# Synchronization of heterogeneous oscillators under network modifications: Perturbation and optimization of the synchrony alignment function

Dane Taylor, Per Sebastian Skardal, Jie Sun

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DANE TAYLOR
Department of Mathematics, UNC Chapel Hill
Department of Mathematics, University at Buffalo, SUNY

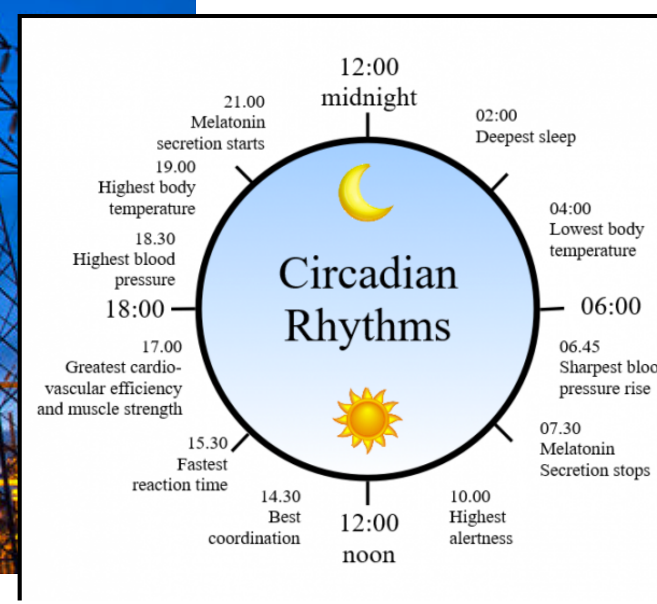
# Synchronization

<u>Engineering applications:</u>

- Power grids, smart grids
- Josephson junctions
- Synthetic cell engineering

<u>Examples in biology:</u>

- Neuronal activity
- Cardiac pacemaker cells
- Circadian rhythms



# Kuramoto Phase-Oscillator Model

-Kuramoto, *Chemical Oscillations, Waves, and Turbulence* (1984)

