Snapshot Attractors and the Transition to Extensive Chaos in Large Mean Field Coupled Systems

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Reference: arXiv 1412.3803

Summary

We consider an example of a globally coupled system of many ($N >> 1$) identical Landau-Stuart oscillators.

Our main points are as follows:

(1) There are two types of attractors in our system: Low dimensional "clumped states" and high dimensional "extensively chaotic" states.

(2) By extensively chaotic we mean that for large N

(attractor dimension) ~ N , and

(# positive Lyapunov exponents) ~ N

(3) Extensive chaos for our system be effectively analyzed by approximating it by snapshot attractors.

(4) We examine dynamical transitions (bifurcations) between extensive chaos and low dimensional clump states.

Model: Landau-Stuart Oscillators

- A system of mean-field-coupled Landau-Stuart oscillators

$$\dot{W}_j = W_j - (1 + iC_2)|W_j|^2 W_j + K(1 + iC_1)(\bar{W} - W_j)$$
$$j = 1, 2, \ldots N,$$

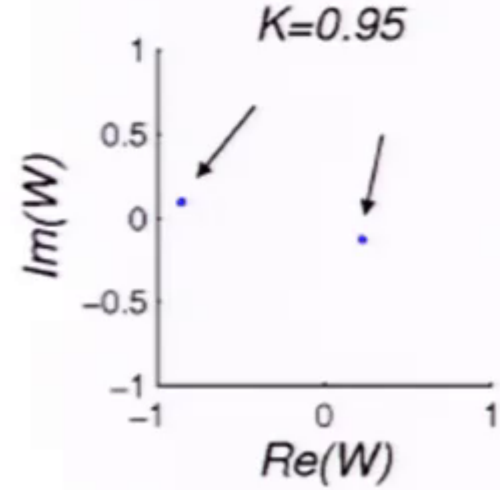
- Since the W's are complex, this is a 2N-dimensional dynamical system,

$$W_j(t) = \rho_j(t) e^{i\theta_j(t)}$$

- The mean field: $\bar{W} = N^{-1} \sum_j W_j$
- We fix C_1=-7.5, C_2=0.90 and examine how the dynamics changes as the coupling constant K varies.

Attractors at intermediate *K* values
(1 > K >0.65)

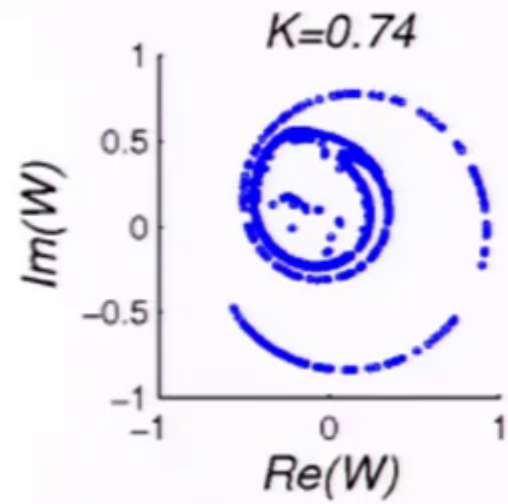
- **Two clump state:**



All oscillators have either one of two *W* values which rigidly rotate in the *W*-plane.
Low dimensional (a fixed point in the rotating frame).

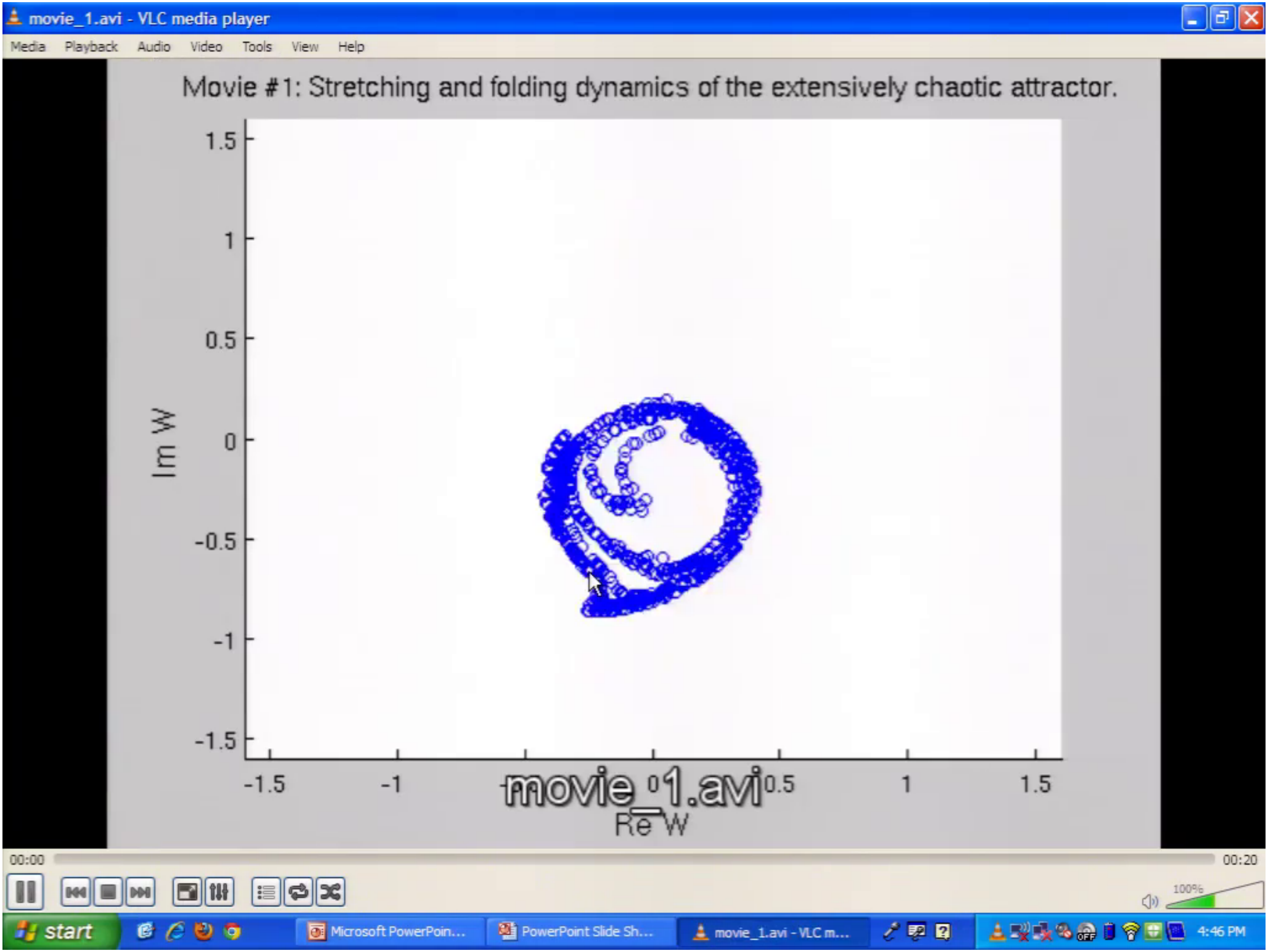
- **Extensive chaos**

The fractal pattern of points in the *W*-plane varies chaotically in time as does the mean field $\overline{W}(t)$.



Matthews, Mirollo, Strogatz, Physica D (1991).
Hakim, Rappel, Phys Rev A (1992).
Nakagawa, Kuramoto, Physica D (1995).

