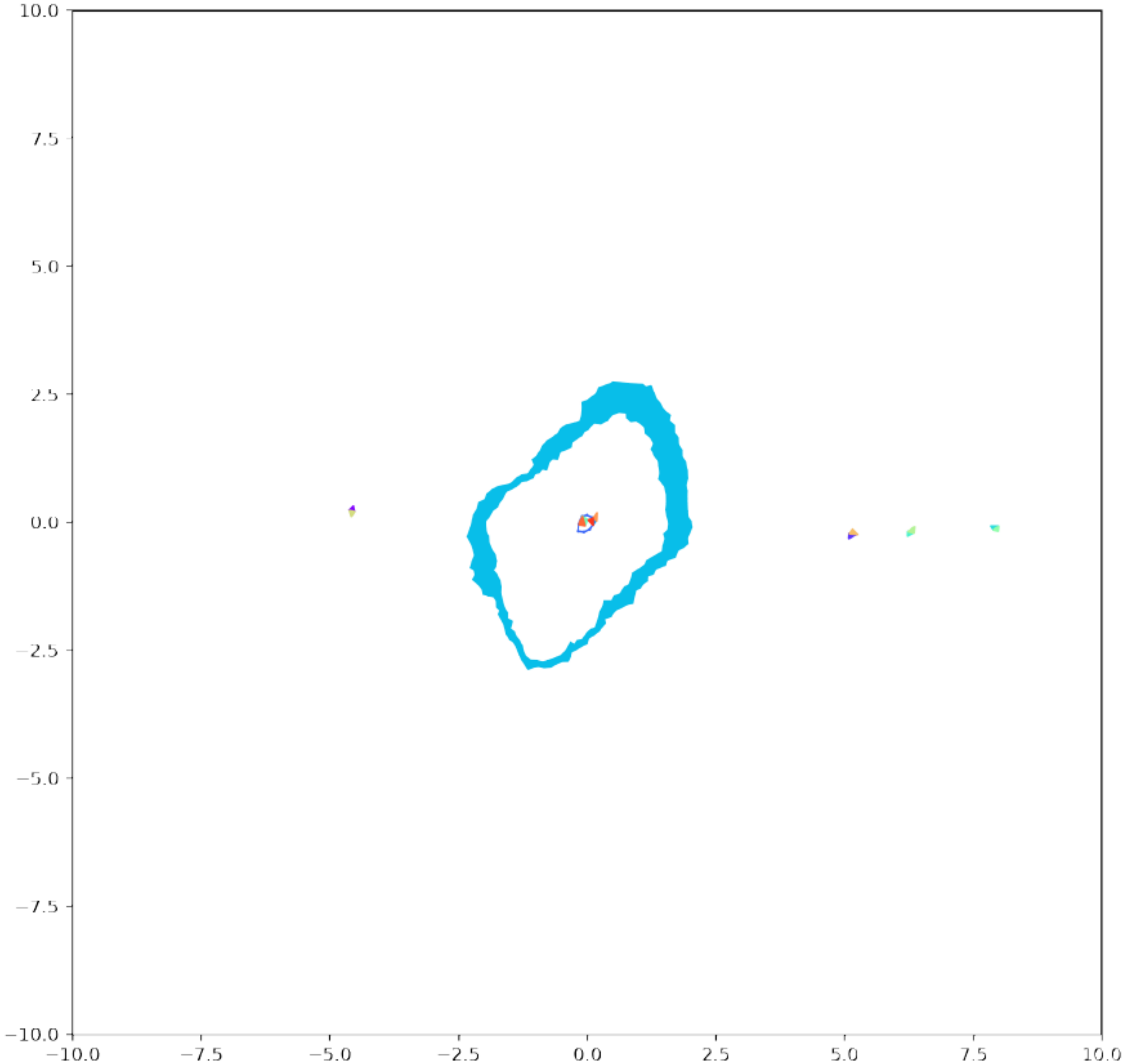
From large to small flattening



From large to small flattening



