Modeling, Simulation, and Control of Differential-Algebraic Port-Hamiltonian Systems

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Mathematics for key technologies

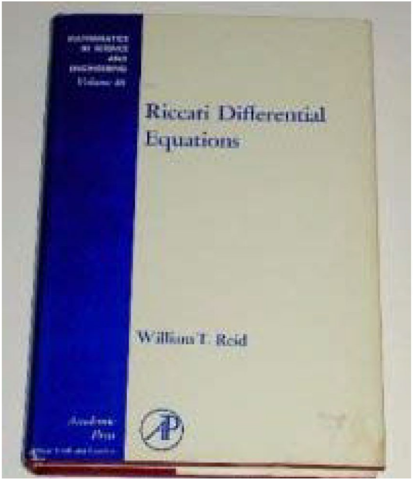
W.T. and Idalia Reid Prize lecture, July 18

Outline

William T. and Idalia Reid

Taken from a wonderful historical overview by John A. Burns at the SIAM Annual meeting 2010.

The Riccati equation book 1972



My first encounter with W.T. Reid and his work happened when I started to work on control theory as a postdoc at Univ. of Wisconsin 1984. A whole new world of mathematics for me.

What are DAEs ?

Differential-algebraic equations (DAEs), are implicit dynamical systems of the form

$$\begin{aligned} 0 &= F(t,\xi,u,\dot{\xi}), \\ y &= G(t,\xi,u), \end{aligned}$$

with $F \in C^0(\mathbb{R} \times \mathbb{D}_\xi \times \mathbb{D}_u \times \mathbb{D}_{\dot{\xi}}, \mathbb{R}^n)$, $G \in C^0(\mathbb{R} \times \mathbb{D}_\xi \times \mathbb{D}_u, \mathbb{R}^p)$.
In the linear case (linearization along solutions)

$$\begin{aligned} E(t)\dot{\xi} &= A(t)\xi + B(t)u + \phi(t), \\ y &= C(t)\xi + D(t)u + \psi(t). \end{aligned}$$

- $\xi : \mathbb{R} \to \Xi$ is the state, finite $\Xi = \mathbb{R}^n$, or infinite dimensional,
- $u : \mathbb{R} \to \mathbb{R}^m$ denotes the control input,
- $y : \mathbb{R} \to \mathbb{R}^p$ denotes the output.

Why DAEs and not ODEs?

(Operator) DAEs provide a unified framework for the analysis, simulation and control of coupled dynamical systems.

- ▷ Automatic (black-box) modelling leads to (operator) DAEs. (Constraints at interfaces).
- ▷ Conservation laws lead to (operator) DAEs. (Conservation of mass, energy, momentum).
- ▷ Coupling of solvers leads to DAEs (discrete time).
- ▷ Control problems are (operator) DAEs (behavior). DAE modeling is standard in multi-physics systems.

Automated modeling with DAEs becomes extremely easy and industrial standard, but

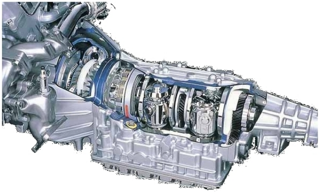
- Operator DAEs integrate and differentiate (in time).
- Numerical simulation methods have instabilities and convergence problems.
- Consistent initialization is difficult.
- The discretized system may be unsolvable even if the DAE is solvable and vice versa.
- Different scales in different components.
- Numerical drift-off due to unresolved hidden constraints.
- Model reduction and (optimal) control is difficult.

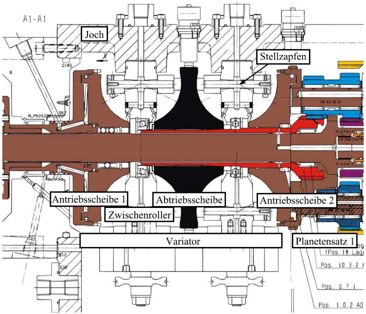
Black-box DAE modeling is great but pushes all difficulties into the analysis/numerics/control.

Outline

Software based control of automatic transmission.
Project with Daimler AG (Dissertation: Peter Hamann).