Movie # 1: Stretching and folding dynamics of the extensively chaotic attractor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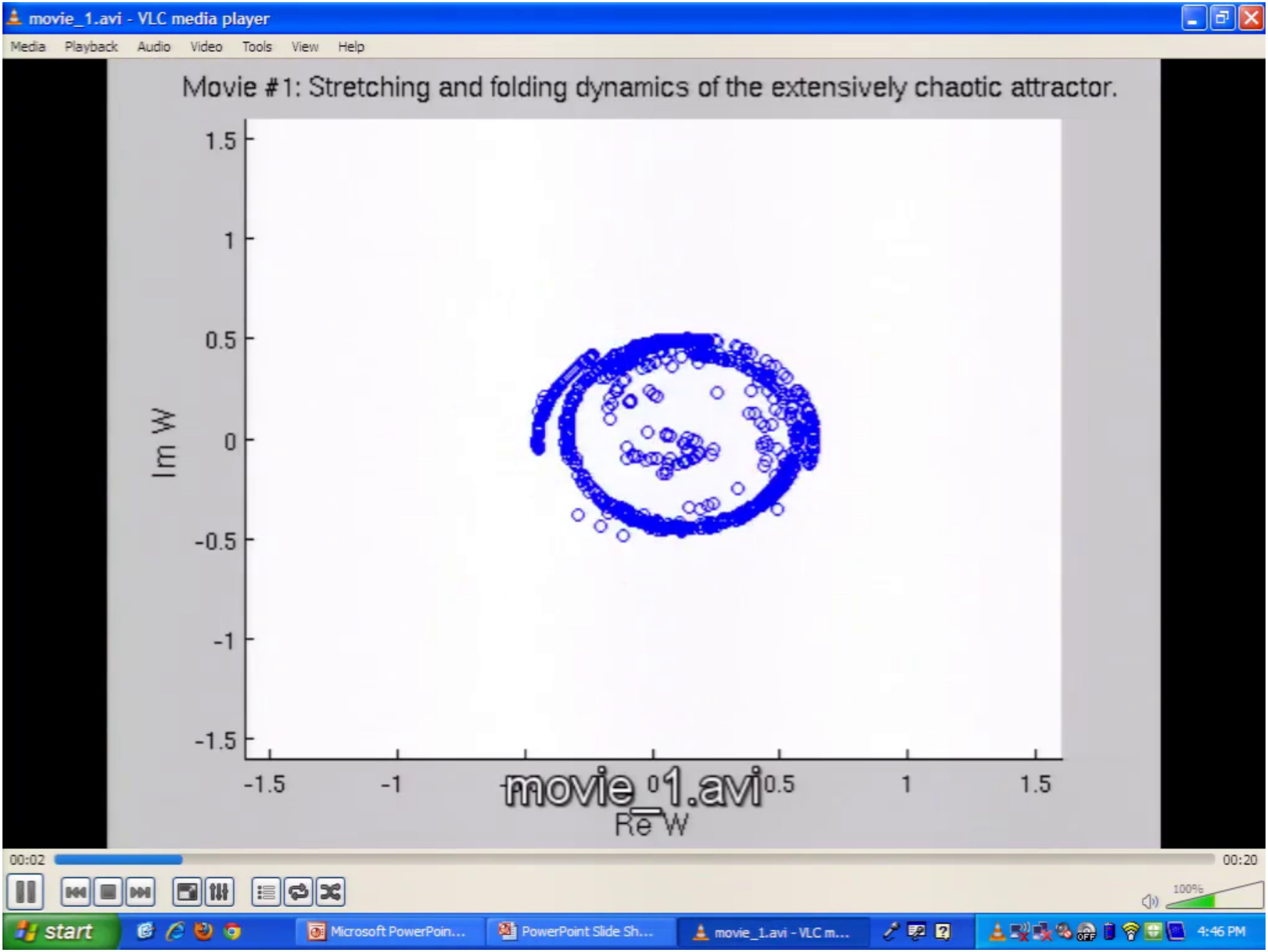
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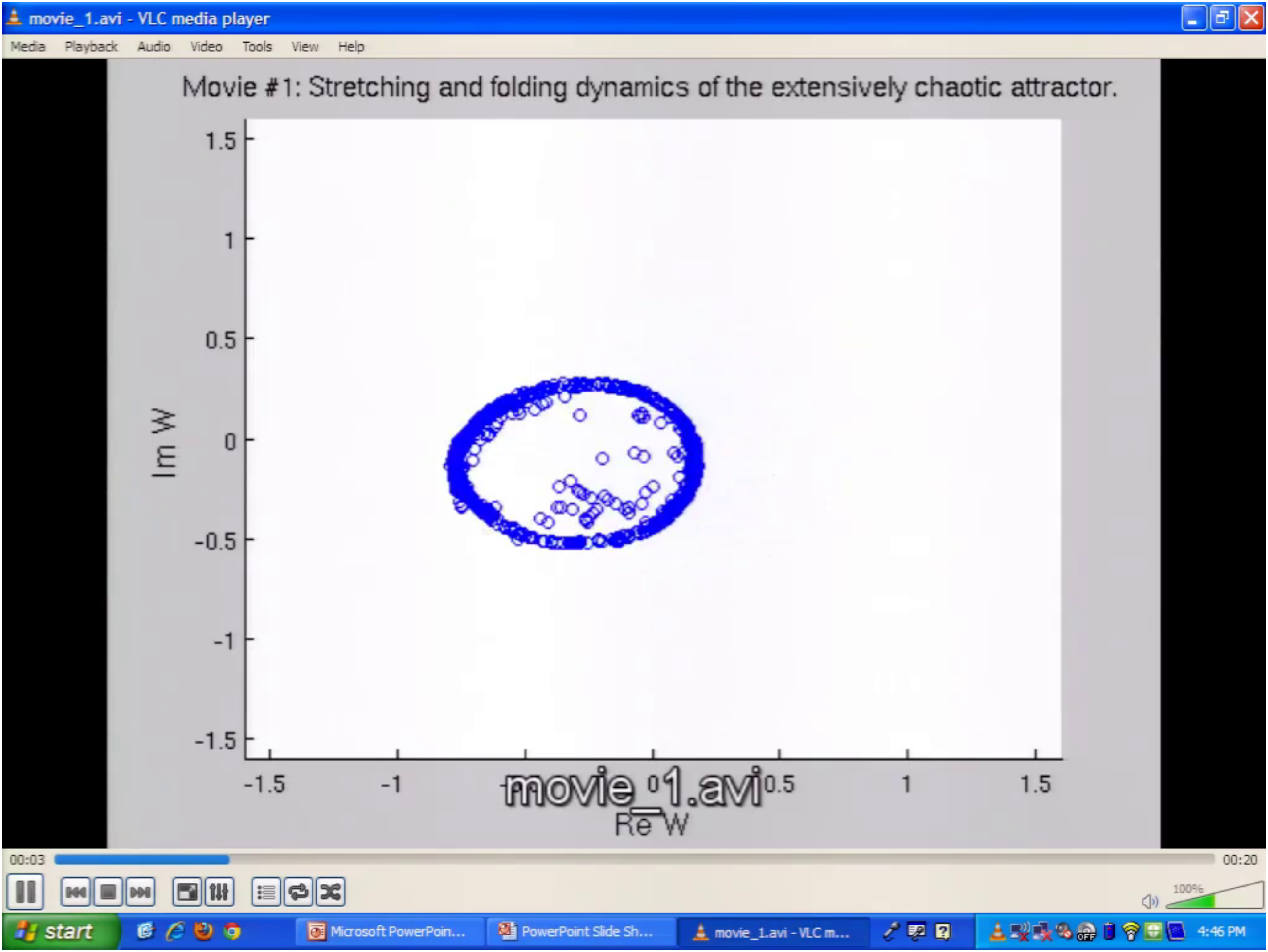
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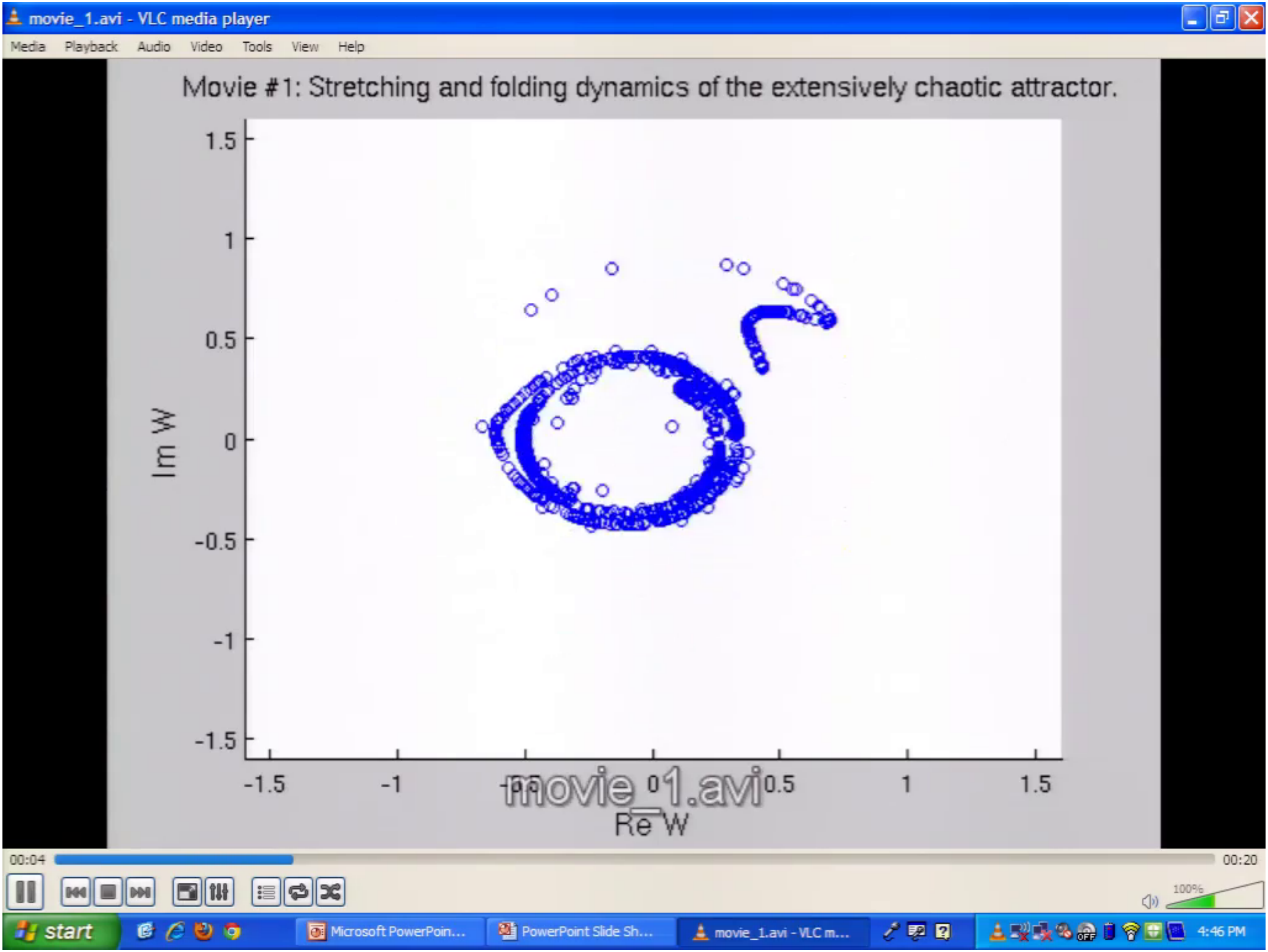
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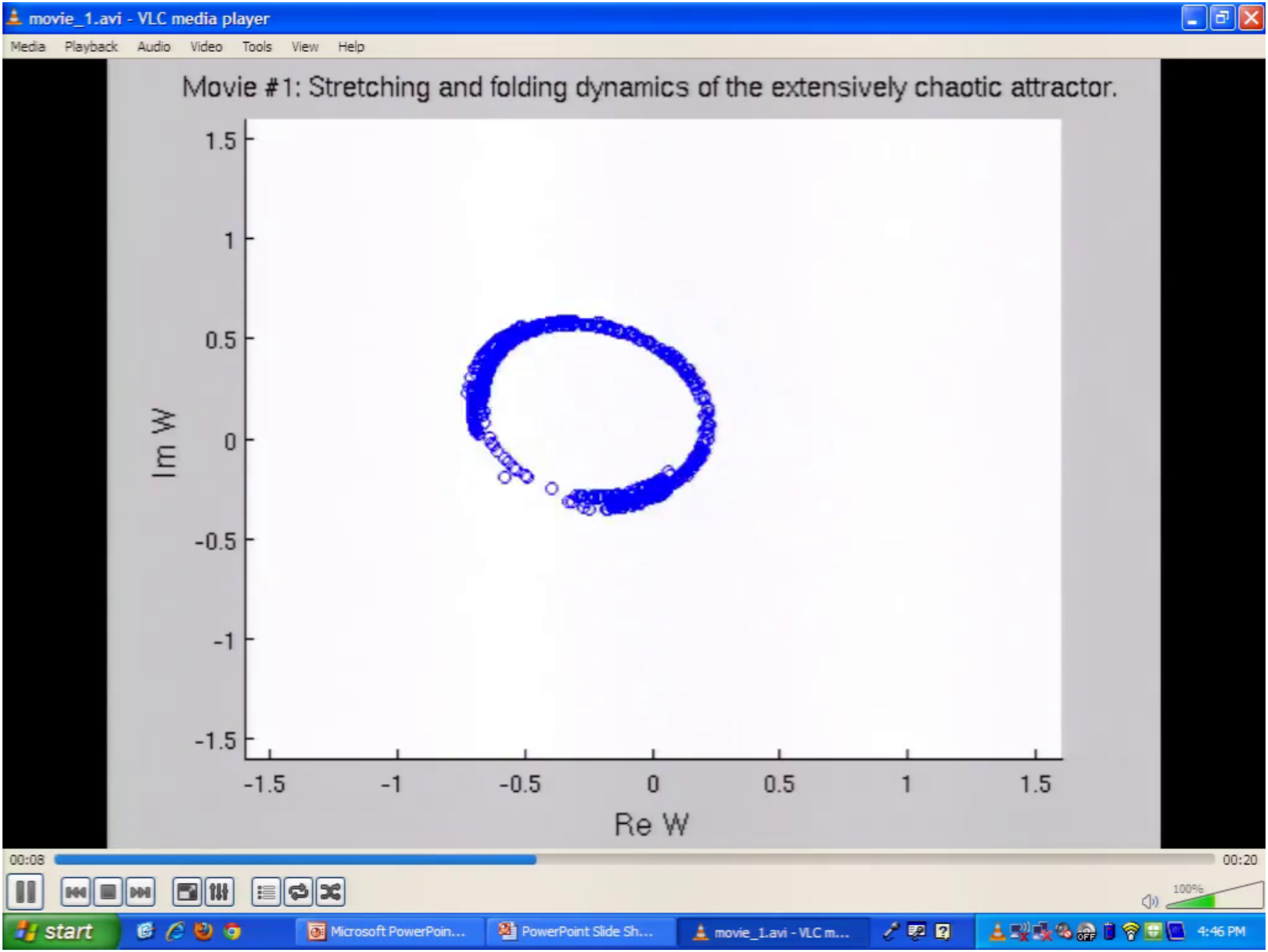