$$\dot{\theta}_n = \omega_n + K \sum_m A_{nm} H(\theta_m - \theta_n)$$

<u>Kuramoto Order Parameter</u>

$$re^{i\psi} = N^{-1} \sum_n e^{i\theta_n}$$

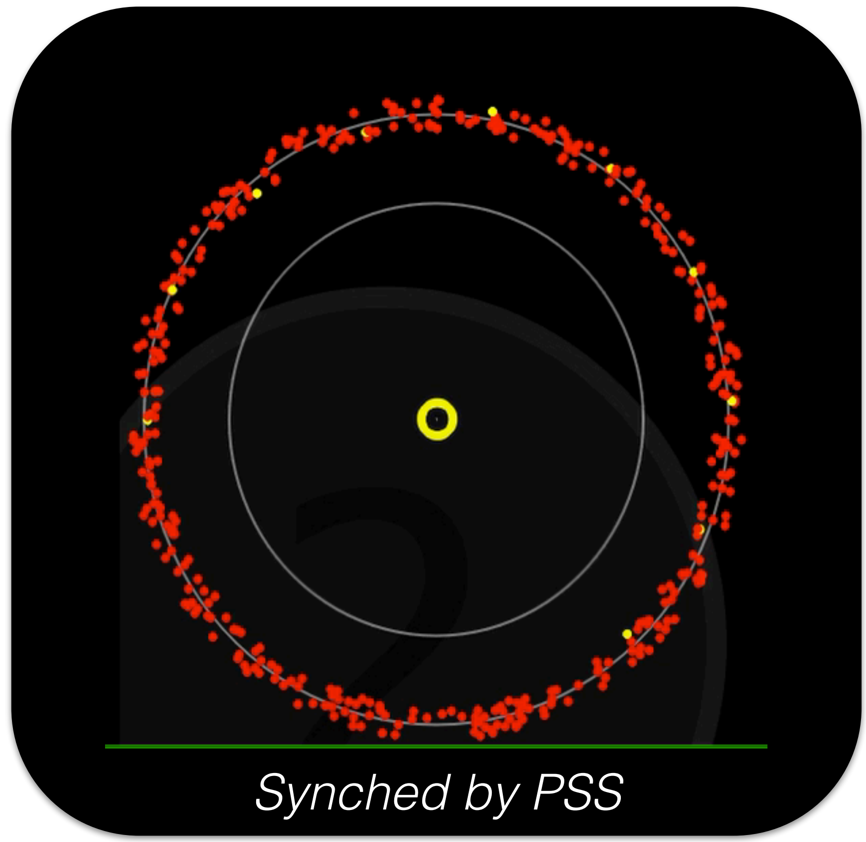
<u>Variance Order Parameter</u>

$$R = 1 - \sigma_\theta^2/2$$

$$\sigma_\theta = N^{-1} \sum_n (\theta_n - \overline{\theta})^2$$

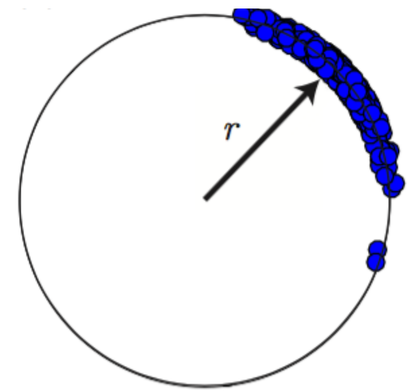
- Order parameters are similar for strong phase synchronization:

$$R \leq r \leq R + \frac{\sum_n (\theta_n - \psi)^4}{24N}$$



# Strong Phase-Locked Synchronization

- Focusing on the phase-locked state, we study the linear approximation



strong phase synchronization

$$\begin{aligned}\dot{\theta}_n &= \omega_n + K \sum_m A_{nm} H(\theta_m - \theta_n) \\ &\approx \omega_n + K H(0) d_n - K H'(0) \sum_m L_{nm} \theta_n \\ &= \hat{\omega}_n - \hat{K} \sum_m L_{nm} \theta_n\end{aligned}$$

where $L_{nm} = \delta_{nm} d_n - A_{nm}$ is the unnormalized Laplacian matrix

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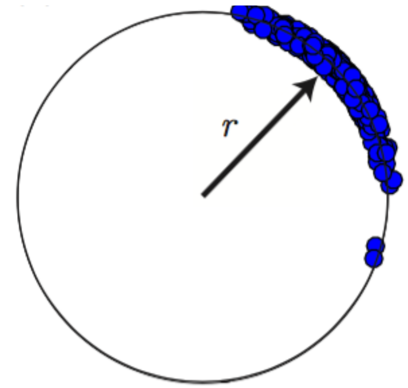


strong phase synchronization

where $L_{nm} = \delta_{nm} d_n - A_{nm}$ is the unnormalized Laplacian matrix

- We call this system **<u>heterogeneous Laplacian dynamics</u>**, and it has the steady-state solution

$$\theta^* = \hat{K}^{-1} L^{\dagger} \hat{\omega} + \overline{\theta}$$

where $L^{\dagger} = \sum_{m=2}^{N} \lambda_m^{-1} v^{(m)} (v^{(m)})^T$ is the Moore-Penrose pseudoinverse

# Synchrony Alignment Function (SAF)

-Skardal, Taylor, Sun. *PRL* 113, 144101 2014

- Describes the phases' variance for heterogeneous Laplacian dynamics

$$
\begin{aligned}
J(\omega, L) &= \frac{1}{N}||L^{\dagger}\omega||^2 \\
&= \frac{1}{N}\sum_{n=2}^{N}\frac{\langle \omega, v^{(n)}\rangle^2}{\lambda_n^2} \\
&= K^2\frac{||\theta^* - \overline{\theta}||^2}{N}
\end{aligned}
$$

- Measures alignment of oscillator frequencies $\{\omega_n\}$ with the network structure
  - Uses full set of eigenvalues $\{\lambda_n\}$ and eigenvectors $\{v^{(n)}\}$ of the network Laplacian matrix $L$