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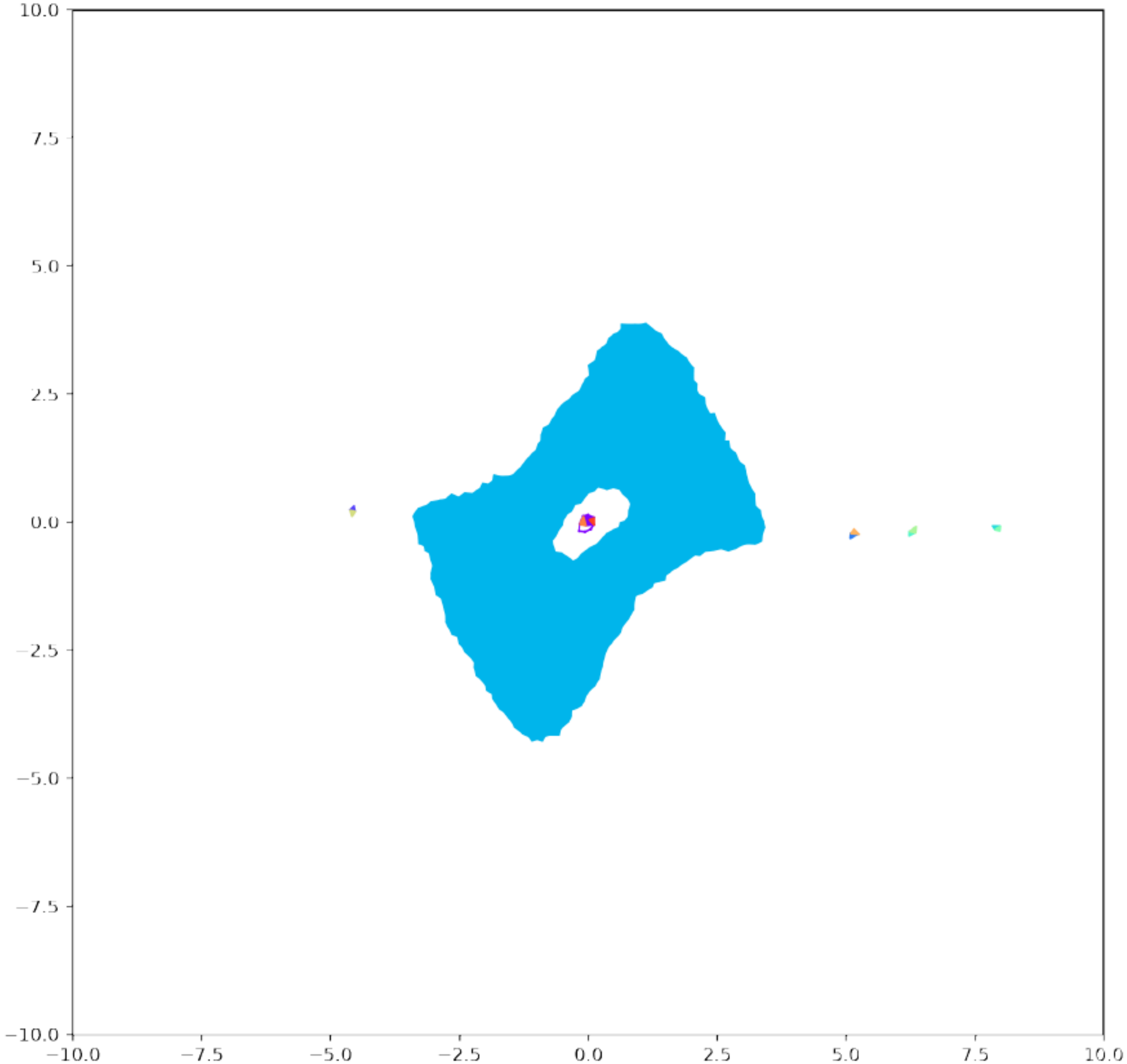