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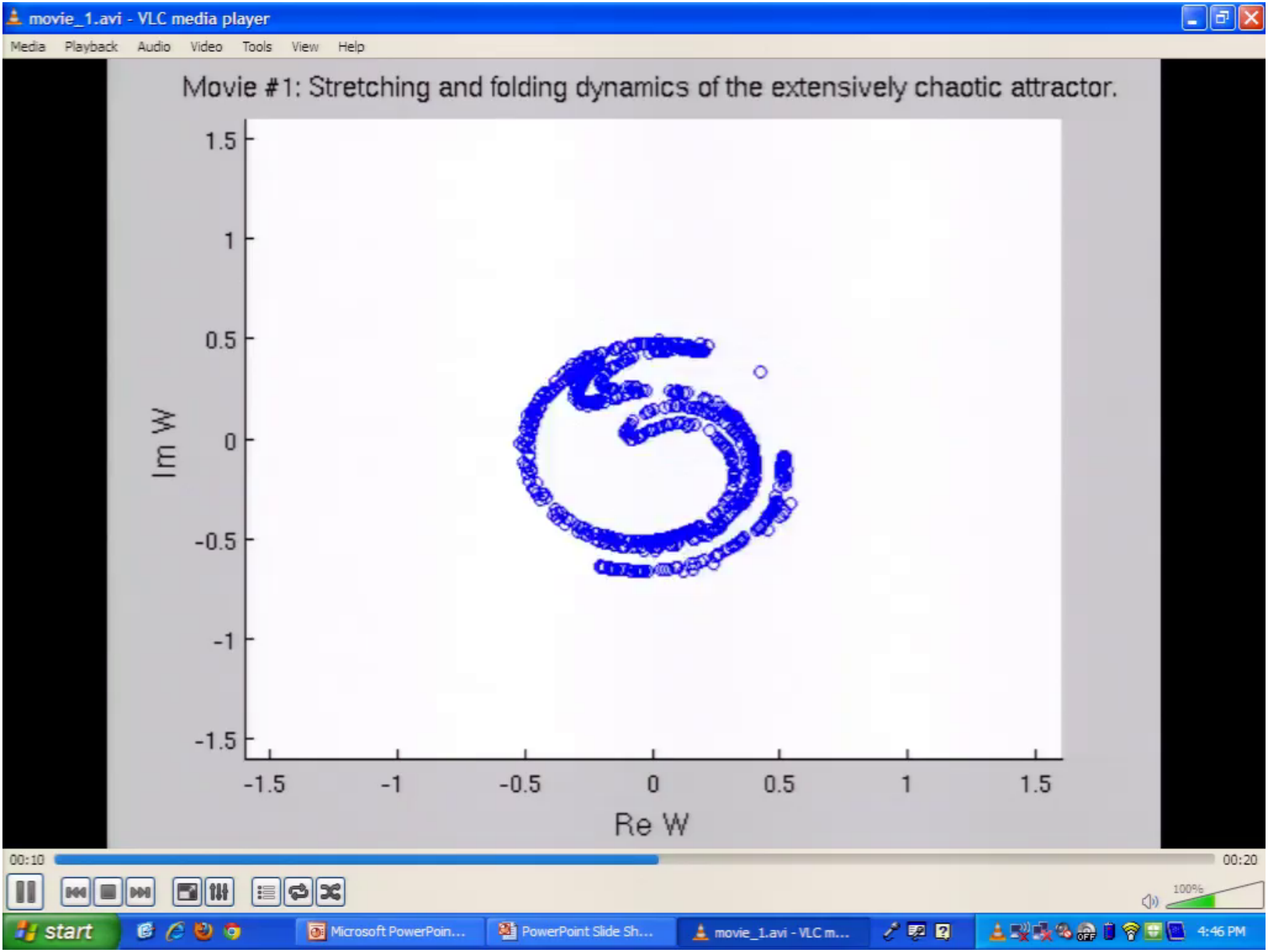
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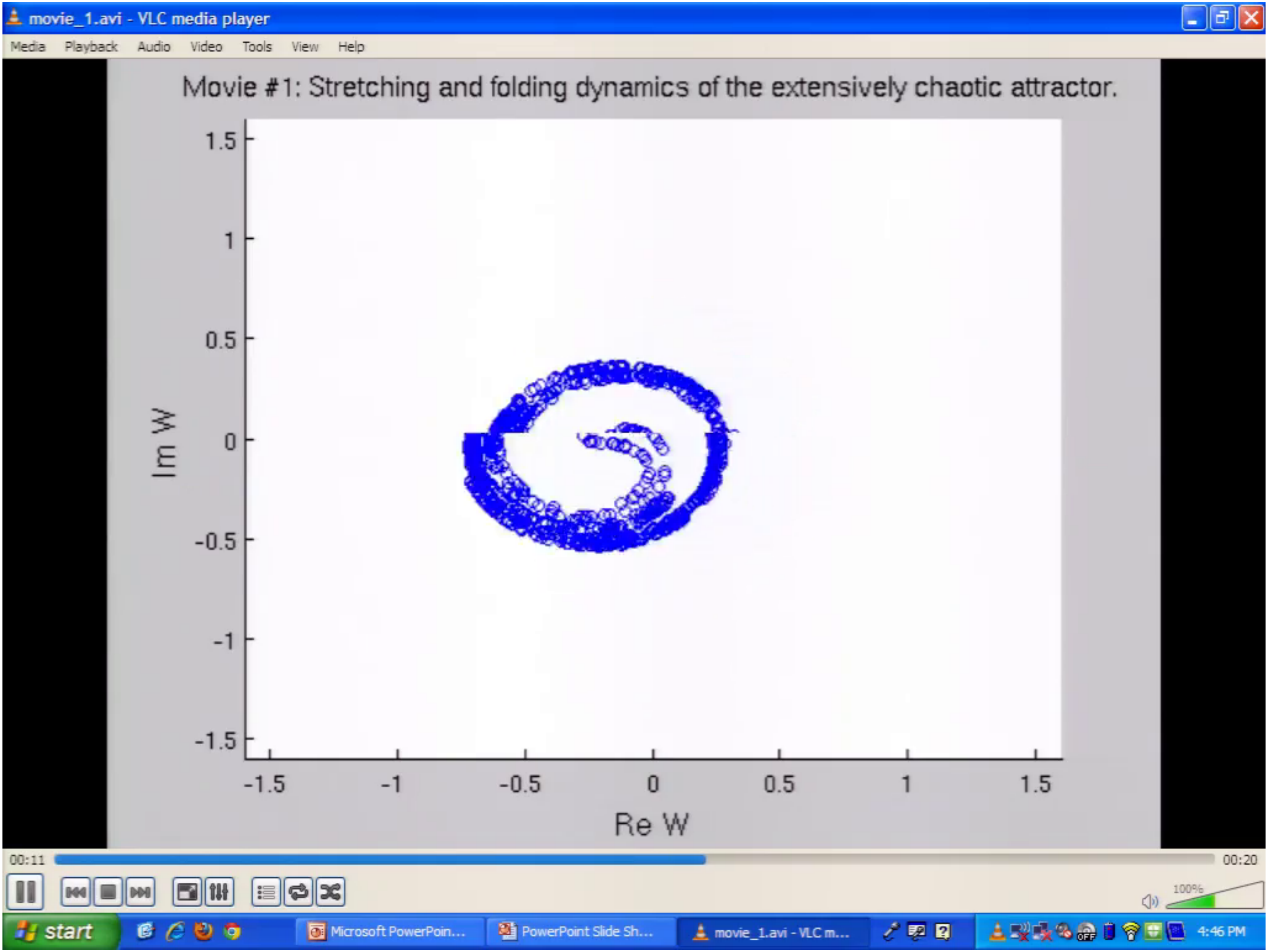
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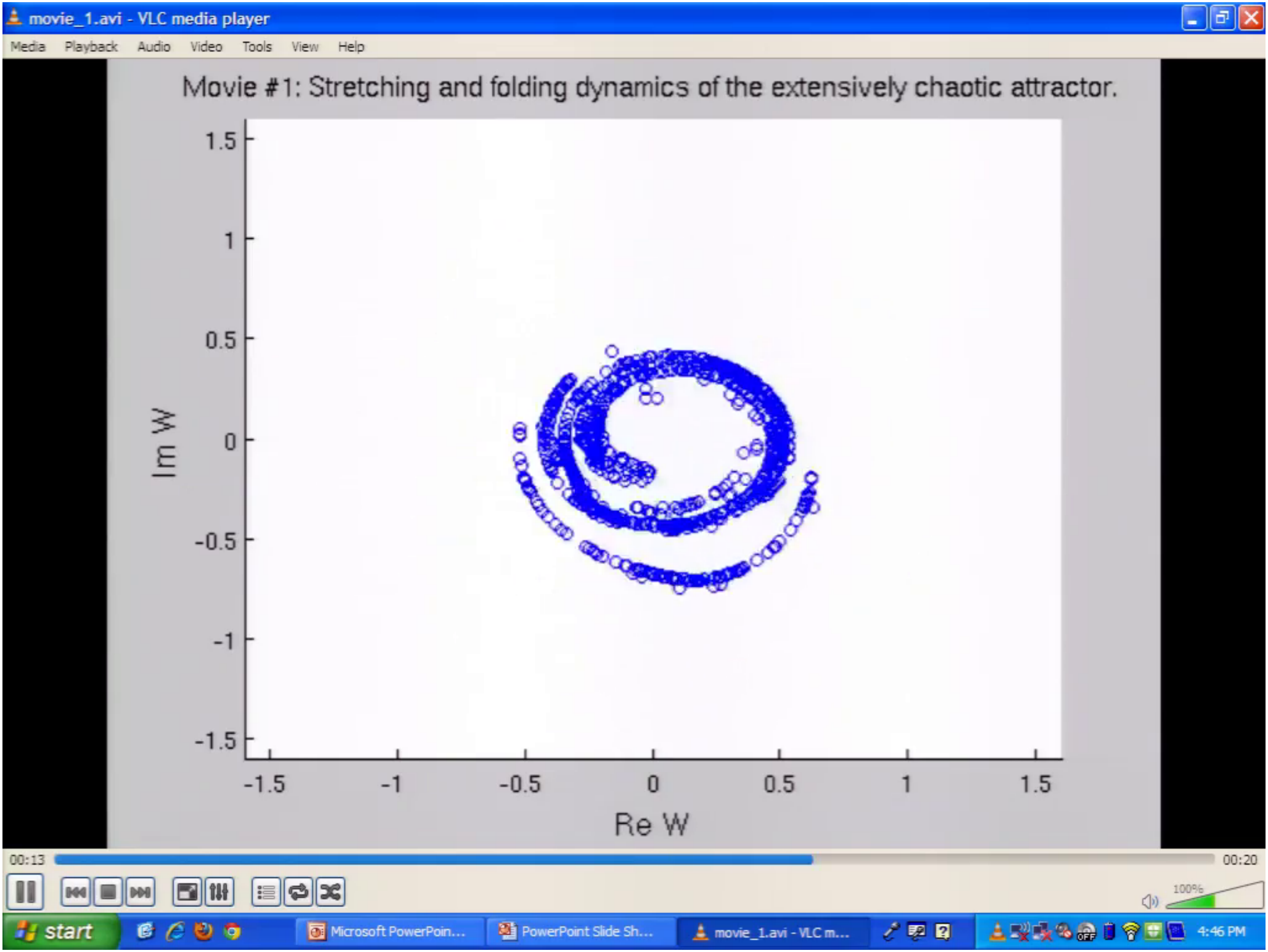