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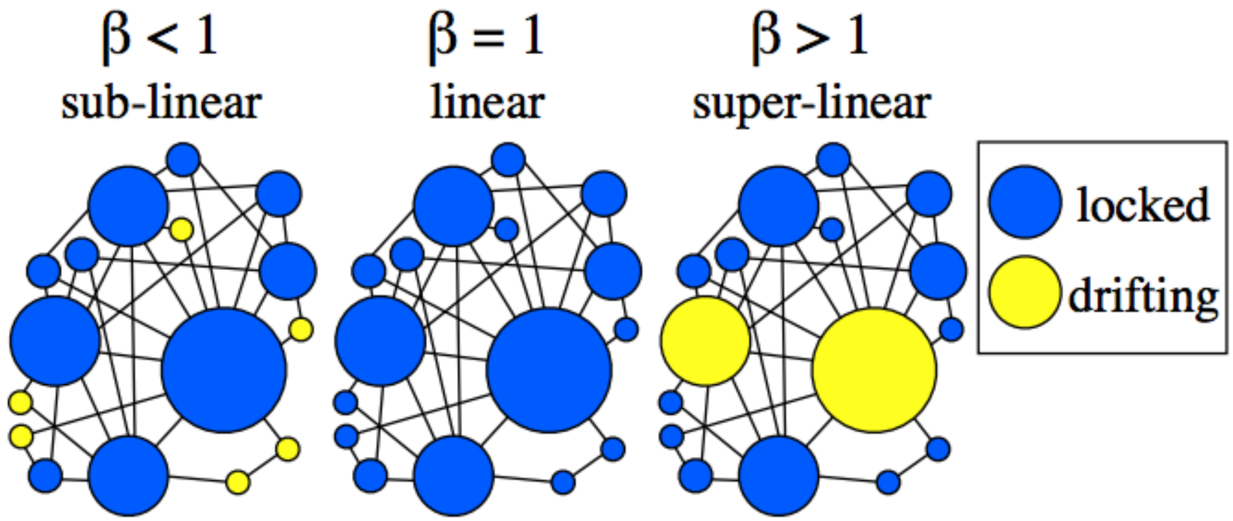
- Measures alignment of oscillator frequencies $\{\omega_n\}$ with the network structure
  - Uses full set of eigenvalues $\{\lambda_n\}$ and eigenvectors $\{v^{(n)}\}$ of the network Laplacian matrix $L$

- The order parameters are given by: $R = 1 - J(\omega, L)/2K^2 \approx r$

- These approximate the order parameters of the Kuramoto system in the regime of strong phase synchronization

# Alignment of Frequencies with Network Dramatically Effects Synchronization

- Degree-frequency correlations can affect the very nature of phase locking, influencing whether high-degree or low-degree nodes first phase lock