Half-toroid model





Technological application, Tasks

- Modeling of multi-physics system: multi-body system, elasticity, hydraulics, friction, ...
- Model in form of a network of subcomponents.
- Real time simulation of transmission.
- Development of control methods for coupled system.
- Model reduction and observer design.
- Real time control of transmission.

Ultimate goals: Decrease fuel consumption, avoid super exact production, improve smoothness of switching

Consider DAE control systems

$$\begin{aligned} 0 &= F(t, \xi, u, \dot{\xi}),\ \xi(t_0) = \xi^0 \\ y &= G(t, \xi, u). \end{aligned}$$

We use a **behavior approach**, forming $z = (x, u)$ and obtain s general non-square DAE

$$\mathcal{F}(t, z, \dot{z}) = 0,\ y = \mathcal{G}(t, z).$$

together with a set of equations for the initial conditions

$$H(z(t_0)) = z^0.$$

System must be regularized (implicit derivatives), the solution manifold (in z space) must be identified for initialization and projection.

▷ P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations — Analysis and Numerical Solution*. EMS Publishing House, Zürich, CH, 2006.

- For regularization, numerical solution, and for the design of controllers, we use derivative arrays.
- Consider the general nonlinear system and form

$$\mathcal{F}_\mu(t, z, \dot{z}, \ldots, z^{(\mu+1)}) = 0,$$

which stacks the original equation and selected derivatives of equations up to level μ in one large system of ℓ equations.
- Here, partial derivatives of $\mathcal{F}_\mu$ with respect to selected variables ζ from $(t, z, \dot{z}, \ldots, z^{(\mu+1)})$ are denoted by $\mathcal{F}_{\mu;\zeta}$.
- The (algebraic) solution set of the derivative array $\mathcal{F}_\mu$ is

$$\mathbb{L}_\mu = \{z_\mu \in \mathbb{I} \times \mathbb{R}^{n+m} \times \ldots \times \mathbb{R}^{n+m} \mid \mathcal{F}_\mu(z_\mu) = 0\}.$$

Local regularization must be possible.

Hypothesis There exist integers μ, r, a, d, and v such that $\mathbb{L}_\mu \neq \emptyset$ and such that for every $z_\mu^0 = (t_0, z^0, \dot{z}^0, \ldots, z^{(\mu+1)^0}) \in \mathbb{L}_\mu$ there exists a (sufficiently small) neighborhood such that:

1. $\mathbb{L}_\mu \subseteq \mathbb{R}^{(\mu+2)(n+m)+1}$ forms a manifold.
2. We have $\operatorname{rank} \mathcal{F}_{\mu;z,\dot{z},\ldots,z^{(\mu+1)}} = r$ on $\mathbb{L}_\mu$.
3. We have $\operatorname{corank} \mathcal{F}_{\mu;z,\dot{z},\ldots,z^{(\mu+1)}} - \operatorname{corank} \mathcal{F}_{\mu-1;z,\dot{z},\ldots,z^{(\mu)}} = v$ on $\mathbb{L}_\mu$.
4. We have $\operatorname{rank} \mathcal{F}_{\mu;\dot{z},\ldots,z^{(\mu+1)}} = r - a$ on $\mathbb{L}_\mu$ such that there exist smooth full rank matrix functions Z_2 and T_2 of size $(\mu+1)\ell \times a$ and $(n+m) \times (n+m-a)$, respectively, satisfying $Z_2^T \mathcal{F}_{\mu;\dot{z},\ldots,z^{(\mu+1)}} = 0$, $\operatorname{rank} Z_2^T \mathcal{F}_{\mu;z} = a$, and $Z_2^T \mathcal{F}_{\mu;z} T_2 = 0$ on $\mathbb{L}_\mu$.
5. We have $\operatorname{rank} \mathcal{F}_{\dot{z}} T_2 = d = \ell - a - v$ on $\mathbb{L}_\mu$ such that there exists a smooth full rank matrix function Z_1 of size $(n+m) \times d$ satisfying $\operatorname{rank} Z_1^T \mathcal{F}_{\dot{z}} T_2 = d$.