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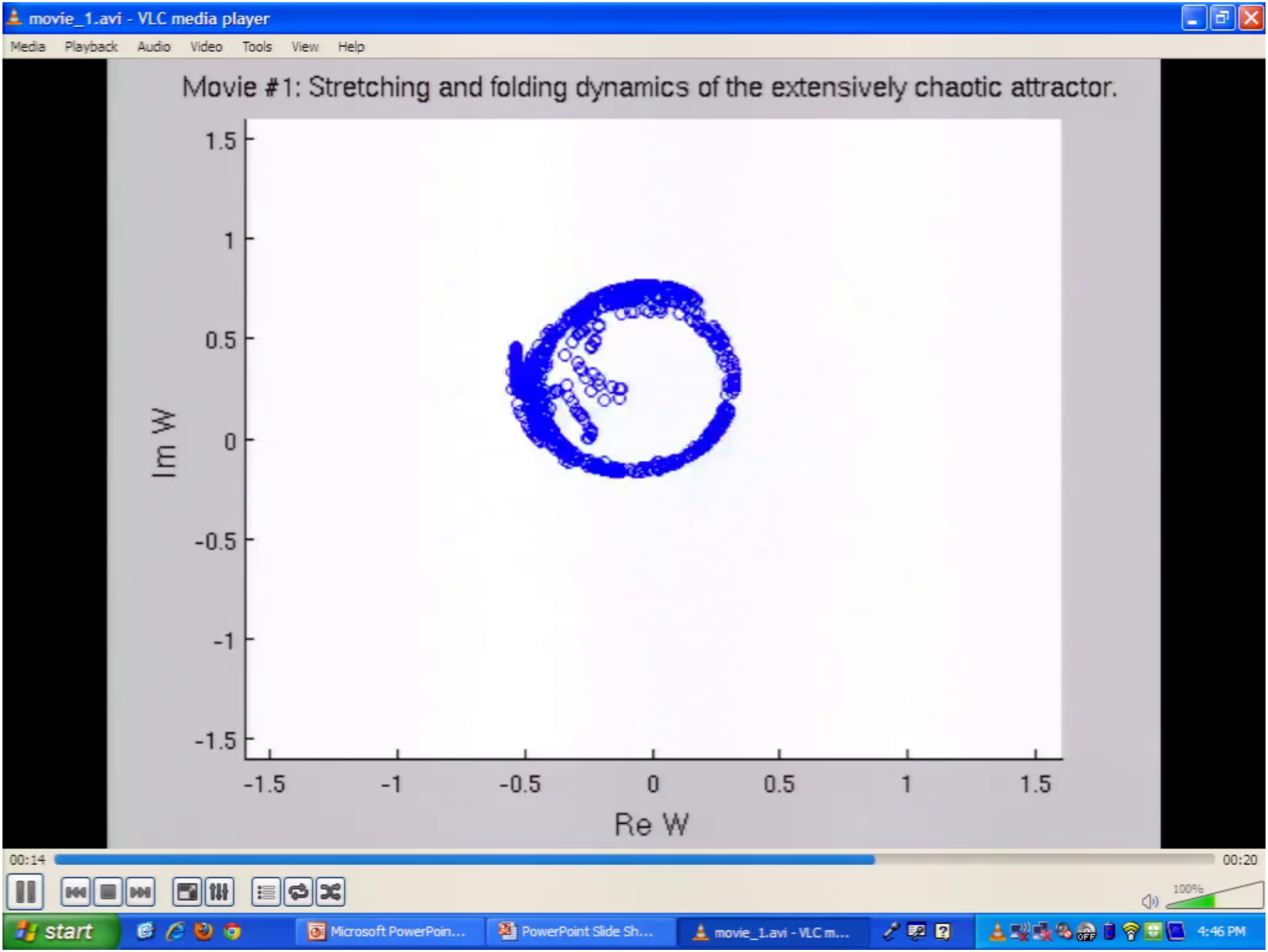
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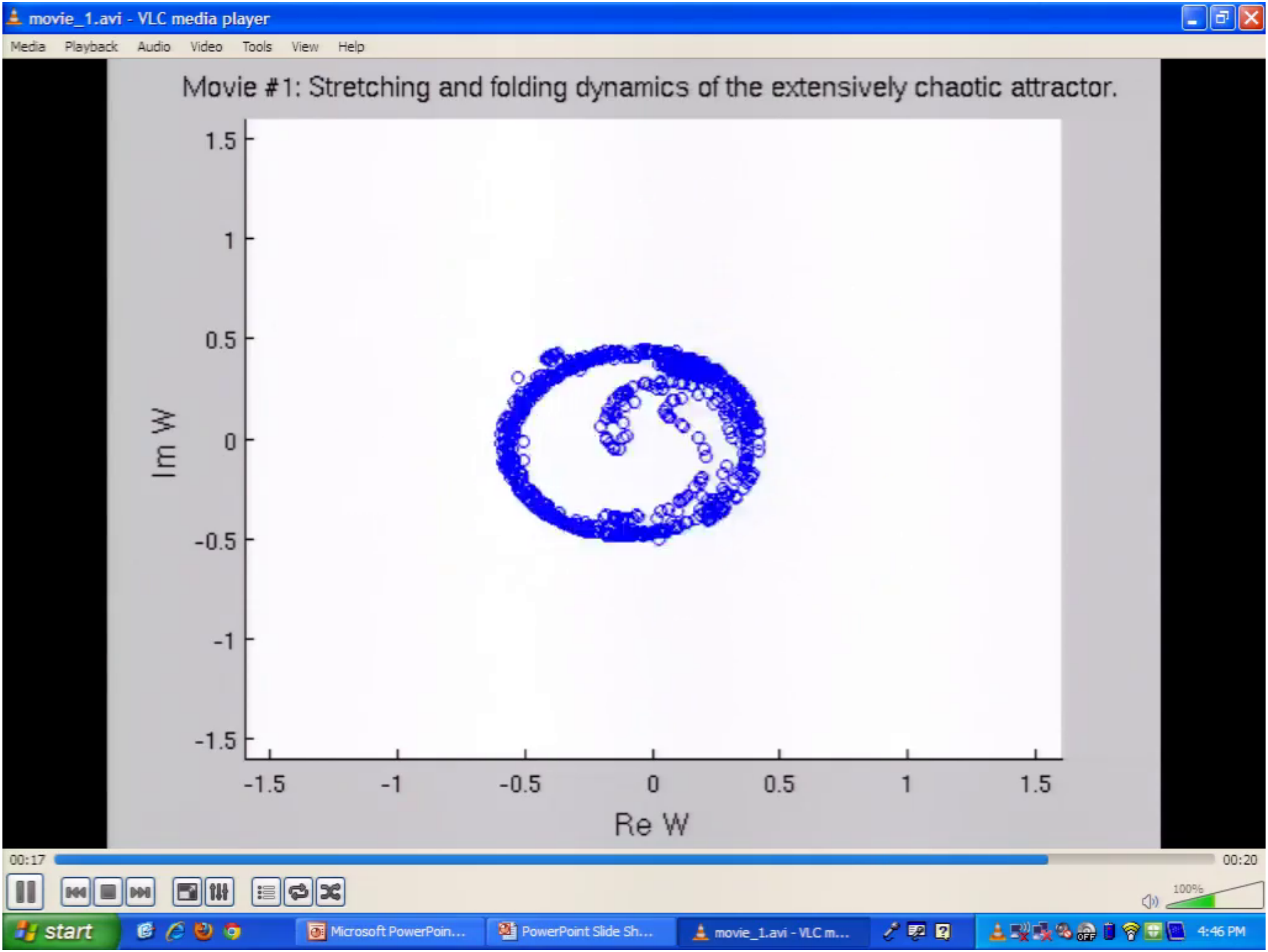
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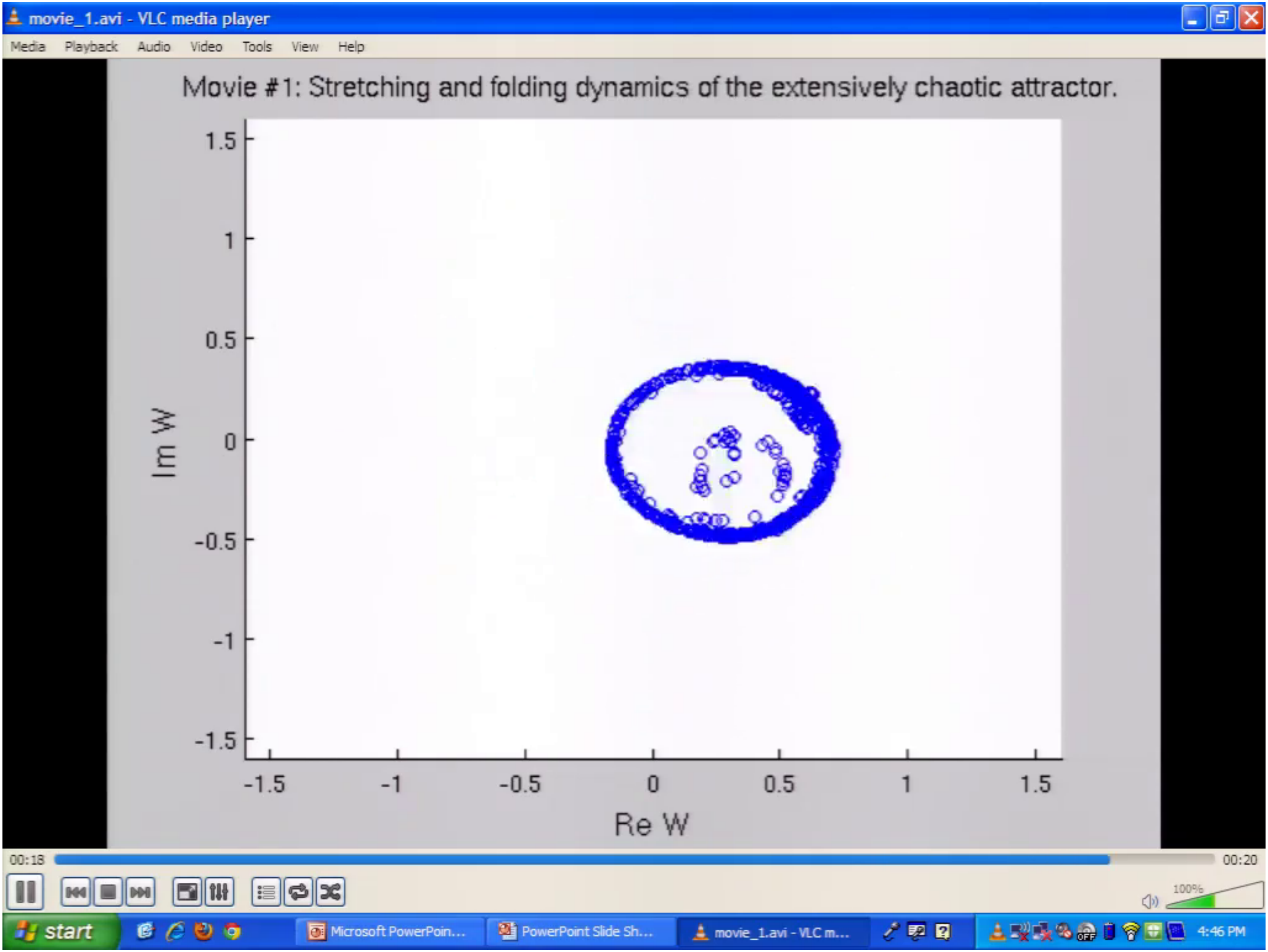