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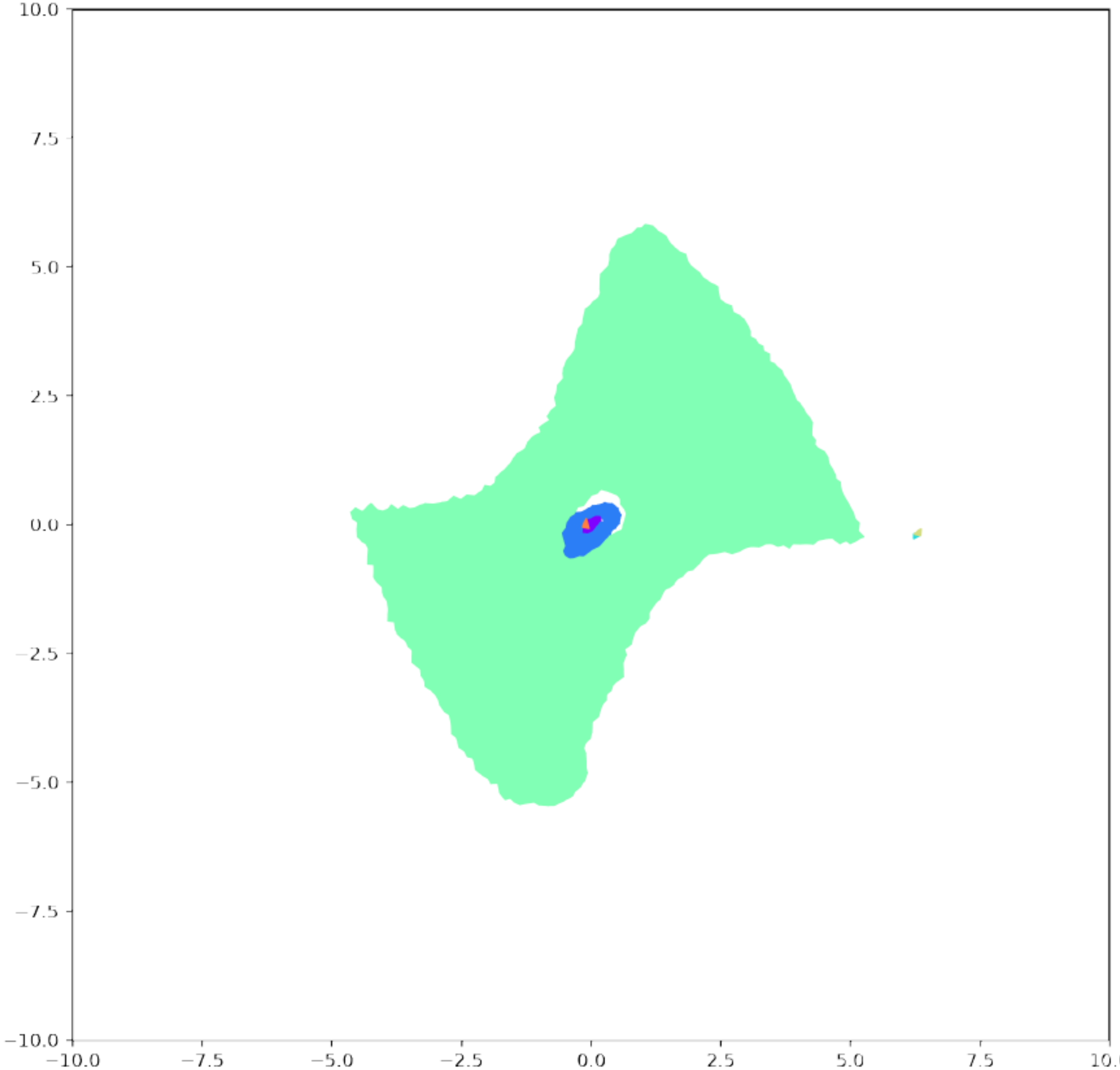