The regularized system

Theorem

The Hypothesis implies locally existence of a regularized system

$$\begin{aligned} \hat{F}_1(t, z, \dot{z}) &= 0, \\ \hat{F}_2(t, z) &= 0, \\ 0 &= 0, \end{aligned}$$

$\hat{F}_1 = Z_1^T \mathcal{F}$ *describes the dynamics of the system,* $\hat{F}_2(t, z) = 0$ *contains all algebraic constraints and defines the solution manifold and the consistency set for initial conditions.*

The solution has not changed (regularized model).

- P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations — Analysis and Numerical Solution*. EMS Publishing House, Zürich, CH, 2006.
- S.L. Campbell, P. Kunkel and V. Mehrmann, *Regularization of linear and nonlinear descriptor systems*. In *Control and Optimization with Differential-Algebraic Constraints*, L.T. Biegler, S.L. Campbell and V. Mehrmann (editors), SIAM, Society of Industrial and Applied Mathematics, Philadelphia, PA, 2012, pp. 17–34

Outline

Optimal control problem

$$\mathcal{J}(x,u) = \mathcal{M}(x(t_f)) + \int_{t_0}^{t_f} \mathcal{K}(t,x,u)\, dt = \min!$$

subject to a DAE constraint already in regularized form

$$F(t,x,\dot{x},u) = 0,\ \ x(t_0) = x_0.$$

x–state, *u*–input. Similar with output equation.
Consider the linear time-varying case.

Cost functional:

$$\mathcal{J}(x,u) = \frac{1}{2}x(t_f)^T M x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} (x^T W x + 2x^T S u + u^T R u)\, dt,$$

$W = W^T \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $S \in C^0(\mathbb{I}, \mathbb{R}^{n,m})$, $R = R^T \in C^0(\mathbb{I}, \mathbb{R}^{m,m})$, $M = M^T \in \mathbb{R}^{n,n}$.

Constraint:

$$E(t)\dot{x} = A(t)x + B(t)u + f(t), \quad x(t_0) = x_0,$$

$E \in C^1(\mathbb{I}, \mathbb{R}^{n,n})$, $A \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $B \in C^0(\mathbb{I}, \mathbb{R}^{n,m})$, $f \in C^0(\mathbb{I}, \mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$.

Here: Determine optimal controls $u \in \mathbb{U} = C^0(\mathbb{I}, \mathbb{R}^m)$.

More general spaces, output controls, operators, and also non-square E, A are possible.

Why don't we just apply the results from Reid's book or the Pontryagin maximum principle?

- For (linear) ODEs the initial value problem has a unique solution $x \in C^1(\mathbb{I}, \mathbb{R}^n)$ for every $u \in \mathbb{U}$, every $f \in C^0(\mathbb{I}, \mathbb{R}^n)$, and every initial value $x_0 \in \mathbb{R}^n$.
- DAEs, where $E(t)$ may be singular, may not be (uniquely) solvable for any $u \in \mathbb{U}$, the regularity of u and the initial conditions are restricted.
- Furthermore, we need solutions $x \in \mathbb{X}$, constrained to a locally constructed manifold.

- P. Kunkel and V. Mehrmann. *Optimal control for unstructured nonlinear differential-algebraic equations of arbitrary index.* Mathematics of Control Signals and Systems, Vol. 20, 227–269, 2008.

Calculus of variations, linear ODEs (E=I)

Opimality conditions (Euler-Lagrange equations).

Theorem

If (x, u) is a solution to the optimal control problem, then there exists a Lagrange multiplier function $\lambda \in C^1(\mathbb{I}, \mathbb{R}^n)$, such that (x, λ, u) satisfy the DAE optimality boundary value problem

$$
\begin{aligned}
\text{(a)} \qquad \dot{x} &= Ax + Bu + f, \; x(t_0) = x_0, \\
\text{(b)} \qquad -\dot{\lambda} &= Wx + Su + A^T\lambda, \; \lambda(t_f) = Mx(t_f), \\
\text{(c)} \qquad 0 &= S^T x + Ru + B^T\lambda.
\end{aligned}
$$