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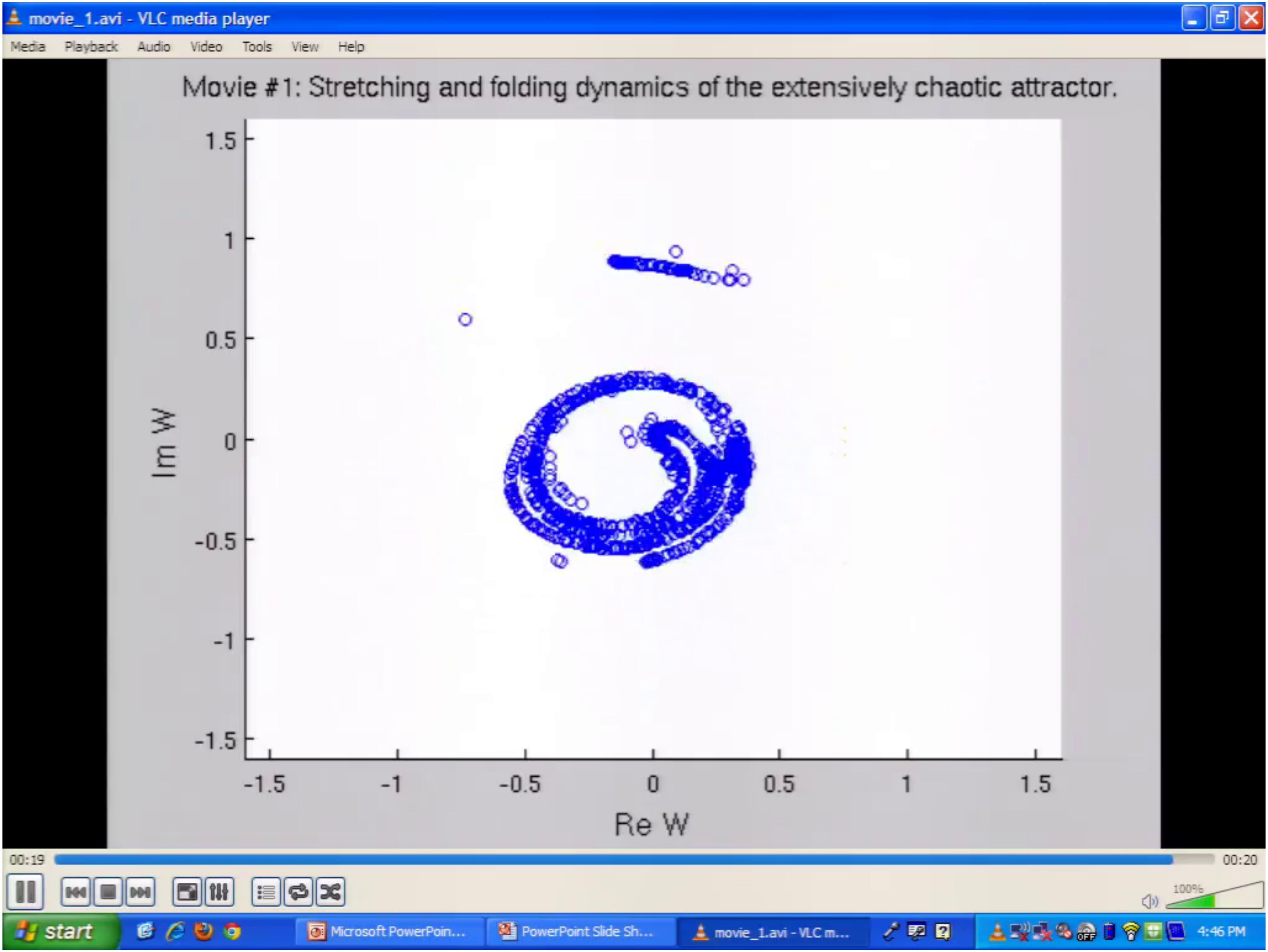
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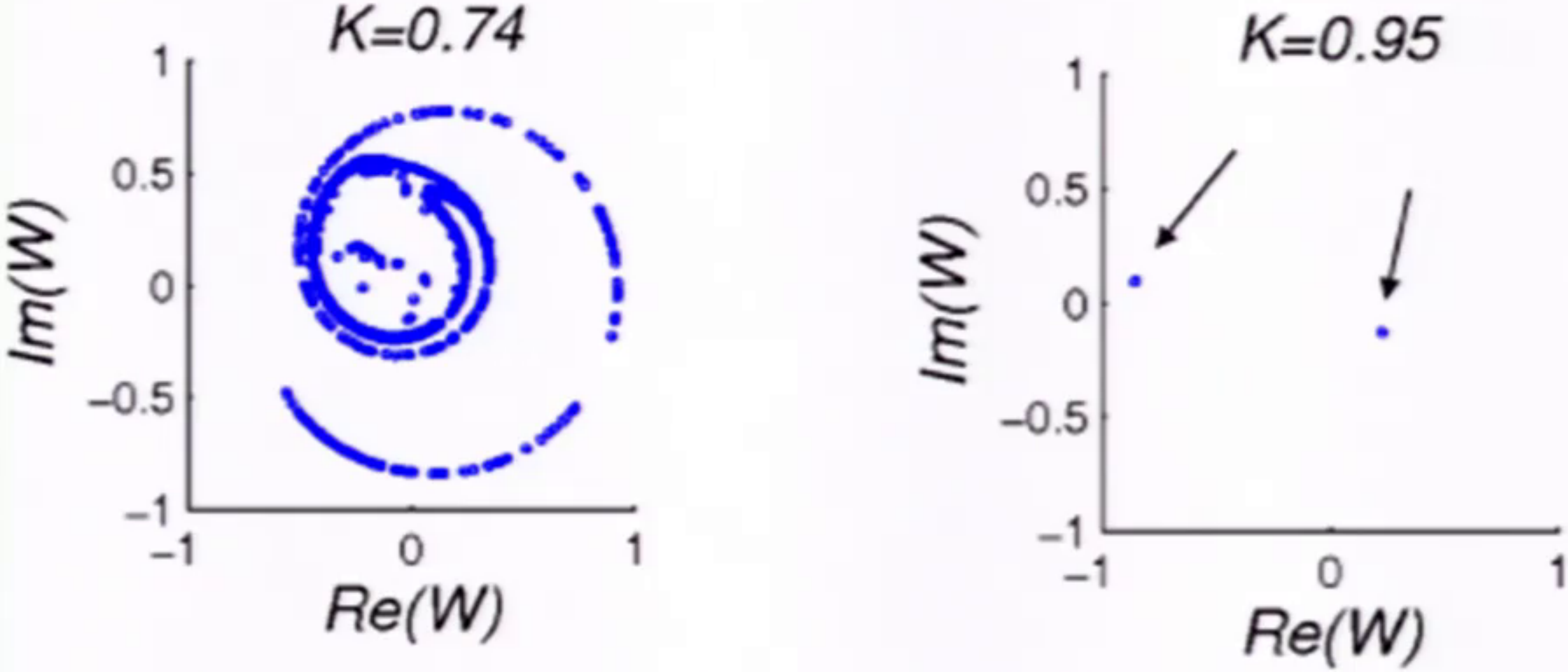
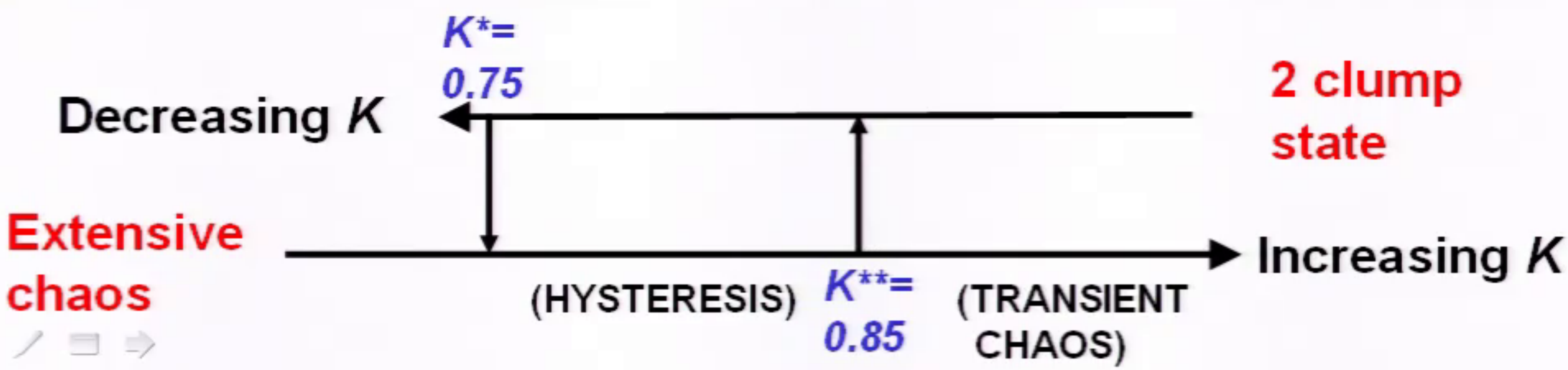