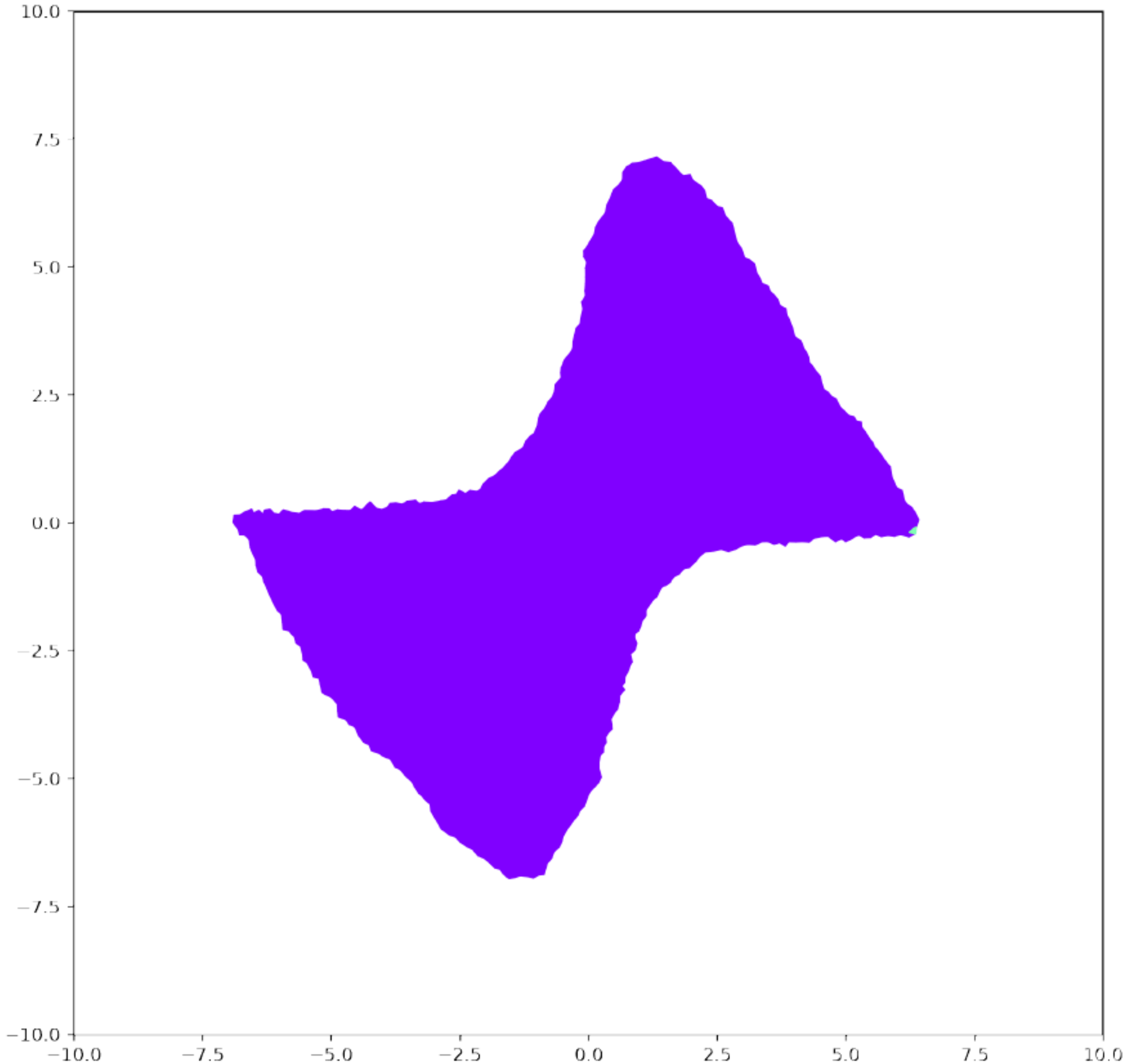
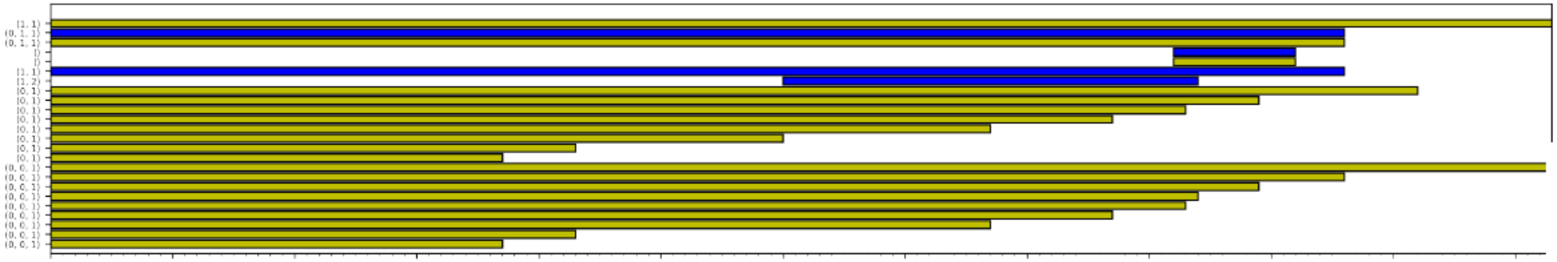