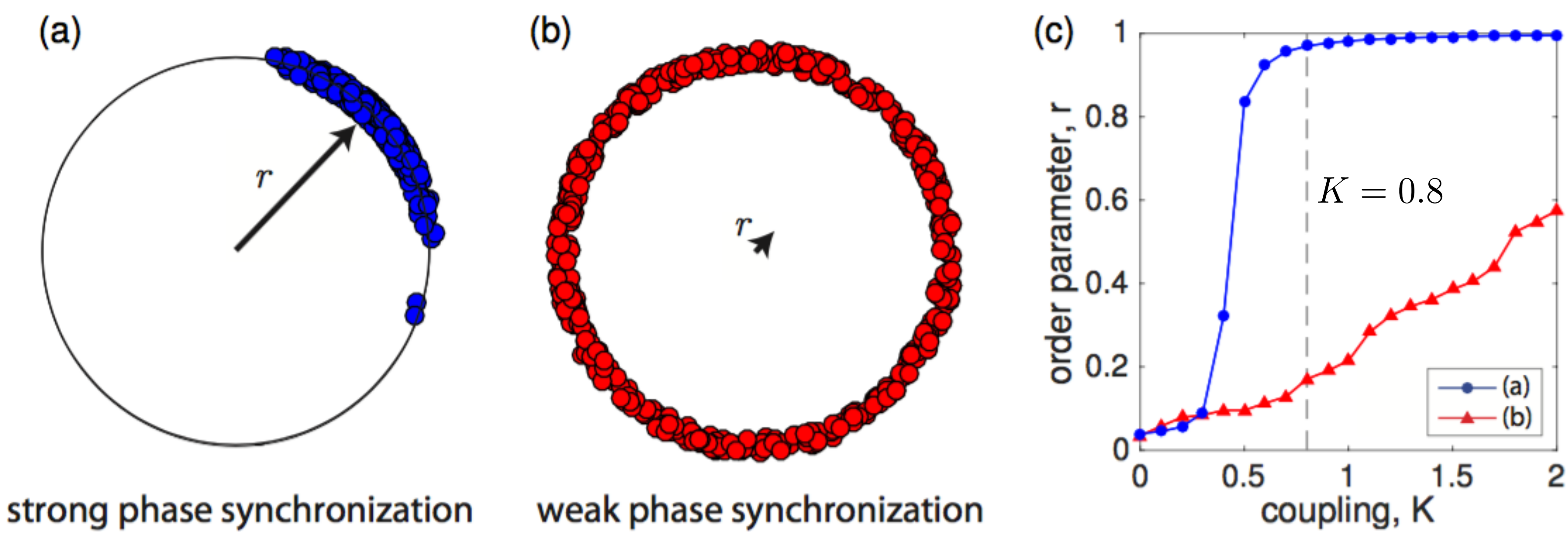



- **Effects of degree-frequency correlations on network synchronization: Universality and full phase-locking.** Skardal, Sun, Taylor & Restrepo. *EPL (Europhysics Letters)* 101, 20001 (2013).

# Alignment of Frequencies with Network Dramatically Effects Synchronization

- Degree-frequency correlations can affect the very nature of phase locking, influencing whether high-degree or low-degree nodes first phase lock

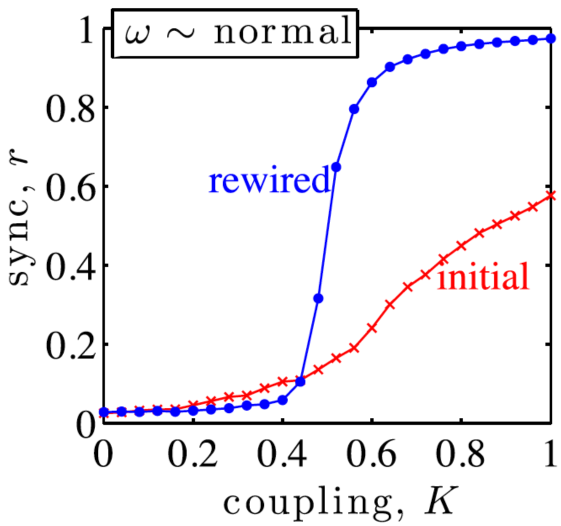
- Alignment also influences level of synchronization
  - Frequencies are either aligned with dominant eigenvector, $\omega = \mathbf{v}^{(N)}$, or given by $\omega = \mathrm{perm}(\mathbf{v}^{(N)})$

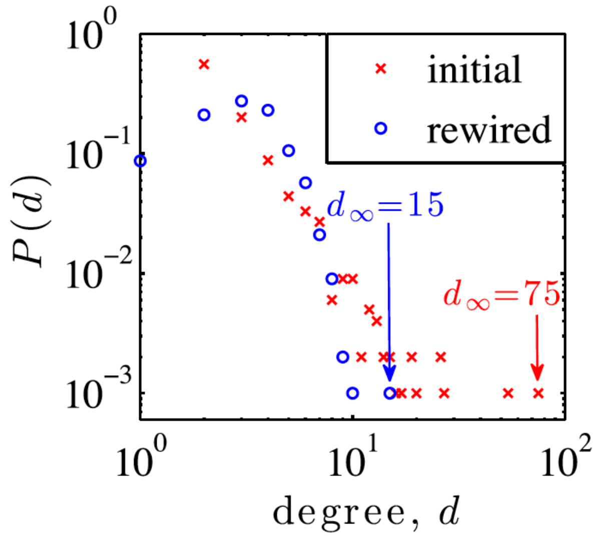


# Example: SAF-Optimized Systems Exhibit Matched Heterogeneity

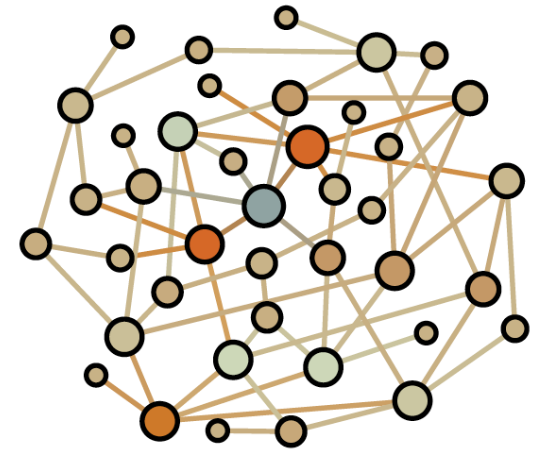
-Skardal, Taylor, Sun. *PRL* 113, 144101 2014