How do attractors evolve with adiabatic variation of K?



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THE TRANSITION TO EXTENSIVE CHAOS AS *K* DECREASES THROUGH *K**~ *0.75*

- As *K* decreases through *K** the clumps appear to explode and there is a relatively rapid transition to extensive chaos (i.e., no long transients of irregularly varying length as we will see occurs as *K* increases past *K***).
- We have a fairly good understanding of the mechanism of this transition. See our paper, arXiv 1412.3803, for this material.

Movie # 2: Explosive transition from clumps to extensive chaos when *K* decreases through *K**

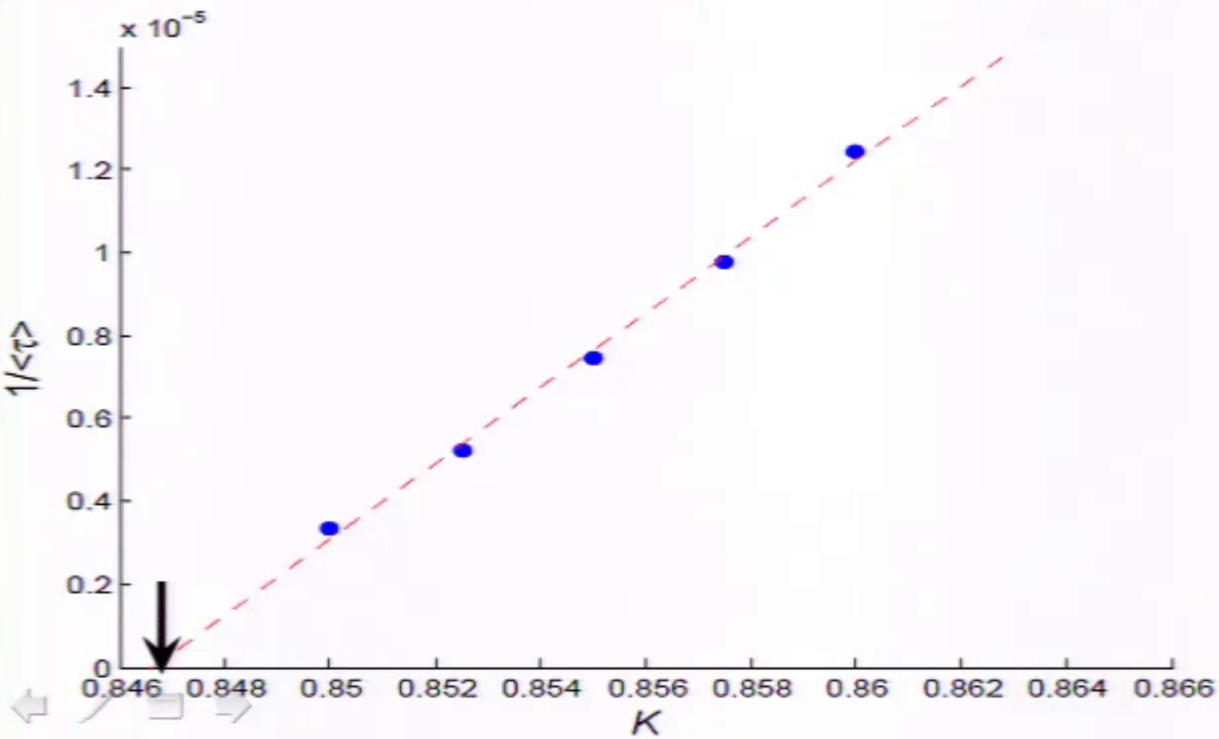
THE TRANSITION TO CLUMPS AS *K* INCREASES PAST *K*** ~ *0.85*

Above $K=K^{**}$ the extensively chaotic attractor is replaced by a chaotic transient.

At fixed $K>K^{**}$, for slightly different conditions (e.g., initial conditions) the lifetime of transient extensively chaotic behavior varies in an apparently random manner.

This average lifetime $<\tau>$ diverges as K approaches K^{**} from above.

This behavior is reminiscent of crisis type transitions in low dimensional chaotic systems (Grebogi, Ott, Yorke, Physica D [1983]).



$$1/<\tau> \sim (K-K^{**})^{\gamma}$$

The numerics are consistent with this formula with a critical exponent value $\gamma = 1$, and K^{**} about 0.847.

Movie # 3: Rapid evolution from an initial condition with oscillator states randomly sprinkled in $|W|<1$ into a transient extensively chaotic state at $K=0.86 > K^{**}$

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Movie # 4: Transition to a two clump state at the end of the extensively chaotic transient state initiated in movie #3

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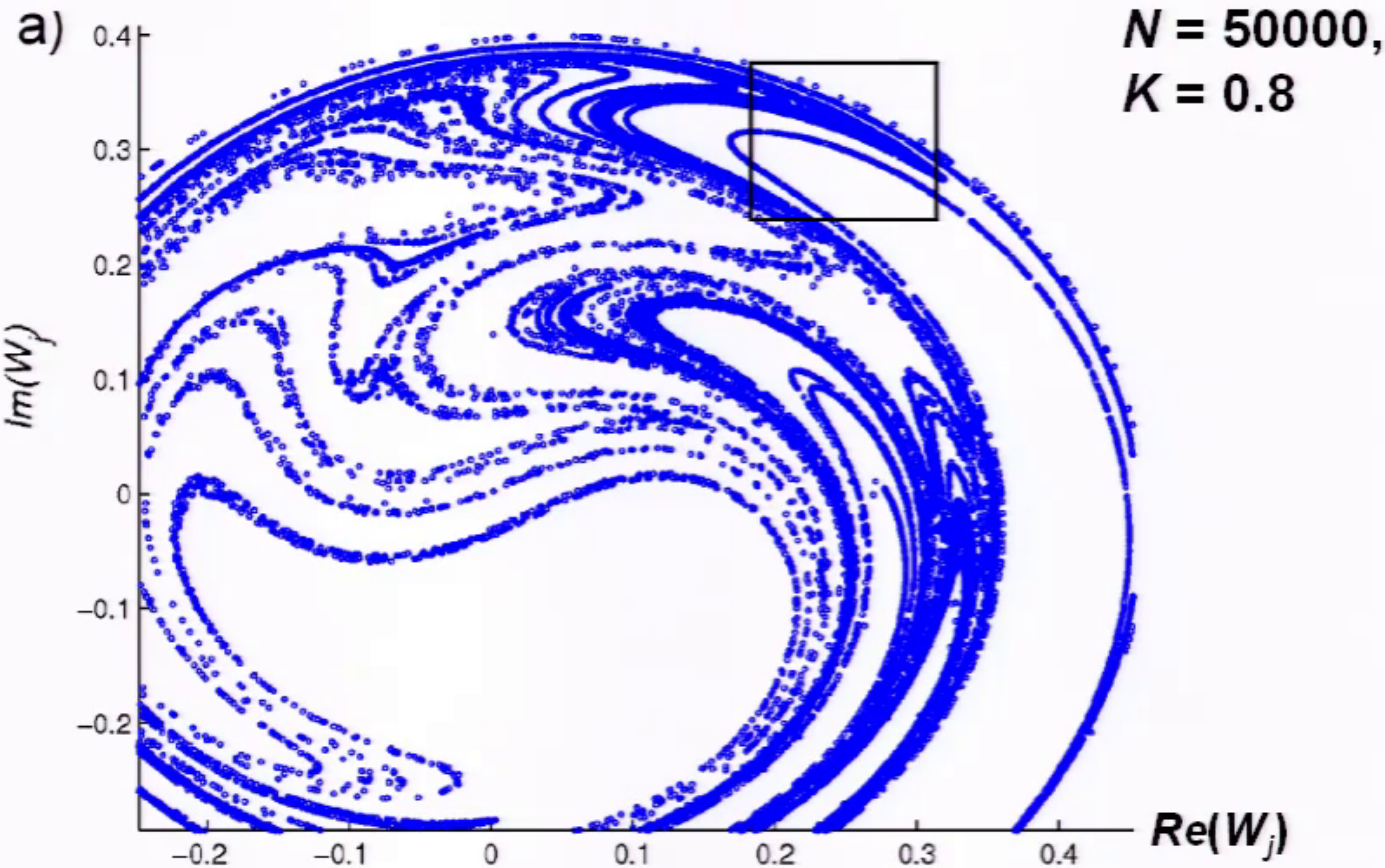
Rest of this talk

Discussion of the character of the extensively chaotic state and why snapshot attractors are relevant to it.