▷ W.T. Reid, Riccati Differential Equations, Academic Press 1972

Naive Idea for DAEs

Replace $\dot{x}$ by $E(t)\dot{x}$ and then do the analysis in the same way. For DAEs the formal optimality system could be

$$
\begin{array}{llcl}
\text{(a)} & E\dot{x} & = & Ax + Bu + f, \; x(t_0) = x_0 \\
\text{(b)} & -\frac{d}{dt}(E^T\lambda) & = & Wx + Su + A^T\lambda, \; (E^T\lambda)(t_f) = Mx(t_f), \\
\text{(b)} & 0 & = & S^Tx + Ru - B^T\lambda.
\end{array}
$$

- In general not true. Counterexamples: Backes 2006
- One has to guarantee that the resulting adjoint equation for λ has a unique solution, but it may not.
- The boundary conditions may not be consistent.

- A. Backes, *Optimale Steuerung der linearen DAE im Fall Index 2.* Dissertation, HU Berlin, Germany, 2006.
- P. Kunkel and V. Mehrmann. *Optimal control for unstructured nonlinear differential-algebraic equations of arbitrary index.* Mathematics of Control Signals and Systems, Vol. 20, 227–269, 2008.

Necessary optimality condition

Theorem

Consider a regularized linear quadratic DAE optimal control problem with a consistent initial condition $x(t_f) \in \operatorname{cokernel} E(t_f)$*. If* $(x, u) \in \mathbb{X} \times \mathbb{U}$ *is a solution to this optimal control problem, then there exists a Lagrange multiplier function* $\lambda \in C^1_{EE^+}(\mathbb{I}, \mathbb{R}^n)$*, such that* (x, λ, u) *satisfy the optimality boundary value problem*

$$
\begin{aligned}
&E\tfrac{d}{dt}(E^+Ex) = (A + E\tfrac{d}{dt}(E^+E))x + Bu + f, \ (E^+Ex)(t_0) = x^0,\\
&-E^T\tfrac{d}{dt}(EE^+\lambda) = Wx + Su + (A + EE^+\dot{E})^T\lambda,\\
&(EE^+\lambda)(t_f) = E^+(t_f)^T Mx(t_f),\\
&0 = S^T x + Ru + B^T\lambda.
\end{aligned}
$$

where E^+ *is the (local) Moore-Penrose inverse of* E*.*

- P. Kunkel and V. Mehrmann. *Optimal control for unstructured nonlinear differential-algebraic equations of arbitrary index.* Mathematics of Control Signals and Systems, Vol. 20, 227–269, 2008.
- P. Kunkel and V. Mehrmann. Optimal control for linear descriptor systems with variable coefficients *Numerical Linear Algebra in Signals, Systems and Control*, P. Van Dooren et al Edts. Lecture Notes in Elect. Engin., Vol 80, Springer, 2011, 313–340.

Solution via formal optimality system

Theorem (Kunkel/M. 2011)

Let all data of the given optimal control problem be sufficiently smooth and let the formal necessary optimality conditions have a solution (x, u, λ)*. Then there exist a function* η *replacing* λ *such that* (x, u, η) *solves the true necessary optimality conditions.*

- Analogous (local) result in general nonlinear case.
- The optimality DAE may need further remodeling.
- Further consistency conditions or smoothness requirements arise.
- Under some extra conditions (invertibility of the weight matrix R, etc) the solution is a feedback control,obtained from the solution of Riccati differential equation.
- In general better to solve the optimality boundary value problem.

Outline

Classical control engineering approach

- Build a prototype.
- Measure the input/ouput behavior.
- Build a (linear) DAE model of the input/output behavior.
- Compute the optimal feedback controller.
- Apply it in the physical system.

This took a while, was very expensive, but worked reasonably well.

However, the pure model based approach failed!

