Numerical Analysis of Coupled Free Flow with a Poroelastic Material

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Modeling and Numerical Methods for Complex Subsurface Flow

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Problem arises in modeling the interaction between a free fluid and a poroelastic material

Free flow

Stokes model

$$\rho_f \dot{\mathbf{u}} - \nabla \cdot \boldsymbol{\sigma}_f = \mathbf{f}_f \text{ in } \Omega_f \times (0, T] \quad (1)$$
$$\nabla \cdot \mathbf{u}_f = 0 \text{ in } \Omega_f \times (0, T] \quad (2)$$

- $\mathbf{u}_f, p_f$ – fluid velocity and pressure, resp.
- ν_f – fluid viscosity.
- $\mathbf{f}_f$ – body forces acting on fluid.
- $\boldsymbol{\sigma}_f = 2\nu_f D(\mathbf{u}_f) - p_f \mathbf{I}$
- $\mathbf{D}(\mathbf{u}_f) = \frac{1}{2}\left(\nabla \mathbf{u}_f + (\nabla \mathbf{u}_f)^T\right)$

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Poroelastic material

Biot Model

$$\rho_s \ddot{\boldsymbol{\eta}} - \nabla \cdot \boldsymbol{\sigma}_p = \mathbf{f}_s \text{ in } \Omega_p \times (0, T] \tag{3}$$

$$(s_0 \, \dot{\phi} + \alpha \nabla \cdot \dot{\boldsymbol{\eta}}) - \nabla \cdot \mathbf{K} \nabla \phi = f_p \quad \text{in } \Omega_p \times (0, T] \,. \tag{4}$$

- $\boldsymbol{\eta}, \phi$ – displacement of structure and pore fluid pressure, resp.
- $f_p, \mathbf{f}_s$ – source/sink and external body force on fluid, resp
- λ_s, μ_s – Lamé constants
- $s_0 > 0, \mathbf{K}$ – storage coefficient and hydraulic conductivity, resp.
- $\alpha > 0$ – Biot-Willis constant
- $\boldsymbol{\sigma}_p = 2\nu_s \mathbf{D}(\boldsymbol{\eta}) + \lambda_s (\nabla \cdot \boldsymbol{\eta}) \mathbf{I} - \alpha \phi \mathbf{I}$

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Poroelastic material

Let $\zeta = \dot{\eta}$ (the velocity of the poroelastic solid material) then:

Biot Model

$$\rho_s \dot{\zeta} - \nabla \cdot \sigma_p = \mathbf{f}_s \text{ in } \Omega_p \times (0, T] \quad (5)$$

$$\eta - \dot{\zeta} = 0 \text{ in } \Omega_p \times (0, T] \quad (6)$$

$$(s_0 \, \dot{\phi} + \alpha \nabla \cdot \dot{\eta}) - \nabla \cdot \mathbf{K} \nabla \phi = f_p \quad \text{in } \Omega_p \times (0, T] \quad (7)$$

- η, ϕ – displacement of structure and pore fluid pressure, resp.
- $f_p, \mathbf{f}_s$ – source/sink and external body force on fluid, resp
- λ_s, μ_s – Lamé constants
- $s_0 > 0, \mathbf{K}$ – storage coefficient and hydraulic conductivity, resp.
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- $\sigma_p = 2\nu_s \mathbf{D}(\eta) + \lambda_s (\nabla \cdot \eta) \mathbf{I} - \alpha \phi \mathbf{I}$

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Interface conditions

- Continuity of flux

$$\mathbf{u} \cdot \mathbf{n}_\Gamma = (\dot{\boldsymbol{\eta}} - \mathbf{K}\nabla\phi) \cdot \mathbf{n}_\Gamma \,.$$

- Balance of stresses

$$\boldsymbol{\sigma}_f \mathbf{n}_\Gamma = \boldsymbol{\sigma}_p \mathbf{n}_\Gamma \,,$$

- Balance of normal stresses:

$$\mathbf{n}_\Gamma \cdot \boldsymbol{\sigma}_f \mathbf{n}_\Gamma = -\phi \,,$$

- Beavers-Joseph Saffman condition

$$\mathbf{n}_\Gamma \cdot \boldsymbol{\sigma}_f \mathbf{t}_\Gamma^l = -\beta(\mathbf{u} - \dot{\boldsymbol{\eta}}) \cdot \mathbf{t}_\Gamma^l \,, \; 1 \leq l \leq d-1$$

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Weak formulation

Function spaces

$$\mathbf{X}_f = \{\mathbf{v} \in \mathbf{H}^1(\Omega_f) : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_f\}, \quad Q_f = L^2(\Omega_f),$$
$$\mathbf{X}_p = \{\boldsymbol{\xi} \in \mathbf{H}^1(\Omega_p) : \boldsymbol{\xi} = \mathbf{0} \text{ on } \Gamma_p^1\}, \quad Q_p = \{r \in H^1(\Omega_p) : r = 0 \text{ on } \Gamma_p^D\}.$$