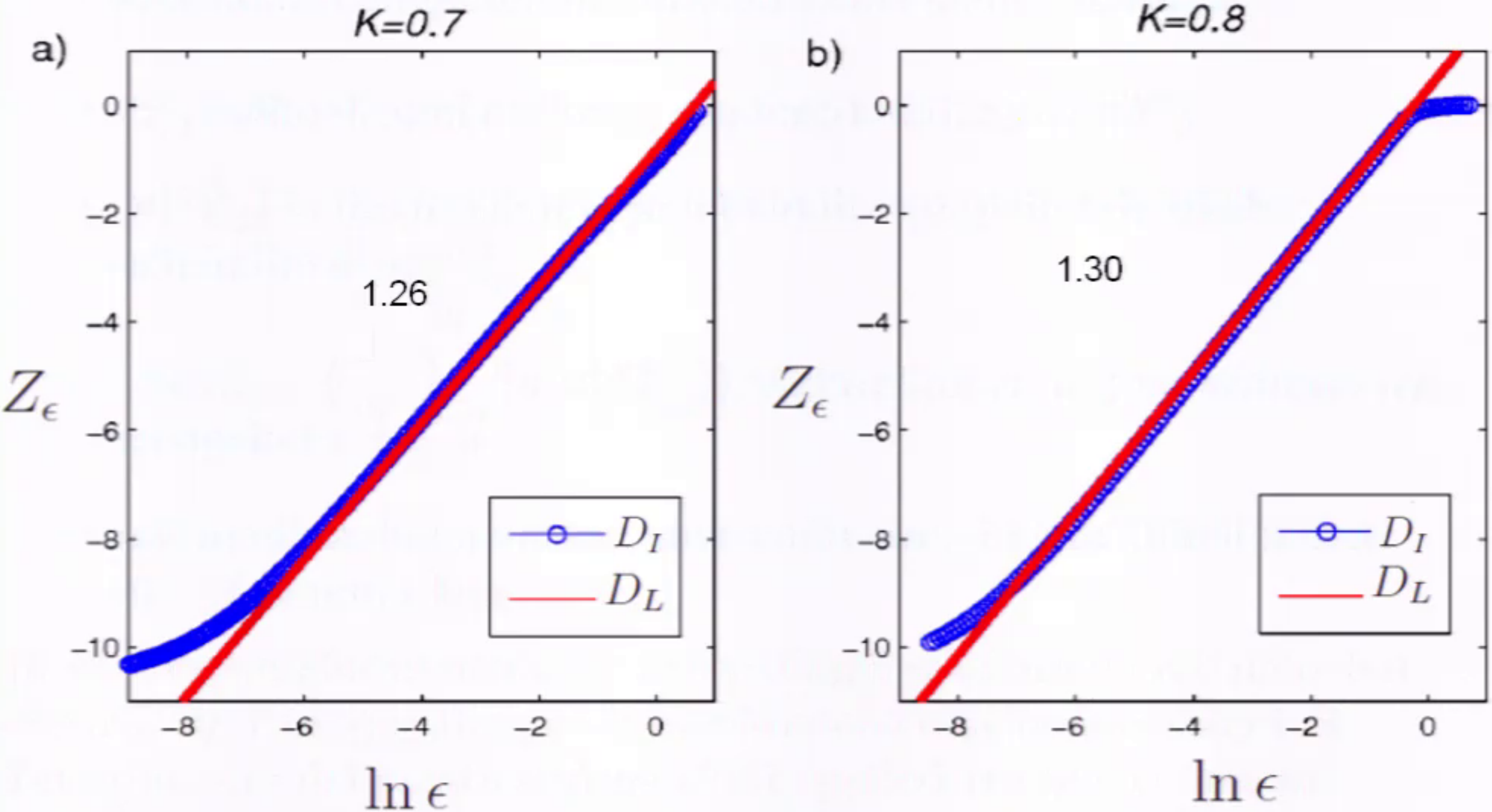
Conclusions.

Fractal distribution of snapshots

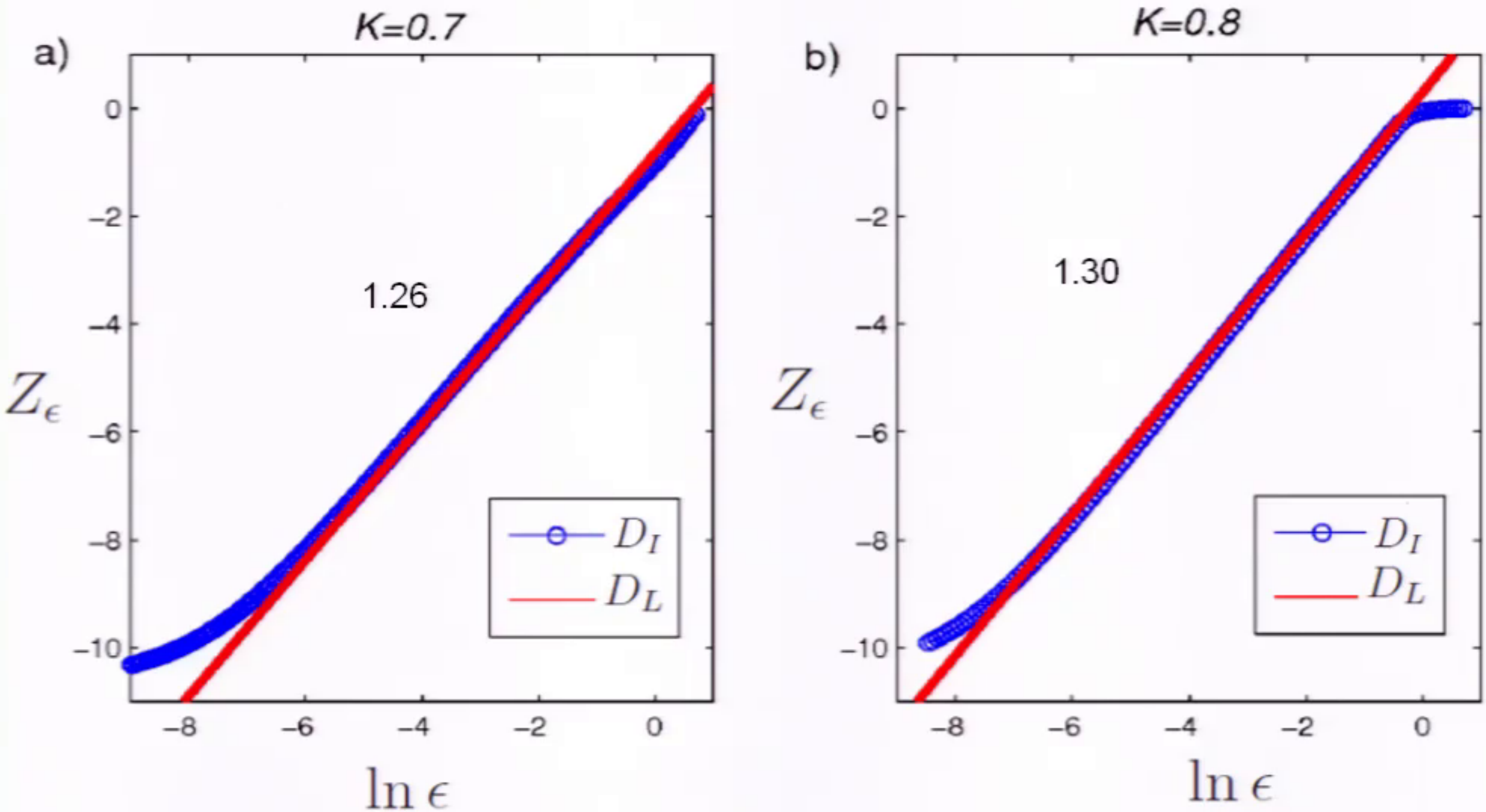


CLAIM: In the limit *N* going to infinity, the fractal dimension *D* of the attractor in the full *2N* dimensional system state space is such that *D/N* approaches the fractal dimension of the snapshots.

Result of Dimension Calculation



Result of Dimension Calculation



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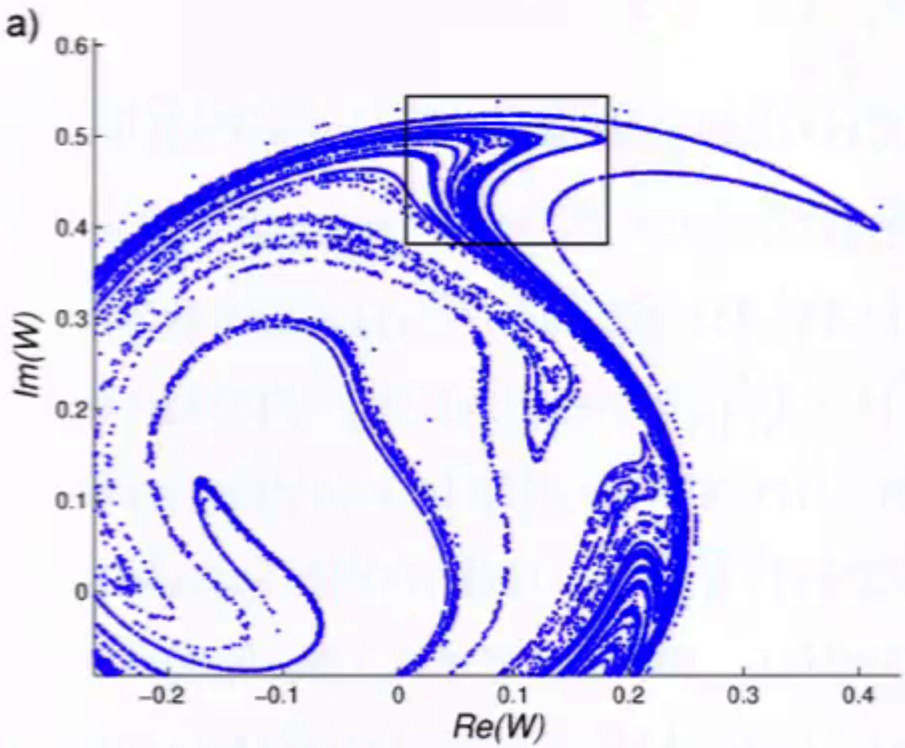
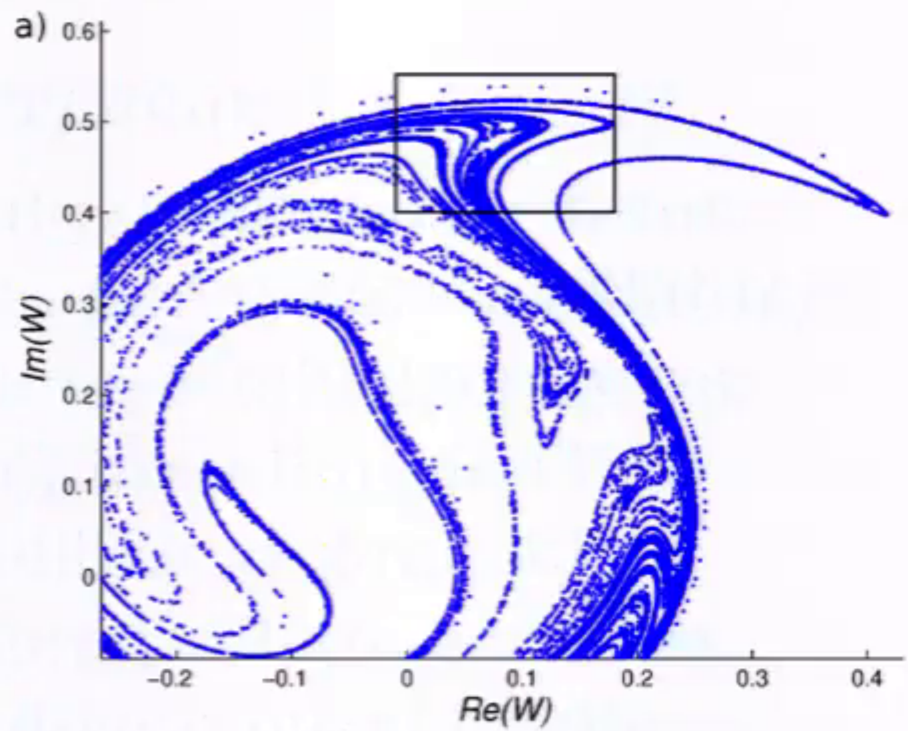
Numerical validation of this perspective