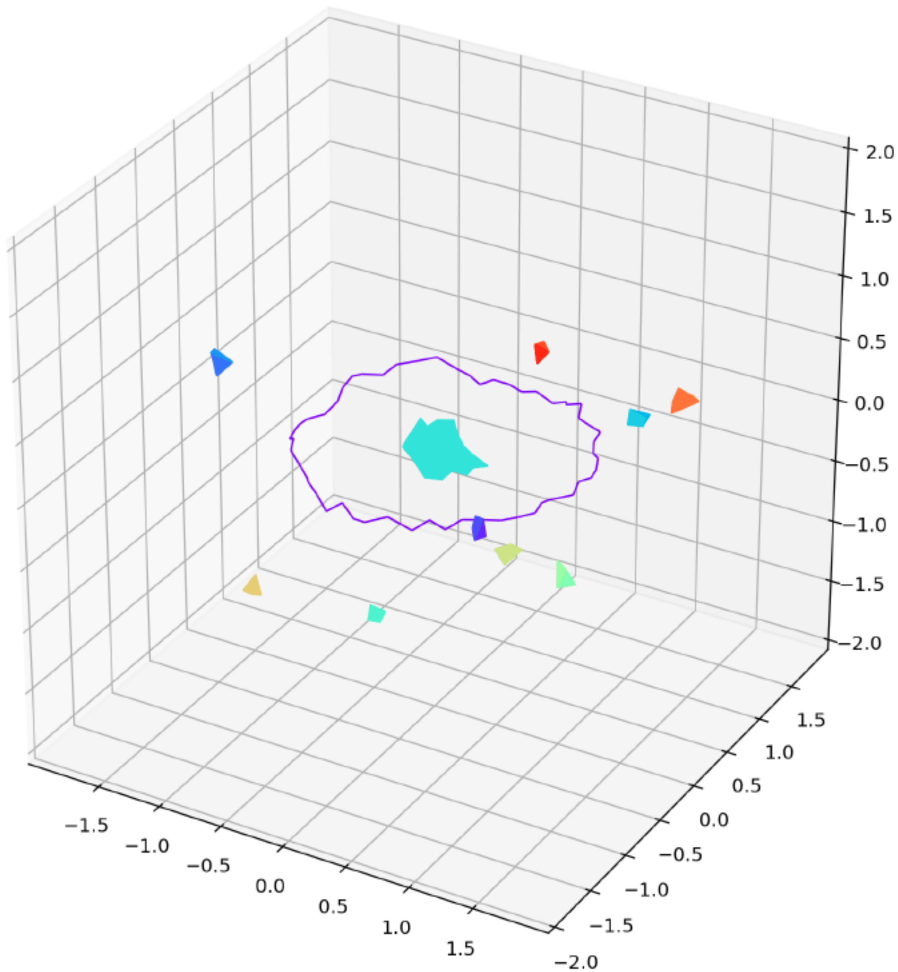
From large to small flattening 24



From large to small flattening 24



Attracting periodic orbit in 3D