Stokes region bilinear forms

$$a_f(\mathbf{v}, \mathbf{w}) = 2\nu_f(\mathbf{D}(\mathbf{u}), \mathbf{D}(\mathbf{w}))_{\Omega_f}, \forall \mathbf{v}, \mathbf{w} \in \mathbf{X}_f$$
$$b_f(\mathbf{v}, q_f) = -(q_f, \nabla \cdot \mathbf{v})_{\Omega_f}, \forall \mathbf{v} \in \mathbf{X}_f, \forall q_f \in Q_f$$

Biot region bilinear forms

$$a_e(\boldsymbol{\eta}, \boldsymbol{\xi}) = (2\nu_s \mathbf{D}(\boldsymbol{\eta}), \mathbf{D}(\boldsymbol{\xi}))_{\Omega_p} + (\lambda_s \nabla \cdot \boldsymbol{\eta}, \nabla \cdot \boldsymbol{\xi})_{\Omega_p}, \forall \boldsymbol{\eta}, \boldsymbol{\xi} \in \mathbf{X}_p$$
$$b_e(\boldsymbol{\xi}, q_p) = \alpha(q_p, \nabla \cdot \boldsymbol{\xi})_{\Omega_p}, \forall \boldsymbol{\xi} \in \mathbf{X}_p, q_p \in Q_p$$
$$a_d(q_p, \psi) = (\mathbf{K}\nabla q_p, \nabla \psi)_{\Omega_p}, \forall q_p, \psi \in Q_p.$$

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Fully coupled weak formulation

Find $(\mathbf{u}, p, \boldsymbol{\eta}, \zeta, \phi) : (0, T) \rightarrow (\mathbf{X}_f \times Q_f \times \mathbf{X}_p \times \mathbf{X}_p \times Q_p)$ s.t $\forall \mathbf{v} \in \mathbf{X}_f$, $q \in Q_f$, $\boldsymbol{\xi} \in \mathbf{X}_p$, $\boldsymbol{\tau} \in \mathbf{X}_p$ and $r \in Q_p$,

$$
\begin{aligned}
&(\rho_f \dot{\mathbf{u}}, \mathbf{v})_{\Omega_f} + a_f(\mathbf{u}, \mathbf{v}) + b_f(\mathbf{v}, p) \\
&+ (\rho_s \boldsymbol{\zeta} - \dot{\boldsymbol{\eta}}, \boldsymbol{\tau})_{\Omega_p} + (\rho_s \dot{\boldsymbol{\zeta}}, \boldsymbol{\xi})_{\Omega_p} + a_e(\boldsymbol{\eta}, \boldsymbol{\xi}) - b_e(\boldsymbol{\xi}, \phi)_{\Omega_p} \\
&+ s_0(\dot{\phi}, r) + b_e(\dot{\boldsymbol{\eta}}, r) + a_d(\phi, r) \\
&+ \langle \phi \mathbf{n}_\Gamma, \mathbf{v} - \boldsymbol{\xi} \rangle_{\Gamma_I} + \textstyle\sum_{l=1}^{d-1} \langle \beta (\mathbf{u} - \dot{\boldsymbol{\eta}}) \cdot \mathbf{t}_\Gamma^l, (\mathbf{v} - \boldsymbol{\xi}) \cdot \mathbf{t}_\Gamma^l \rangle_{\Gamma_I} + \langle (\dot{\boldsymbol{\eta}} - \mathbf{u}) \cdot \mathbf{n}_\Gamma, r \rangle_{\Gamma_I} \\
&\qquad = (\mathbf{f}_f, \mathbf{v})_{\Omega_f} + (\mathbf{f}_s, \boldsymbol{\xi})_{\Omega_p} + (f_p, r)_{\Omega_p},
\end{aligned}
$$

$$b_f(\mathbf{u}, q) = 0,$$

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Fully discrete coupled scheme

- $(\mathbf{X}_f^h, Q_f^h)$ and $(\mathbf{X}_p^h, Q_p^h)$ are the Taylor-Hood element $(\mathbb{P}_2, \mathbb{P}_1)$
- Backward Euler in time on $\{t_n\}_{n=1}^N$:

$$\mathcal{D}_{\Delta t} g^n := \frac{g^n - g^{n-1}}{\Delta t}$$

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- Backward Euler in time on $\{t_n\}_{n=1}^N$:

$$\mathcal{D}_{\Delta t} g^n := \frac{g^n - g^{n-1}}{\Delta t}$$

Given $(\mathbf{u}_h^0, \boldsymbol{\eta}_h^0, \boldsymbol{\zeta}_h^0, \phi_h^0) \in \mathbf{X}_f^h \times \mathbf{X}_p^h \times \mathbf{X}_p^h \times Q_p^h$,
find $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\eta}_h^n, \boldsymbol{\zeta}_h^n, \phi_h^n) \in \mathbf{X}_f^h \times Q_f^h \times \mathbf{X}_p^h \times \mathbf{X}_p^h \times Q_p^h$, for $1 \le n \le N$ s.t