Self-consistent mean field



External mean field



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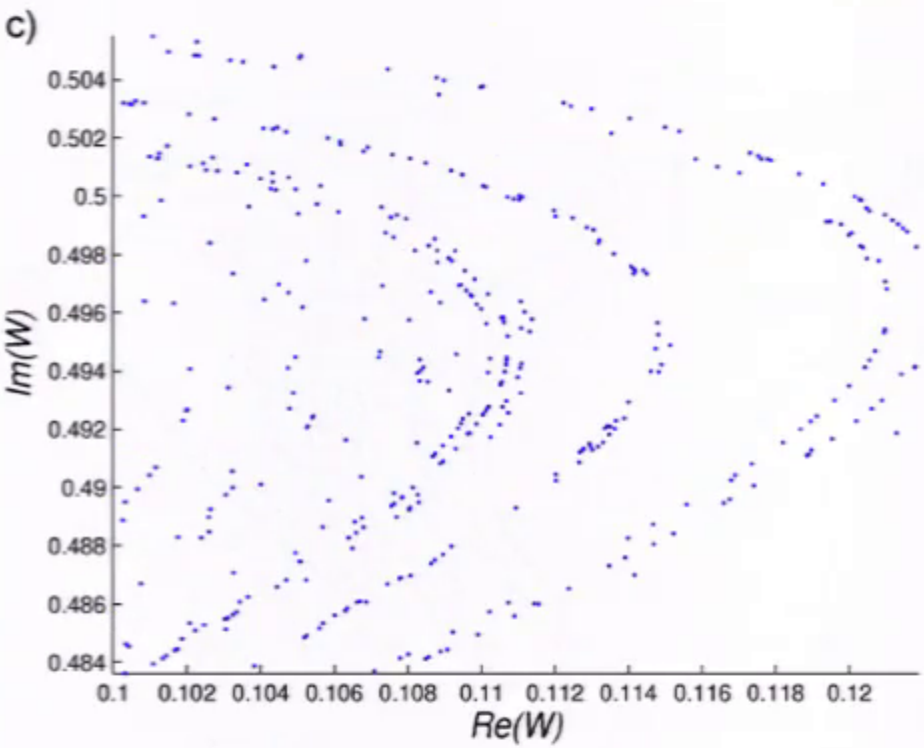
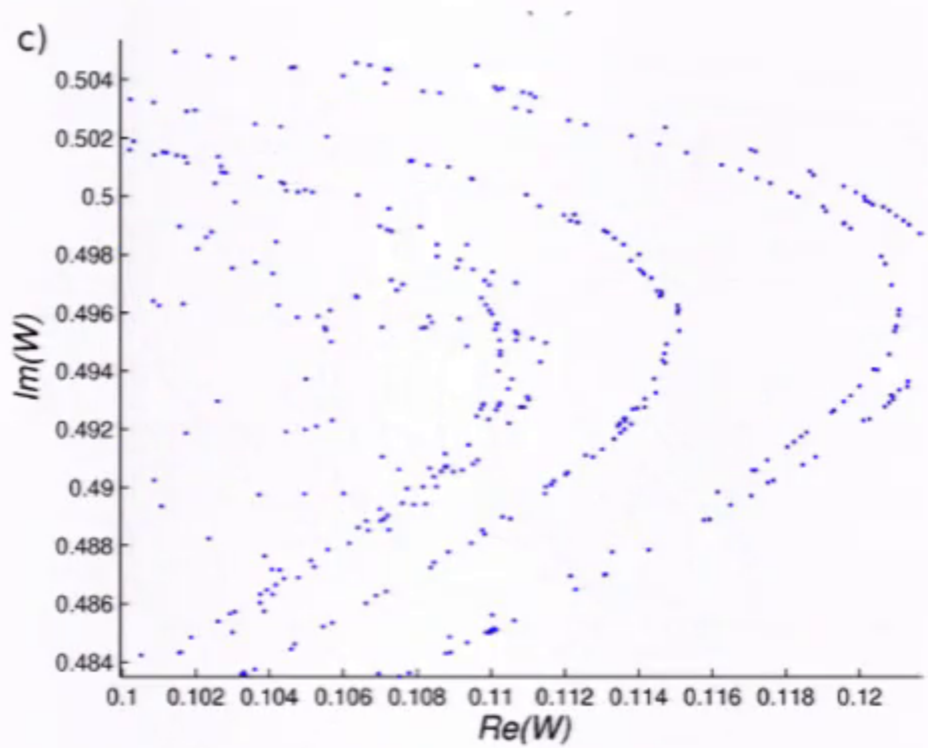
Further Blow-up

Self-consistent mean field



External mean field



Lyapunov Dimension
(Previous talk by Young)

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- Consider an M dimensional system with Lyapunov exponents

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M$$

- Q is the largest integer such that $\sum_{q=1}^{Q} \lambda_q \geq 0.$
- Lyapunov dimension: $D_L = Q + \frac{1}{|\lambda_{Q+1}|} \sum_{q=1}^{Q} \lambda_q.$

Kaplan-Yorke conjecture: $D_I = D_L$

J. L. Kaplan and J. A. Yorke (1979)

For random dynamical systems: L. S. Young and F. Ladrappier (1988)

Application of Kaplan-Yorke Dimension Formula to Snapshots

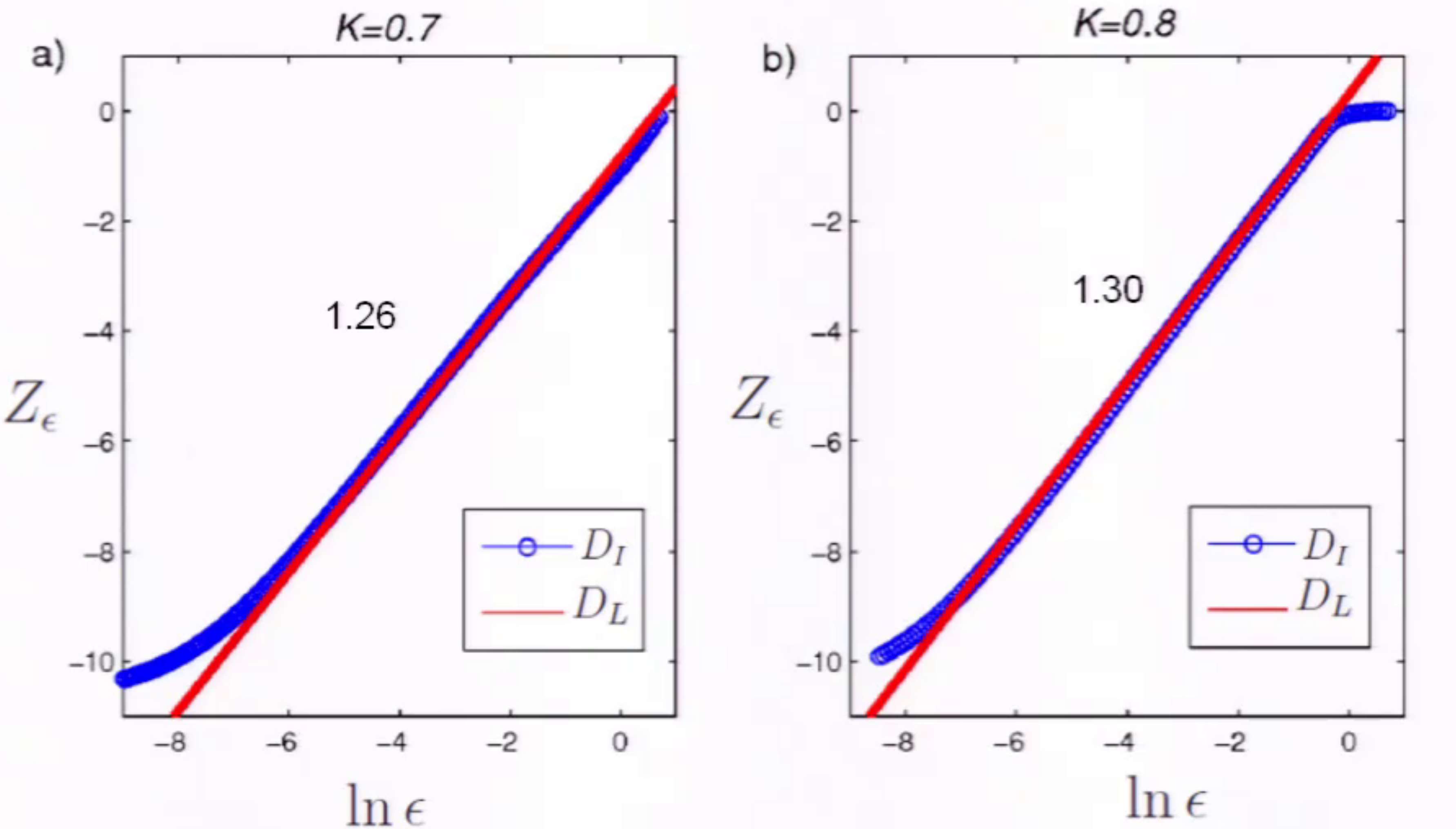
- For large *N*, $\bar{W}(\mathrm{t})$ is regarded as externally imposed
- In this limit, there are two Lyapunov exponents for each oscillator.

$$\lambda_1 > 0 > \lambda_2 \qquad \lambda_1 + \lambda_2 < 0$$

$$D_L = 1 + \lambda_1 / |\lambda_2|$$

for all *j* = 1,2,…,*N*

Results



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b)
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Z_ϵ
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D_I
D_L
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−2
0

Extensivity

Our hypothesis that, in the large N limit, $\overline{W}(t)$ can be regarded as imposed implies that, for the full 2N dimensional system, there are essentially N Lyapunov exponents $\lambda_1 > 0$ and N Lyapunov exponents $\lambda_2 < 0$.

One indication of extensivity:

$$\text{(\# positive } \lambda\text{'s)} \sim N$$

Applying the Kaplan-Yorke formula to this situation we have that the dimension of the attractor in the full $2N$ dimensional space at $N >> 1$ satisfies

$$N \lambda_1 + (D - N) \lambda_2 = 0$$

or

$$D/N = 1 + (\lambda_1 / |\lambda_2|) \quad \text{(Extensive)}$$

Final comment: We believe that, with appropriate modifications, most of this can be extended to the case of nonidentical oscillators with some spread in their parameters. [See our paper on arXiv.]

Summary of Main Points

- High dimensional extensively chaotic attractors of mean field coupled systems can be viewed as a collection of chaotically driven uncoupled units and analyzed using a snapshot attractor approach.
- Low dimensional attractors may coexist with extensively chaotic attractors accompanied by subcritical bifurcations between them.
- Ref.: arXiv 1412.3803.

Possible Relevance to Fluid Turbulence of the Transition to Extensive Chaos

In fluids, as forcing increases, transitions from steady, to low dimensional dynamics, to turbulence are observed (e.g., Brandstater and Swinney [1987]).

Turbulence takes place on an extensively chaotic attractor. E.g., Constantin, Foias, Temam [Physica D, 1988] show that (Dimension) ~ (Container Volume).

Also it has been shown both numerically (e.g., papers by B. Eckhardt et al.) and experimentally, that in pipe flow, low dimensional and turbulent attractors coexist.

Our work provides a simple, understandable model for coexistence and transitions between extensive and nonextensive dynamics.