**homogeneous frequencies**




**homogeneous network**



-color indicates frequency

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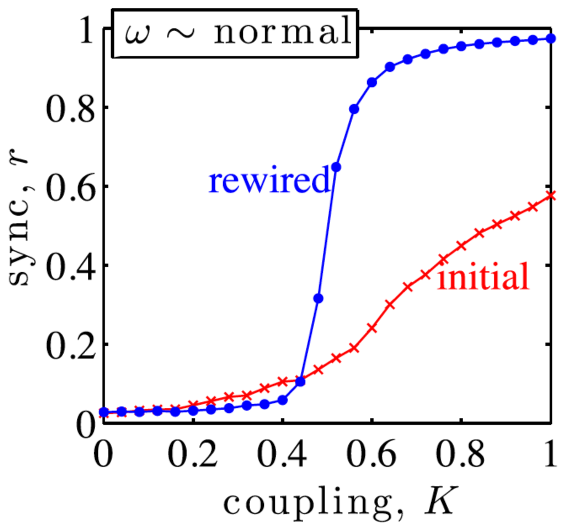


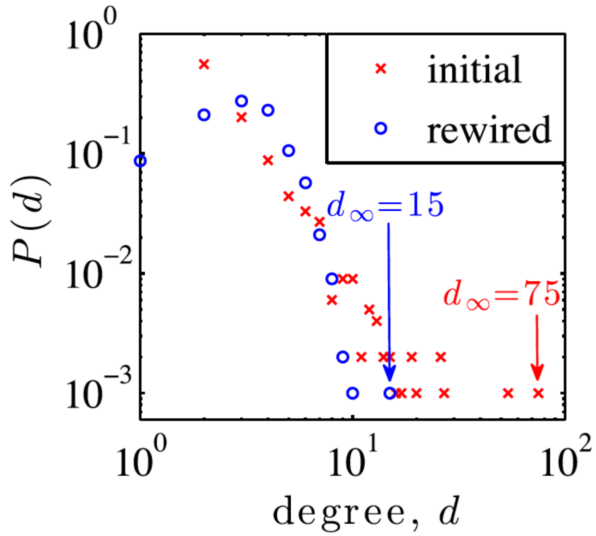


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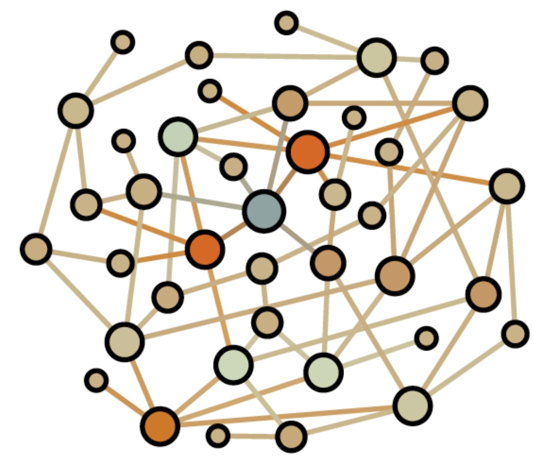

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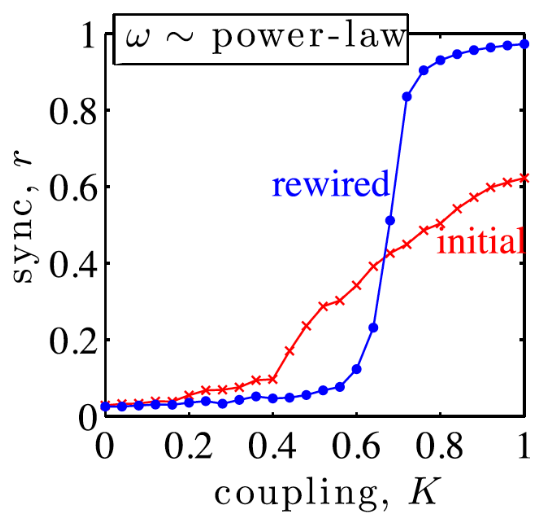