$$\begin{aligned}
&\rho_f(\mathcal{D}_{\Delta t}\mathbf{u}_h^n, \mathbf{v})_{\Omega_f} + a_f(\mathbf{u}_h^n, \mathbf{v}) + b_f(\mathbf{v}, p_h^n) \\
&+ \rho_s(\boldsymbol{\zeta}_h^n - \mathcal{D}_{\Delta t}\boldsymbol{\eta}_h^n, \boldsymbol{\tau})_{\Omega_p} + \rho_s(\mathcal{D}_{\Delta t}\boldsymbol{\zeta}_h^n, \boldsymbol{\xi})_{\Omega_p} + a_e(\boldsymbol{\eta}_h^n, \boldsymbol{\xi}) - b_e(\boldsymbol{\xi}, \phi_h^n) \\
&+ s_0(\mathcal{D}_{\Delta t}\phi_h^n, r)_{\Omega_p} + b_e(\mathcal{D}_{\Delta t}\boldsymbol{\eta}_h^n, r) + a_d(\phi_h^n, r) \\
&+ \langle \phi_h^n \mathbf{n}_\Gamma, \mathbf{v} - \boldsymbol{\xi}\rangle_{\Gamma_I} + \textstyle\sum_{l=1}^{d-1} \langle \beta(\mathbf{u}_h^n - \mathcal{D}_{\Delta t}\boldsymbol{\eta}_h^n)\cdot \mathbf{t}_\Gamma^l, (\mathbf{v} - \boldsymbol{\xi})\cdot \mathbf{t}_\Gamma^l\rangle_{\Gamma_I} \\
&+ \langle (\mathcal{D}_{\Delta t}\boldsymbol{\eta}_h^n - \mathbf{u}_h^n)\cdot \mathbf{n}_\Gamma, r\rangle_{\Gamma_I} = (\mathbf{f}_f^n, \mathbf{v})_{\Omega_f} + (\mathbf{f}_s^n, \boldsymbol{\xi})_{\Omega_p} + (f_p^n, r)_{\Omega_p}, \\
&b_f(\mathbf{u}_h^n, q) = 0,
\end{aligned}$$

for all $(\mathbf{v}, q, \boldsymbol{\xi}, \boldsymbol{\chi}, r) \in \mathbf{X}_f^h \times Q_f^h \times \mathbf{X}_p^h \times \mathbf{X}_p^h \times Q_p^h$.

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Poroelasitc locking

- The discretization of the poroelastic problem using standard finite elements leads to *non-physical oscillations* in the pore fluid pressure under
  - low permeability
  - small time steps
  - low compressibility

Poroelasitc locking

- The discretization of the poroelastic problem using standard finite elements leads to *non-physical oscillations* in the pore fluid pressure under
  - low permeability
  - small time steps
  - low compressibility
- Using *inf-sup* stable spaces diminishes the oscillations but they are not completely removed

Stabilization through Fluid Pressure Laplacian (FPL)

- Introduced by Truty and Zimmermann (2006)[1] and further analyzed by Aguilar et. al (2008)[2] and Rodrigo et al (2016) [3]

[1] A. Truty and T. Zimmermann. Stabilized mixed finite element formulations for materially nonlinear partially saturated two-phase media. *Computer Methods in Applied Mechanics and Engineering, 195(13):1517–1546, 2006*

[2] Aguilar G., Gaspar F., Lisbona F., and Rodrigo C. Numerical stabilization of Biot's consolidation model by a perturbation on the flow equation. *International Journal for Numerical Methods in Engineering, 75(11):1282–1300, 2008*

[3] C. Rodrigo, F.J. Gaspar, X. Hu, and L.T. Zikatanov. Stability and monotonicity for some discretizations of the Biots consolidation model. *Computer Methods in Applied Mechanics and Engineering, 298:183–204, 2016*

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- Introduced by Truty and Zimmermann (2006)[1] and further analyzed by Aguilar et. al (2008)[2] and Rodrigo et al (2016) [3]
- Stabilization technique is equivalent to adding a stabilization term

$$a_{stab}(q_p, \psi) = \epsilon \frac{h^2}{\lambda_s + 2\nu_s} \left(\mathcal{D}_{\Delta t} \nabla q_p, \nabla \psi \right)_{\Omega_p}, \forall q_p, \psi \in Q_p^h$$

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- The success of this method depends on a careful choice of ϵ,**however there is no technique for deriving an optimal choice in 2D**.
- $\epsilon = \frac{1}{6}$ has been shown to be optimal in the 1D case (Rodrigo et al).