Pressure control

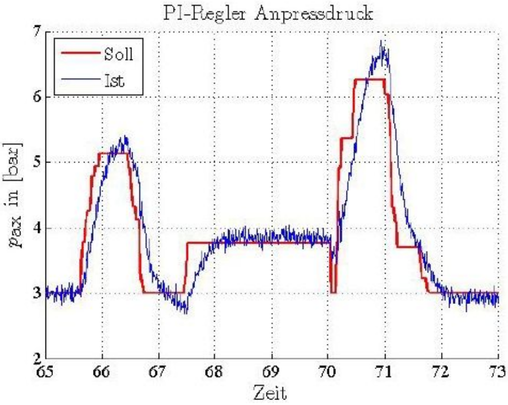


Figure: Pressure Control

Outline

Energy based modeling

- **Choose representations of models so that coupling of different physical domains works across many scales.**
- Use energy as common quantity of different physical systems.
- We want a representation that is good for model coupling, that is good for discretization, and that is close to physics.
- Is there such a Jack of all trades?
- A system theoretic way to deal with such energy based modeling is that of **port-Hamiltonian systems**.

- P. C. Breedveld. *Modeling and Simulation of Dynamic Systems using Bond Graphs*, pages 128–173. EOLSS Publishers Co. Ltd./UNESCO, Oxford, UK, 2008.
- B. Jacob and H. Zwart. *Linear port-Hamiltonian systems on infinite-dimensional spaces*. Operator Theory: Advances and Applications, 223. Birkhäuser/Springer Basel CH, 2012.
- A. J. van der Schaft, D. Jeltsema, Port-Hamiltonian systems: network modeling and control of nonlinear physical systems. In *Advanced Dynamics and Control of Structures and Machines*, CISM Courses and Lectures, Vol. 444. Springer Verlag, New York, N.Y., 2004.
- A. J. van der Schaft, Port-Hamiltonian differential-algebraic systems. In *Surveys in Differential-Algebraic Equations I*, 173-226. Springer-Verlag, 2013. Port-Hamiltonian systems theory: An introductory overview.

Classical nonlinear port-Hamiltonian (pH) ODE/PDE systems

$$
\begin{aligned}
\dot{x} &= (J(x,t) - R(x,t))\,\nabla_x \mathcal{H}(x) + (B(x,t) - P(x,t))u(t),\\
y(t) &= (B(x,t) + P(x,t))^T \nabla_x \mathcal{H}(x) + (S(x,t) + N(x,t))u(t),
\end{aligned}
$$

- $\mathcal{H}(x)$ is the *Hamiltonian*: it describes the distribution of internal energy among the energy storage elements;
- $J = -J^T$ describes the *energy flux* among energy storage elements within the system;
- $R = R^T \geq 0$ describes *energy dissipation/loss* in the system;
- $B \pm P$: *ports* where energy enters and exits the system;
- $S + N$, $S = S^T$, $N = -N^T$, direct *feed-through* input to output.
- In the infinite dimensional case J, R, B, P, S, N are *operators* that map into appropriate function spaces.

- Port-Hamiltonian systems generalize *Hamiltonian systems*.
- *Conservation of energy* replaced by *dissipation inequality*

$$\mathcal{H}(x(t_1)) - \mathcal{H}(x(t_0)) \leq \int_{t_0}^{t_1} y(t)^T u(t)\, dt,$$

- Port-Hamiltonian systems are closed under *power-conserving interconnection*. Models can be coupled in *modularized* way.
- Minimal constant coefficient pH systems are *stable and passive.*
- Port-Hamiltonian structure allows to preserve physical properties in *Galerkin projection, model reduction*.
- Physical properties encoded in *algebraic structure* of coefficients and in *geometric structure* associated with flow.
- Systems are *easily extendable* to incorporate multiphysics components: chemical reaction, thermodynamics, electrodynamics, mechanics, etc. *Open/closed systems.*

Definition (C. Beattie, V. M., H. Xu, H. Zwart 2017)

A linear variable coefficient DAE of the form

$$\begin{aligned} E\dot{x} &= [(J-R)Q - EK]\,x + (B-P)u, \\ y &= (B+P)^T Qx + (S+N)u, \end{aligned}$$

with $E, A, Q, R = R^T, K \in C^0(\mathbb{I}, \mathbb{R}^{n,n})$, $B, P \in C^0(\mathbb{I}, \mathbb{R}^{n,m})$, $S + N \in C^0(\mathbb{I}, \mathbb{R}^{m,m})$ is called *port-Hamiltonian DAE (pHDAE)* if :

i) $\mathcal{L} := Q^T E \frac{d}{dt} - Q^T JQ - Q^T EK$ is *skew-adjoint*.

ii) $Q^T E = E^T Q$ is bounded from below by a constant symmetric H_0.

iii) $W := \begin{bmatrix} Q^T RQ & Q^T P \\ P^T Q & S \end{bmatrix} \geq 0, \; t \in \mathbb{I}$.