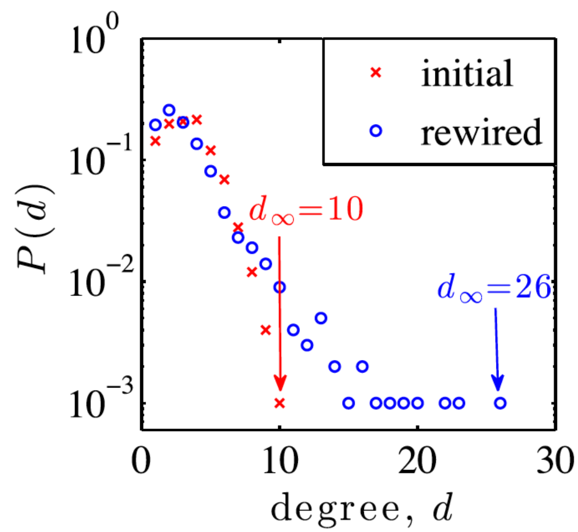


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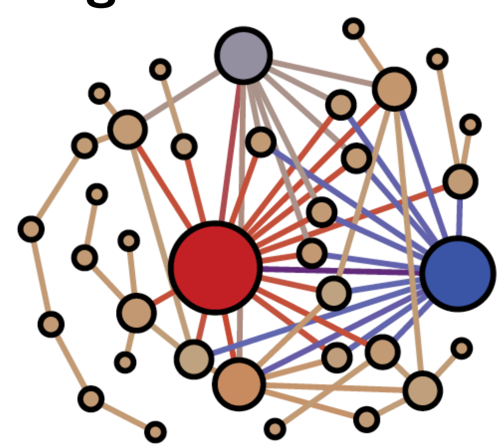

# Motivation

- **These network optimization experiments were based on Markov Chain Monte Carlo (MCMC) methods in which thousands of perturbations were proposed as possible improvements:**
  - perturbations that decreased the SAF were implemented
  - perturbations that increased the SAF were rejected
- **MCMC is inefficient since perturbations are chosen at random**
- **We will develop efficient algorithms by approximating how network modifications affect the SAF**
  - Efficient in two ways:
    (1) small number of rewires needed to enhance synchronization
    (2) small computational complexity of rewiring algorithms

# Outline

- We develop a first-order spectral perturbation analysis for how network modifications such as rewiring affect the SAF
- We use these perturbations to rank network edges (and potential new edges) according to their importance to synchronization
  - Rankings take into account the oscillators' heterogeneous frequencies!
- Based on these rankings, we develop rewiring algorithms to efficiency enhance a systems ability to synchronize

**Synchronization of heterogeneous oscillators under network modifications: Perturbation and optimization of the synchrony alignment function**

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# Perturbation Analysis of the SAF

- Let symmetric matrix $\epsilon\Delta L$ denote a change to the Laplacian matrix encoding a modification to an undirected network and consider the SAF

$$J(\omega, L) = \frac{1}{N}\sum_{n=2}^{N}\frac{\langle\omega, v^{(n)}\rangle^2}{\lambda_n^2}$$

- The perturbed SAF is given by

$$J(\omega, L+\epsilon\Delta L) = J(\omega, L) + \epsilon J'(0) + \mathcal{O}(\epsilon^2)$$

where

$$J'(0) = \frac{2}{N}\sum_{n=2}^{N}\left(\frac{\boldsymbol{\omega}^T\boldsymbol{v}^{(n)}}{\lambda_n^3}\right)\left(\sum_{m=2}^{N}\frac{[\boldsymbol{\omega}^T\boldsymbol{v}^{(m)}][(\boldsymbol{v}^{(m)})^T\Delta L\boldsymbol{v}^{(n)}]}{(1-\lambda_m/\lambda_n)-\delta_{nm}}\right)$$