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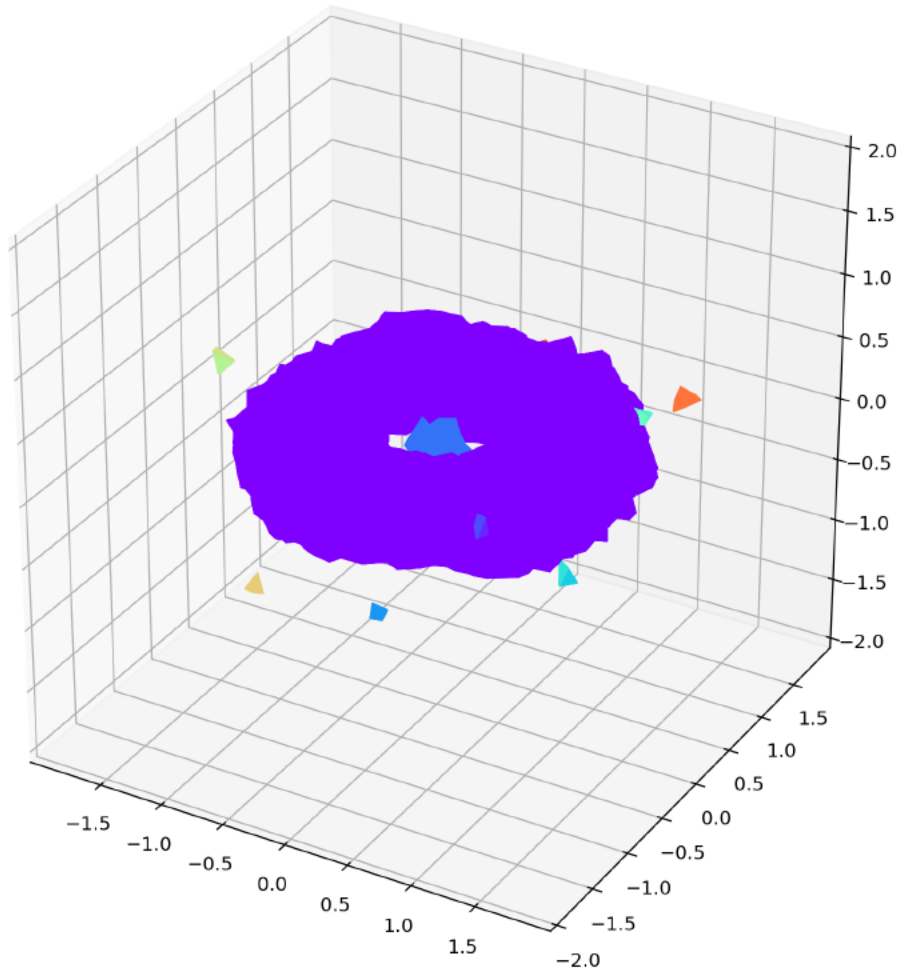
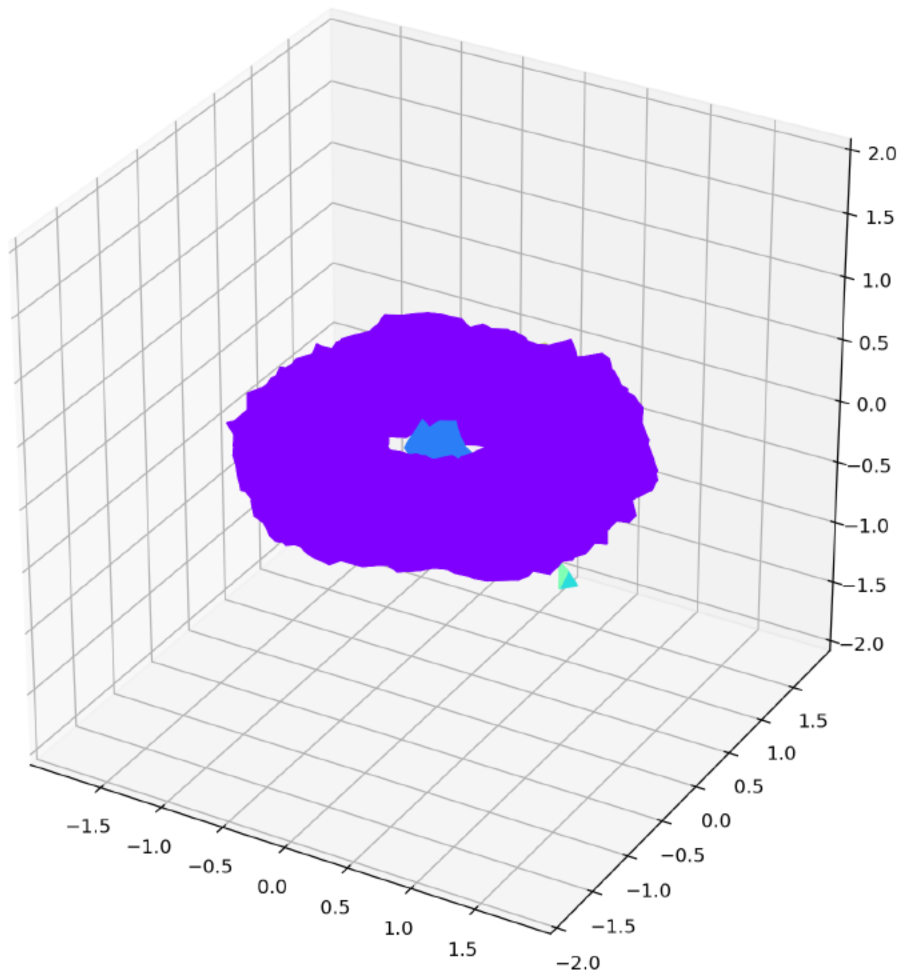