New Hamiltonian defined as $\mathcal{H}(x) := \frac{1}{2} x^T Q^T Ex : C^1(\mathbb{I}, \mathbb{R}^n) \to \mathbb{R}$.

- Nonlinear formulation available.
- *Dissipation inequality* still holds.
- pHDAE systems closed under *power-conserving interconnection*. Models can be coupled in *modularized* way.
- pHDAE structure invariant under time varying basis changes.
- Canonical forms in constant and variable coefficient case.
- Port-Hamiltonian structure preserved under constraint preserving *Galerkin projection, model reduction*.
- Representation is very robust to structured perturbations

- C. Beattie, V. M., H. Xu, and H. Zwart, *Linear port-Hamiltonian descriptor systems*. https://arxiv.org/pdf/1705.09081.pdf
- C. Beattie, V. Mehrmann, and P. Van Dooren, *Robust port-Hamiltonian representations of passive systems*. http://arxiv.org/abs/1801.05018
- N. Gillis, V. Mehrmann, and P. Sharma, *Computing nearest stable matrix pairs*. Numerical Linear Algebra with Applications, 2018. https://arxiv.org/pdf/1704.03184.pdf
- C. Mehl, V. M., and M. Wojtylak, *Linear algebra properties of dissipative Hamiltonian descriptor systems*. http://arxiv.org/abs/1801.02214
- L. Scholz, Condensed Forms for linear Port-Hamiltonian Descriptor Systems. Preprint 09-2017, Institut f. Mathematik, TU Berlin, 2017.

Current work:

- Rewrite models as networks of pHDAE systems. Energy networks (gas, power).
- Design new structure preserving data assimilation and model reduction techniques.
- Analysis, numerical simulation, discretization, adaptivity in space, time, model hierarchy.
- Optimal control techniques based on pHDAE structure.

- P. Domschke, A. Dua, J.J. Stolwijk, J. Lang, and V. M., *Adaptive Refinement Strategies for the Simulation of Gas Flow in Networks using a Model Hierarchy*, *Electronic Transactions Numerical Analysis*, 2018.
- H. Egger, T. Kugler, B. Liljegren-Sailer, N. Marheineke, and V. M., *On structure preserving model reduction for damped wave propagation in transport networks*, SIAM Journal Scientific Computing, Vol. 40, A331–A365, 2018. V. M., R. Morandin, S. Olmi, and E. Schöll, *Qualitative Stability and Synchronicity Analysis of Power Network Models in Port-Hamiltonian form*, 2017. https://arXiv:1712.03160
- V. M., M. Schmidt, and J. Stolwijk, *Model and Discretization Error Adaptivity within Stationary Gas Transport Optimization*, Vietnam J. of Mathmatics, 2018. http://arxiv.org/abs/1712.02745, 2017.
- J.J. Stolwijk and V. M. *Error analysis and model adaptivity for flows in gas networks*. Anal. Stiintifice ale Univ. Ovidius Constanta. Seria Matematica, 2018.

Conclusions

- DAEs (pHDAEs) are a very important mathematical concept.
- They are ideal to model complex multiphysics systems.
- Mathematical theory and development of numerical methods triggered by real world applications.
- Optimal control theory still needs further attention.
- Modeling, simulation and optimization techniques need to be improved.
- A new paradigm is evolving.

The heritage of William T. Reid is very much alive!

Thank you very much
for receiving the Reid Prize
for your attention
and my sponsors for their support

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Details: http://www.math.tu-berlin.de/?id=76888