- This result uses the classical spectral perturbation results:

$$\lambda_n(\epsilon) = \lambda_n + \epsilon\ (\boldsymbol{v}^{(n)})^T\Delta L\boldsymbol{v}^{(n)} + \mathcal{O}(\epsilon^2).$$

$$\boldsymbol{v}^{(n)}(\epsilon) = \boldsymbol{v}^{(n)} + \sum_{m\neq n}\frac{(\boldsymbol{v}^{(m)})^T\Delta L\boldsymbol{v}^{(n)}}{\lambda_n-\lambda_m}\boldsymbol{v}^{(m)}. + \mathcal{O}(\epsilon^2)$$

# Perturbation Analysis of the SAF

- The addition (+) and removal (-) of an edge $(p, q)$ yields the perturbation matrix

$$\Delta L_{ij}^{(pq)} = \begin{cases} \pm 1, & (i,j) \in \{(p,p),(q,q)\} \\ \mp 1, & (i,j) \in \{(p,q),(q,p)\} \\ 0, & otherwise, \end{cases}$$

- Using $(\boldsymbol{v}^{(m)})^T \Delta L^{(pq)} \boldsymbol{v}^{(n)} = (\boldsymbol{v}_p^{(m)} - \boldsymbol{v}_q^{(m)})(\boldsymbol{v}_p^{(n)} - \boldsymbol{v}_q^{(n)})$., , we obtain the following first-order approximation for adding a set $\mathcal{E}^{(+)}$ of edges and removing a set $\mathcal{E}^{(-)}$ of edges

$$J(\boldsymbol{\omega}, L(\epsilon)) = J(\boldsymbol{\omega}, L) + \sum_{(p,q)\in\mathcal{E}^{(+)}} \epsilon Q_{pq} - \sum_{(p,q)\in\mathcal{E}^{(-)}} \epsilon Q_{pq} + \mathcal{O}(\epsilon^2)$$

where

$$Q_{pq} = \frac{2}{N} \sum_{n=2}^{N} \left( \frac{\boldsymbol{\omega}^T \boldsymbol{v}^{(n)}}{\lambda_n^3} \right) \left( \sum_{m=1}^{N} \frac{[\boldsymbol{\omega}^T \boldsymbol{v}^{(m)}][(\boldsymbol{v}_p^{(m)} - \boldsymbol{v}_q^{(m)})(\boldsymbol{v}_p^{(n)} - \boldsymbol{v}_q^{(n)})]}{(1-\lambda_m/\lambda_n) - \delta_{nm}} \right)$$

# Ranking Edges for SAF Optimization

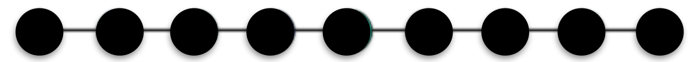
- The value $Q_{pq}$ indicates the importance of edge $(p, q)$ to the SAF and strongly-synchronized phase-locked oscillators
- Let $\mathcal{E} \subseteq \{1, \dots, N\} \times \{1, \dots, N\}$ denote the set of network edges
- We rank edges $\mathcal{E}$ so that the top-ranked edge $(p, q)$ has the smallest (most negative) value $Q_{pq}$ so that its removal most increases $J(\omega, L)$
- We also rank potential new edges $\{1, \dots, N\} \times \{1, \dots, N\} \setminus \mathcal{E}$ so that the top-ranked new edge $(p, q)$ corresponds to the smallest value $Q_{pq}$

Edge rankings allow us to efficiently modify networks to tune their synchronization properties.

Importantly, these rankings reflect the oscillators' heterogeneous dynamics!