[1] A. Truty and T. Zimmermann. Stabilized mixed finite element formulations for materially nonlinear partially saturated two-phase media. *Computer Methods in Applied Mechanics and Engineering, 195(13):1517–1546, 2006*

[2] Aguilar G., Gaspar F., Lisbona F., and Rodrigo C. Numerical stabilization of Biot's consolidation model by a perturbation on the flow equation. *International Journal for Numerical Methods in Engineering, 75(11):1282–1300, 2008*

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FPL stabilized scheme

- $(\mathbf{X}_f^h, Q_f^h)$ and $(\mathbf{X}_p^h, Q_p^h)$ are the Taylor-Hood element $(\mathbb{P}_2, \mathbb{P}_1)$
- Backward Euler in time on $\{t_n\}_{n=1}^N$:

$$\mathcal{D}_{\Delta t} g^n := \frac{g^n - g^{n-1}}{\Delta t}$$

Given $(\mathbf{u}_h^0, \boldsymbol{\eta}_h^0, \boldsymbol{\zeta}_h^0, \phi_h^0) \in \mathbf{X}_f^h \times \mathbf{X}_p^h \times \mathbf{X}_p^h \times Q_p^h$,
find $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\eta}_h^n, \boldsymbol{\zeta}_h^n, \phi_h^n) \in \mathbf{X}_f^h \times Q_f^h \times \mathbf{X}_p^h \times \mathbf{X}_p^h \times Q_p^h$, for $1 \le n \le N$ s.t

$$\begin{aligned}
&\rho_f(\mathcal{D}_{\Delta t}\mathbf{u}_h^n, \mathbf{v})_{\Omega_f} + a_f(\mathbf{u}_h^n, \mathbf{v}) + b_f(\mathbf{v}, p_h^n) \\
&\quad + \rho_s(\boldsymbol{\zeta}_h^n - \mathcal{D}_{\Delta t}\boldsymbol{\eta}_h^n, \boldsymbol{\tau})_{\Omega_p} + \rho_s(\mathcal{D}_{\Delta t}\boldsymbol{\zeta}_h^n, \boldsymbol{\xi})_{\Omega_p} + a_e(\boldsymbol{\eta}_h^n, \boldsymbol{\xi}) - b_e(\boldsymbol{\xi}, \phi_h^n) \\
&\quad + s_0(\mathcal{D}_{\Delta t}\phi_h^n, r)_{\Omega_p} + b_e(\mathcal{D}_{\Delta t}\boldsymbol{\eta}_h^n, r) + a_d(\phi_h^n, r) + \boxed{a_{stab}(\phi_h^n, r)} \\
&\quad + \langle \phi_h^n \mathbf{n}_\Gamma, \mathbf{v} - \boldsymbol{\xi}\rangle_{\Gamma_I} + \textstyle\sum_{l=1}^{d-1} \langle \beta(\mathbf{u}_h^n - \mathcal{D}_{\Delta t}\boldsymbol{\eta}_h^n)\cdot \mathbf{t}_\Gamma^l, (\mathbf{v} - \boldsymbol{\xi})\cdot \mathbf{t}_\Gamma^l\rangle_{\Gamma_I} \\
&\quad + \langle (\mathcal{D}_{\Delta t}\boldsymbol{\eta}_h^n - \mathbf{u}_h^n)\cdot \mathbf{n}_\Gamma, r\rangle_{\Gamma_I} = (\mathbf{f}_f^n, \mathbf{v})_{\Omega_f} + (\mathbf{f}_s^n, \boldsymbol{\xi})_{\Omega_p} + (f_p^n, r)_{\Omega_p}, \\
&b_f(\mathbf{u}_h^n, q) = 0,
\end{aligned}$$

for all $(\mathbf{v}, q, \boldsymbol{\xi}, \boldsymbol{\chi}, r) \in \mathbf{X}_f^h \times Q_f^h \times \mathbf{X}_p^h \times \mathbf{X}_p^h \times Q_p^h$.

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Stability

Let

$$E(\mathbf{v}, \boldsymbol{\eta}, \zeta, \phi) := \frac{\rho_f}{2}\|\mathbf{v}\|^2_{\Omega_f} + \nu_s\|\mathbf{D}\boldsymbol{\eta}\|^2_{\Omega_p} + \frac{\lambda_s}{2}\|\nabla\cdot\boldsymbol{\eta}\|^2_{\Omega_p} + \frac{\rho_s}{2}\|\boldsymbol{\zeta}\|^2_{\Omega_p} + \frac{s_0}{2}\|\phi\|^2_{\Omega_p}$$

$$\|(\mathbf{v}, \phi)\|_{\mathbf{X}_f\times Q_p} := \left(2\nu_f\|\mathbf{D}\mathbf{v}\|^2_{\Omega_f} + \|\mathbf{K}^{1/2}\nabla\phi\|^2_{\Omega_p}\right)^{1/2}.$$

Assume that $\mathbf{f}_s = \mathbf{0}$, $\mathbf{u}_h^0 = \mathbf{0}$, $\boldsymbol{\eta}_h^0 = \mathbf{0}$, $\boldsymbol{\zeta}_h^0 = \mathbf{0}$ and $\phi_h^0 = 0$. Suppose that $\{(\mathbf{u}_h^k, p_h^k, \boldsymbol{\eta}_h^k, \phi_h^k)\}_{1\le k\le N}$ is the solution at time step k. Then, for all $1 \le k \le N$,

$$E(\mathbf{u}_h^k, \boldsymbol{\eta}_h^k, \boldsymbol{\zeta}_h^k, \phi_h^k) + \Delta t\sum_{i=1}^{k}\|(\mathbf{u}_h^i, \phi_h^i)\|^2_{\mathbf{X}_f\times Q_p} \le \Delta t\mathcal{C}_k^2$$

where $\mathcal{C}$ depends on data and Sobolev inequality constants.