Attracting periodic orbit in 3D



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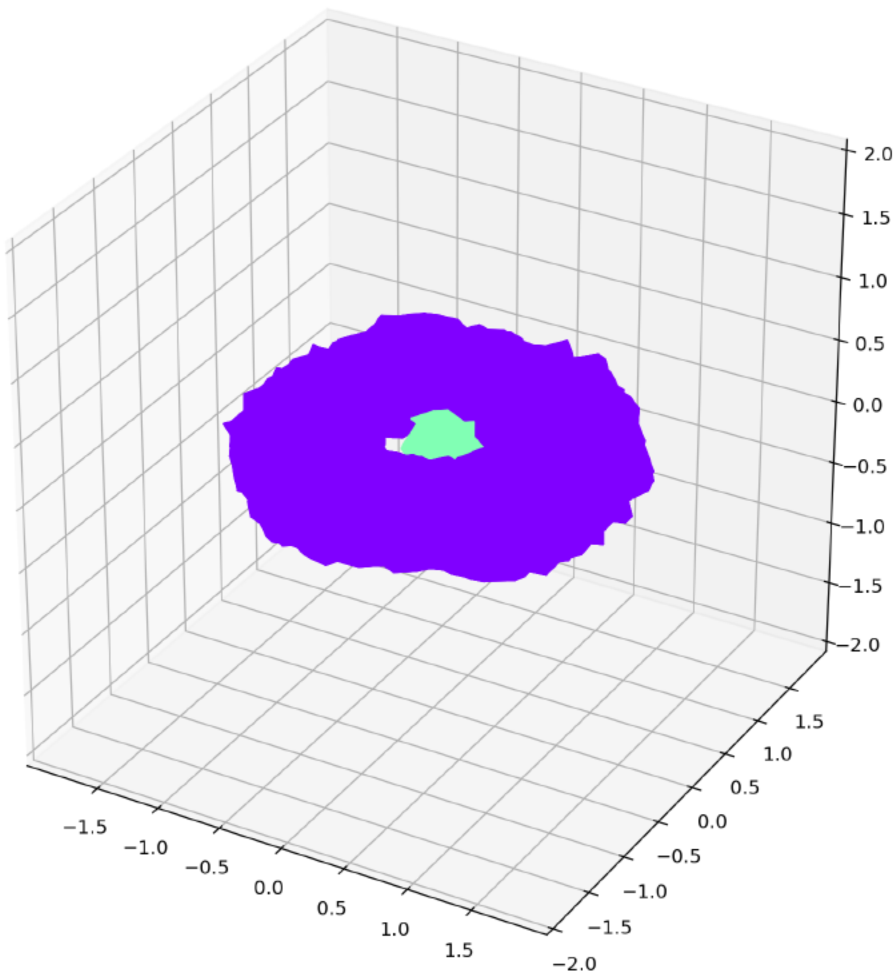
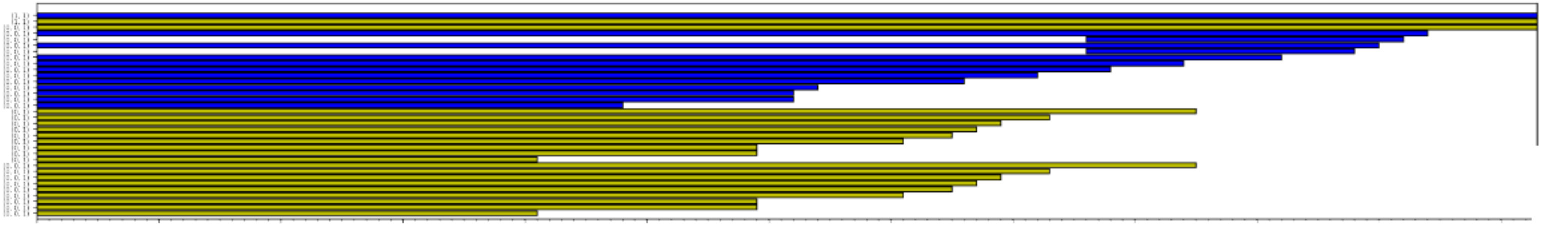
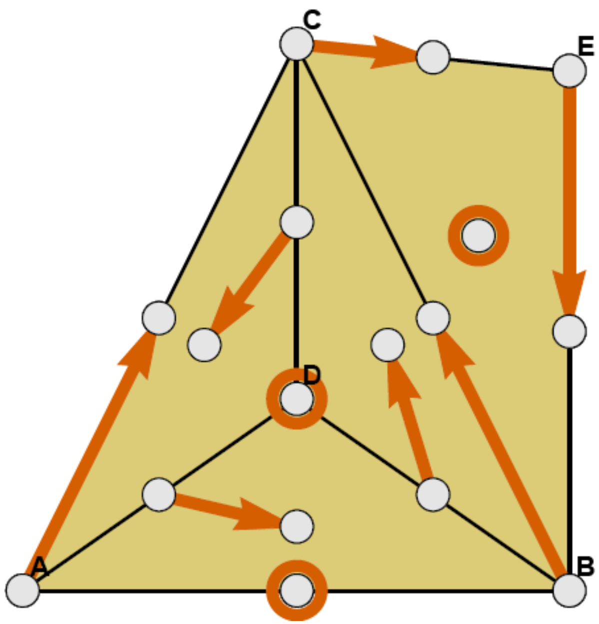
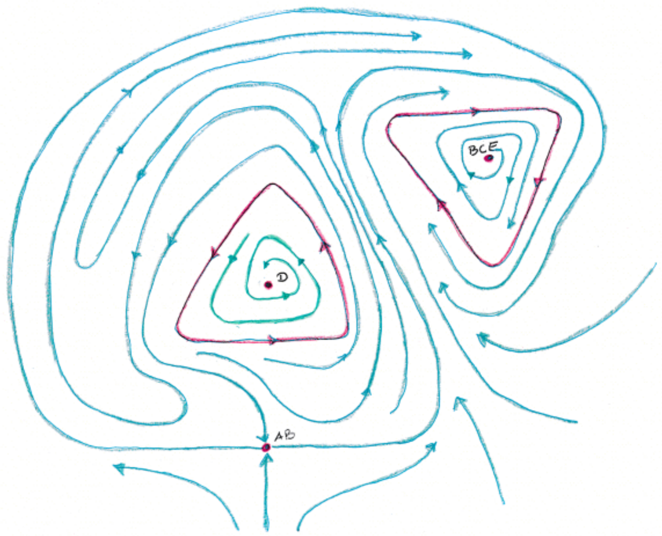


Attracting periodic orbit in 3D



Relation to classical theory





Theorem. (B. Batko, T. Kaczynski, MM, Th. Wanner) There exists an usc, acyclic valued, homotopic to identity, multivalued map $F : X \rightrightarrows X$ and a Morse decomposition $M = \{ M_p \mid p \in P \}$ of the induced multivalued dynamical system such that for any convex I in P the Conley indexes of $\mathcal{M}(I)$ and $M(I)$ coincide.

References

- T. Kaczynski, M. Mrozek, and Th. Wanner, Towards a Formal Tie Between Combinatorial and Classical Vector Field Dynamics, *Journal of Computational Dynamics* (**2016**).
- M. Mrozek, Conley-Morse-Forman theory for combinatorial multivector fields on Lefschetz complexes, *Foundations of Computational Mathematics* (**2017**).
- B. Batko, T. Kaczynski, M. Mrozek, and Th. Wanner, Towards a Formal Tie Between Combinatorial and Classical Vector Field Dynamics. Part II, in preparation.
- T. Dey, M. Juda, T. Kapela, J. Kubica, M. Mrozek, Persistent Homology of Morse Decompositions in Combinatorial Dynamics, in preparation.

Thank you for your attention!