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Convergence analysis

- $\Omega_f = (0,1) \times (1,2)$ and $\Omega_p = (0,1) \times (0,1)$ with $\Gamma_I = (0,1) \times \{1\}$.
- **Stokes velocity** and **displacement**: Dirichlet boundary on Γ_f and on Γ_p
- **Darcy pore fluid pressure**: Neumann on $\Gamma_p^N = \{0,1\} \times (0,1)$ and Dirichlet on $\Gamma_p^D = (0,1) \times \{0\}$.
- **Parameters:** $\rho_f, \nu_f, \rho_s, \nu_s, \lambda_s, s_0, \alpha, \beta = 1$ and $\mathbf{K} = \mathbb{I}$
- **Manufactured smooth solution:** [4]

$$
\begin{aligned}
\mathbf{u}(\mathbf{x},t) &= \Big(\pi \cos(\pi t)(-3.0x + \cos(\pi y)), \pi \cos(\pi t)(y + 1.0)\Big), \\
p(\mathbf{x},t) &= e^t \sin(\pi x)\cos(\pi y) + 2.0\pi \cos(\pi t), \\
\boldsymbol{\eta}(\mathbf{x},t) &= \Big(\sin(\pi t)(-3.0x + \cos(\pi y)), \sin(\pi t)(y + 1.0)\Big), \\
\phi(\mathbf{x},t) &= e^t \sin(\pi x)\cos(\pi y).
\end{aligned}
$$

[4] I. Ambartsumyan, E. Khattatov, I. Yotov, and P. Zunino. A Lagrange multiplier method for a Stokes-Biot fluid-poroelastic structure interaction model. *Numerische Mathematik*, Apr 2018

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Convergence analysis: Spatial errors in Ω_f

h	$\lvert\lvert p - p_h \rvert\rvert_{\Omega_f}$	rate	$\lvert\lvert \mathbf{u} - \mathbf{u}_h \rvert\rvert_{\Omega_f}$	rate	$\lvert\lvert \mathbf{D}(\mathbf{u} - \mathbf{u}_h) \rvert\rvert_{\Omega_f}$	rate
$\frac{1}{2}$	$3.327e-01$		$4.113e-02$		$4.744e-01$	
$\frac{1}{4}$	$5.652e-02$	2.55	$5.445e-03$	2.91	$1.164e-01$	2.02
$\frac{1}{8}$	$1.149e-02$	2.29	$7.050e-04$	2.95	$2.852e-02$	2.03
$\frac{1}{16}$	$2.693e-03$	2.09	$8.951e-05$	2.96	$7.089e-03$	2.00
$\frac{1}{32}$	$6.761e-04$	2.00	$1.122e-05$	3.00	$1.772e-03$	2.00

Errors and spatial convergence rates in Ω_f with $T = 10^{-4}$ and $\Delta t = 10^{-6}$.

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Convergence analysis: Spatial errors in Ω_p

h	$\|\|\phi - \phi_h\|\|_{\Omega_p}$	rate	$\|\|\nabla(\phi - \phi_h)\|\|_{\Omega_p}$	rate	$\|\|\mathbf{D}(\boldsymbol{\eta} - \boldsymbol{\eta}_h)\|\|_{\Omega_p}$	rate
$\frac{1}{2}$	$2.113e-01$		$1.998e+00$		$4.792e-05$	
$\frac{1}{4}$	$3.649e-02$	2.54	$9.382e-01$	1.09	$1.175e-05$	2.02
$\frac{1}{8}$	$7.530e-03$	2.28	$4.469e-01$	1.06	$2.879e-06$	2.02
$\frac{1}{16}$	$1.734e-03$	2.12	$2.199e-01$	1.02	$7.129e-07$	2.01
$\frac{1}{32}$	$4.185e-04$	2.05	$1.093e-01$	1.01	$1.780e-07$	2.00

Errors and spatial convergence rates in Ω_p with $T = 10^{-4}$ and $\Delta t = 10^{-6}$.

$\|\|\zeta - \zeta_h\|\|_{\Omega_p}$	rate
$4.078e-02$	
$5.426e-03$	2.91
$7.035e-04$	2.94
$8.937e-05$	2.97
$1.130e-05$	2.98

Errors and spatial convergence rates in Ω_p with $T = 10^{-4}$ and $\Delta t = 10^{-6}$.

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Convergence analysis: Temporal errors in Ω_f

Δt	$\|\|p_{h,\Delta t} - p_{h,\frac{\Delta t}{2}}\|\|_{\Omega_f}$	rate	$\|\|\mathbf{u}_{h,\Delta t} - \mathbf{u}_{h,\frac{\Delta t}{2}}\|\|_{\Omega_f}$	rate
$\frac{1}{16}$	$1.032e+00$		$1.677e-02$	
$\frac{1}{32}$	$5.353e-01$	0.94	$8.767e-03$	0.94
$\frac{1}{64}$	$2.729e-01$	0.97	$4.495e-03$	0.96
$\frac{1}{128}$	$1.378e-01$	0.99	$2.272e-03$	0.98
$\frac{1}{256}$	$6.922e-02$	1.00	$1.140e-03$	0.99

Errors and temporal convergence rates in Ω_f with $T = 1.0$ and $h = \frac{1}{16}$.

$\|\|\mathbf{D}(\mathbf{u}_{h,\Delta t} - \mathbf{u}_{h,\frac{\Delta t}{2}})\|\|_{\Omega_f}$	rate
$1.022e-01$	
$5.425e-02$	0.91
$2.810e-02$	0.94
$1.428e-02$	0.98
$7.187e-03$	1.00

Errors and temporal convergence rates in Ω_f with $T = 1.0$ and $h = \frac{1}{16}$.

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Convergence analysis: Temporal errors in Ω_p

Δt	$\|\|\phi_{h,\Delta t} - \phi_{h,\frac{\Delta t}{2}}\|\|_{\Omega_p}$	rate	$\|\|\mathbf{D}(\boldsymbol{\eta}_{h,\Delta t} - \boldsymbol{\eta}_{h,\frac{\Delta t}{2}})\|\|_{\Omega_p}$	rate
$\frac{1}{16}$	$2.043e-01$		$1.784e-01$	
$\frac{1}{32}$	$1.105e-01$	0.89	$1.118e-01$	0.67
$\frac{1}{64}$	$5.762e-02$	0.93	$6.377e-02$	0.80
$\frac{1}{128}$	$2.943e-02$	0.96	$3.427e-02$	0.90
$\frac{1}{256}$	$1.487e-02$	0.98	$1.780e-02$	0.95

Errors and temporal convergence rates in Ω_p with $T = 1.0$ and $h = \frac{1}{16}$.

$\|\|\zeta_{h,\Delta t} - \zeta_{h,\frac{\Delta t}{2}}\|\|_{\Omega_p}$	rate
$4.121e-01$	
$2.354e-01$	0.81
$1.274e-01$	0.87
$6.656e-02$	0.94
$3.407e-02$	0.97

Errors and temporal convergence rates in Ω_p with $T = 1.0$ and $h = \frac{1}{16}$.

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Convergence analysis: Ω_f

h	$\|\|p - p_h\|\|_{\Omega_f}$	rate	$\|\|\mathbf{u} - \mathbf{u}_h\|\|_{\Omega_f}$	rate	$\|\|\mathbf{D}(\mathbf{u} - \mathbf{u}_h)\|\|_{\Omega_f}$	rate
$\frac{1}{4}$	$2.110e+00$		$3.334e-02$		$2.342e-01$	
$\frac{1}{8}$	$5.461e-01$	1.95	$8.717e-03$	1.94	$6.208e-02$	1.91
$\frac{1}{16}$	$1.377e-01$	1.99	$2.200e-03$	1.99	$1.574e-02$	1.97
$\frac{1}{32}$	$3.448e-02$	2.00	$5.499e-04$	2.00	$3.940e-03$	2.00
$\frac{1}{64}$	$8.623e-03$	2.00	$1.373e-04$	2.00	$9.845e-04$	2.00

Errors and convergence rates in Ω_f with $T = 1.0$ and $\Delta t = h^2$.

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Convergence analysis: Ω_p

h	$\|\|\phi - \phi_h\|\|_{\Omega_p}$	rate	$\|\|\nabla(\phi - \phi_h)\|\|_{\Omega_p}$	rate	$\|\|\mathbf{D}(\boldsymbol{\eta} - \boldsymbol{\eta}_h)\|\|_{\Omega_p}$	rate
$\frac{1}{4}$	$4.347e-01$		$2.498e+00$		$4.252e-01$	
$\frac{1}{8}$	$1.181e-01$	1.88	$1.200e+00$	1.05	$1.351e-01$	1.65
$\frac{1}{16}$	$3.023e-02$	1.97	$5.946e-01$	1.01	$3.632e-02$	1.89
$\frac{1}{32}$	$7.602e-03$	1.99	$2.966e-01$	1.00	$9.260e-03$	1.97
$\frac{1}{64}$	$1.903e-03$	2.00	$1.482e-01$	1.00	$2.326e-03$	2.00

Errors and convergence rates in Ω_p with $T = 1.0$ and $\Delta t = h^2$.

$\|\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|\|_{\Omega_p}$	rate
$9.053e-01$	
$2.622e-01$	1.79
$6.856e-02$	1.93
$1.735e-02$	1.98
$4.351e-03$	2.00

Errors and convergence rates in Ω_p with $T = 1.0$ and $\Delta t = h^2$.

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Free flow over clamped a poroelastic material

- **Computational Domain :** $\Omega = \Omega_f \cup \Omega_p$

$$\boxed{\Omega_f = (0,1) \times (1,2) \text{ and } \Omega_p = (0,1) \times (0,1) \text{ and } \Gamma_I = (0,1) \times \{1\}}$$

- Ω_f: **Stokes velocity boundary**

$$\mathbf{u}(x,y) = \begin{cases} (0, -10^{-2}\sin(\pi x))^T & \text{if } y = 2 \\ (0,0)^T & \text{if } x = 0, 1. \end{cases}$$

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- Ω_p: **Pore pressure boundary**

$$\phi = g_D(x,y) \text{ on } \Gamma_p^D$$

- Ω_p: **Displacement boundary**

$$\Gamma_p = \Gamma_p^1 \cup \Gamma_p^2 \text{ where } \Gamma_p^1 \text{ is the left boundary } (x = 0) \text{ and } \Gamma_p^2 \text{ is the rest}$$

$$\boldsymbol{\eta} = \mathbf{0} \text{ on } \Gamma_p^1 \times (0, T],$$

$$\boldsymbol{\sigma}_p \cdot \mathbf{n}_p = \mathbf{0} \text{ on } \Gamma_p^2 \times (0, T],$$

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Free flow over a clamped poroelastic material

- **Material parameters and source functions**

Ω_f :

$$\rho_f = 1.0, \quad \nu_f = 1.0, \quad \mathbf{f}_f = \mathbf{0}$$

Ω_p :[5]

$$\rho_f = 1.0, \quad \nu_f = 1.0, \quad \mathbf{f}_f = \mathbf{0}, \quad \alpha = 1.0, \quad \rho_s = 2.0, \quad \nu_s = 3.57 \times 10^3,$$
$$\lambda_s = 1.4 \times 10^4, \quad s_0 = 1.0 \times 10^{-5}, \quad \beta = \frac{1.0}{\sqrt{\kappa}} \text{ where } \mathbf{K} = \kappa \mathbf{I}$$
$$\mathbf{K} = 10^{-7}\mathbb{I}, \quad \mathbf{f}_s = \mathbf{0}, \quad f_p = 0$$

- **Initial conditions**

$$\mathbf{u} = \boldsymbol{\eta} = \boldsymbol{\zeta} = \mathbf{0}, \quad \phi = 0$$

[5] M. Wheeler, G. Xue, and I. Yotov. *Coupling multipoint flux mixed finite element methods with continuous Galerkin methods for poroelasticity.* Comp Geosci, 18(1):5775, Feb 2014

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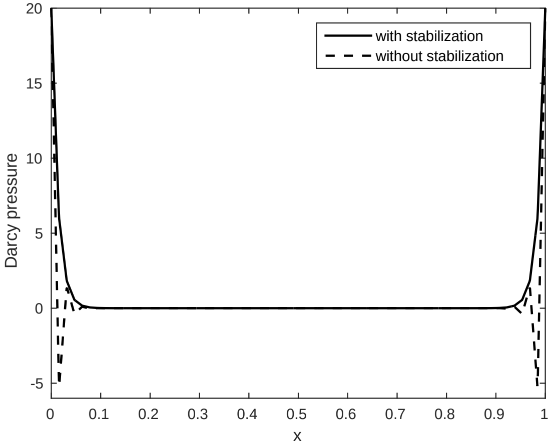
Free flow over a clamped poroelastic material

Pore pressure boundary $g_D(x,y) = 20$
Cross section of pore fluid pressure in Ω_p along $y = 0.125$ at $t_1 = 1.0 \times 10^{-6}$, $h = \frac{1}{64}$



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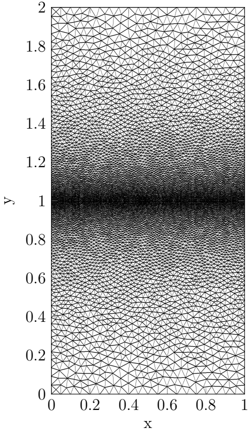
Transient behavior

$$g_{\mathrm{D}}(x, y) = 0 \text{ on } \Gamma_p^D \text{ and set } \mathbf{K} = 10^{-7}\mathbb{I} \text{ in } \Omega_p$$

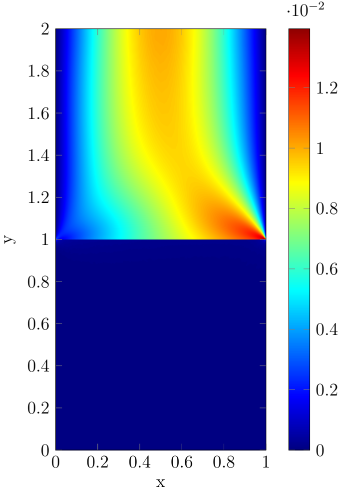
$$\Delta t = 5.0 \times 10^{-3} \text{ on unstructured mesh}$$



Norm velocity at $T = 5.0$

$$g_D(x,y) = 0 \text{ on } \Gamma_p^D \text{ and set } \mathbf{K} = 10^{-7}\mathbb{I} \text{ in } \Omega_p$$

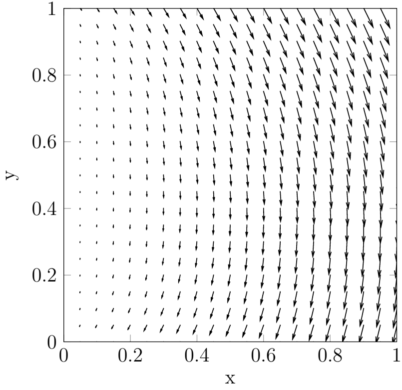
$$\Delta t = 5.0 \times 10^{-3}$$



Displacement vector field at $T = 5.0$

$$g_{\mathrm{D}}(x, y) = 0 \text{ on } \Gamma_p^D \text{ and set } \mathbf{K} = 10^{-7}\mathbb{I} \text{ in } \Omega_p$$

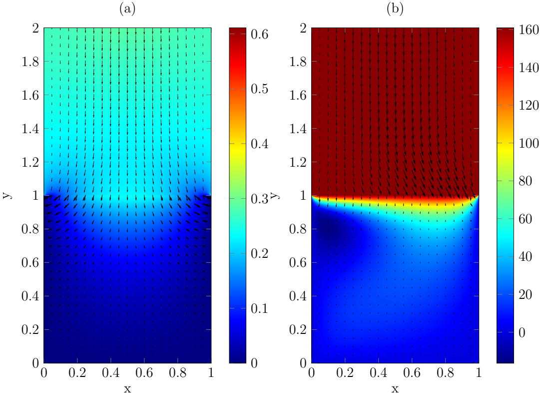
$$\Delta t = 5.0 \times 10^{-3}$$



Effect of permeability on pressure and velocity

$$g_{\mathrm{D}}(x,y) = 0 \text{ on } \Gamma_p^D, \quad \mathbf{K} = 10^{-2}\mathbb{I} \text{ (a) } \mathbf{K} = 10^{-7}\mathbb{I} \text{ (b)}$$

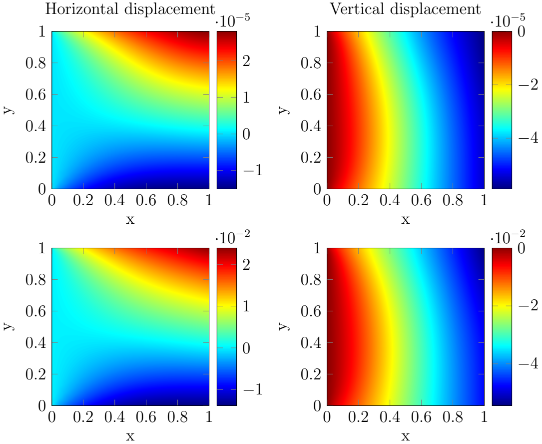
$$\Delta t = 5.0 \times 10^{-3} \text{ on unstructured mesh}$$



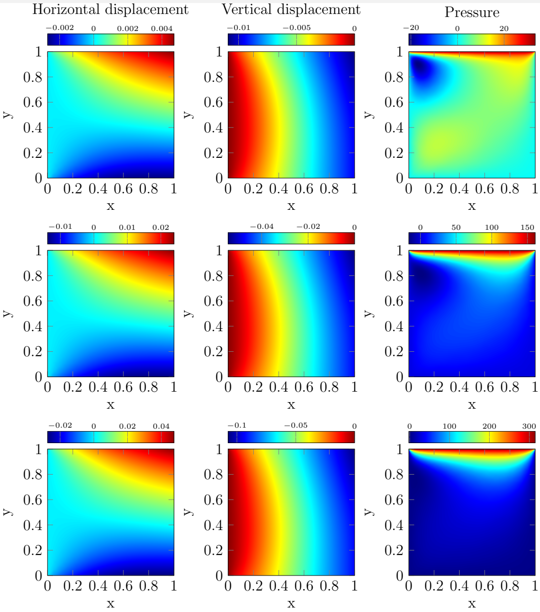
Effect of permeability on displacement

$$g_{\mathrm{D}}(x,y) = 0 \text{ on } \Gamma_p^D, \quad \mathbf{K} = 10^{-2}\mathbb{I} \text{ (top)}, \quad \mathbf{K} = 10^{-7}\mathbb{I} \text{ (bottom)}$$

$$\Delta t = 5.0 \times 10^{-3} \text{ on unstructured mesh}$$



Transient behavior T=1 (top) T=5 (middle) T= 10 (bottom)



Conclusion

- Presented a fully coupled scheme for the coupled Stokes-Biot equations
- Numerical study of convergence for smooth problems
- Fluid Pressure Laplacian stabilization technique removes unphysical oscillations for realistic material parameters

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