# Example Illustrating Importance of Oscillator Frequencies for Edge Ranking

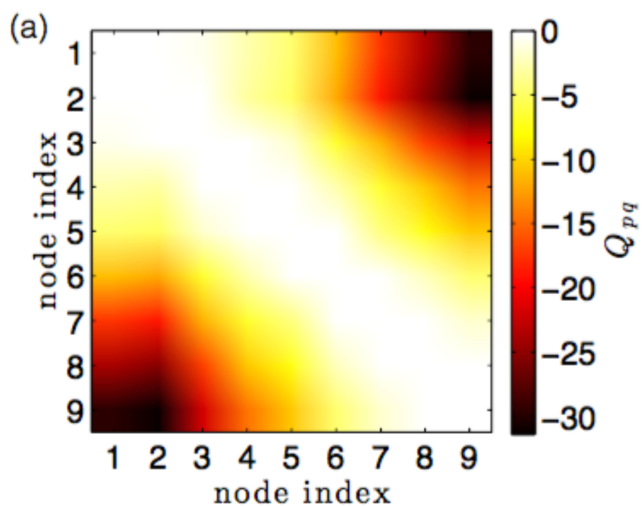
- We add edges to a chain network



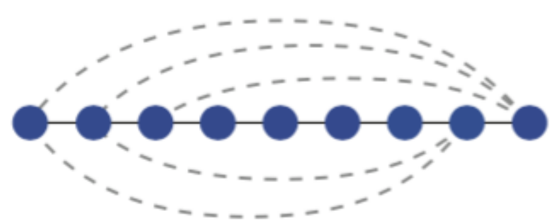
  - In one case, the oscillator frequencies are nonidentical, but similar

homogeneous frequencies

$Q_{pq}$ values:




top-ranked edges:



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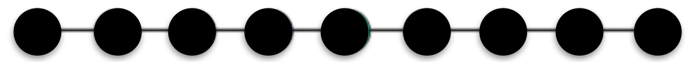
- We add edges to a chain network 
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homogeneous frequencies | heterogeneous frequencies

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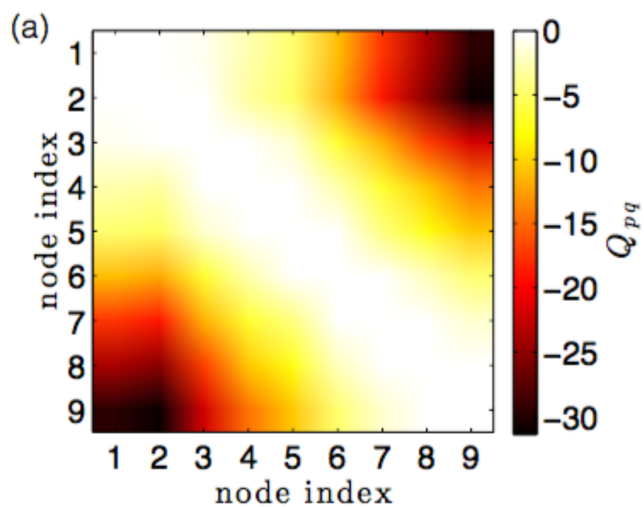


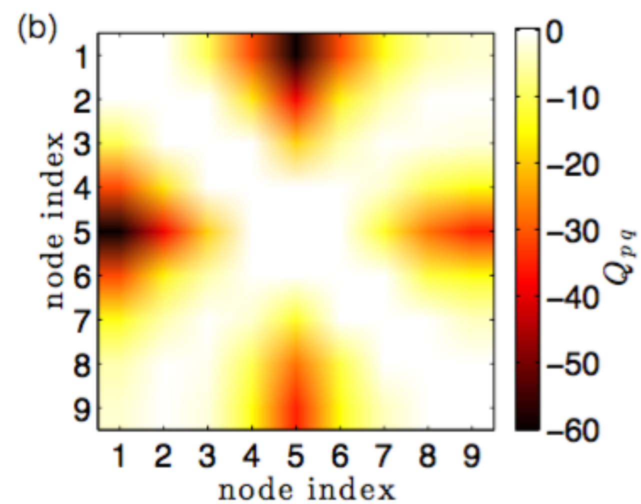


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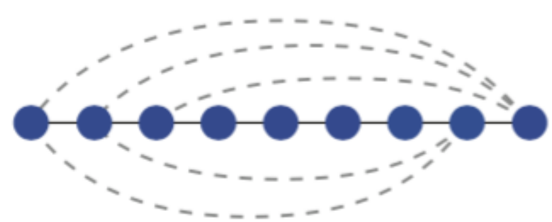

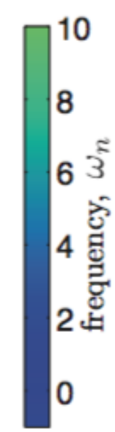

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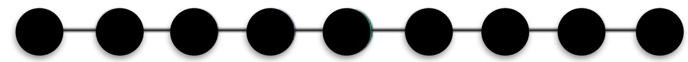
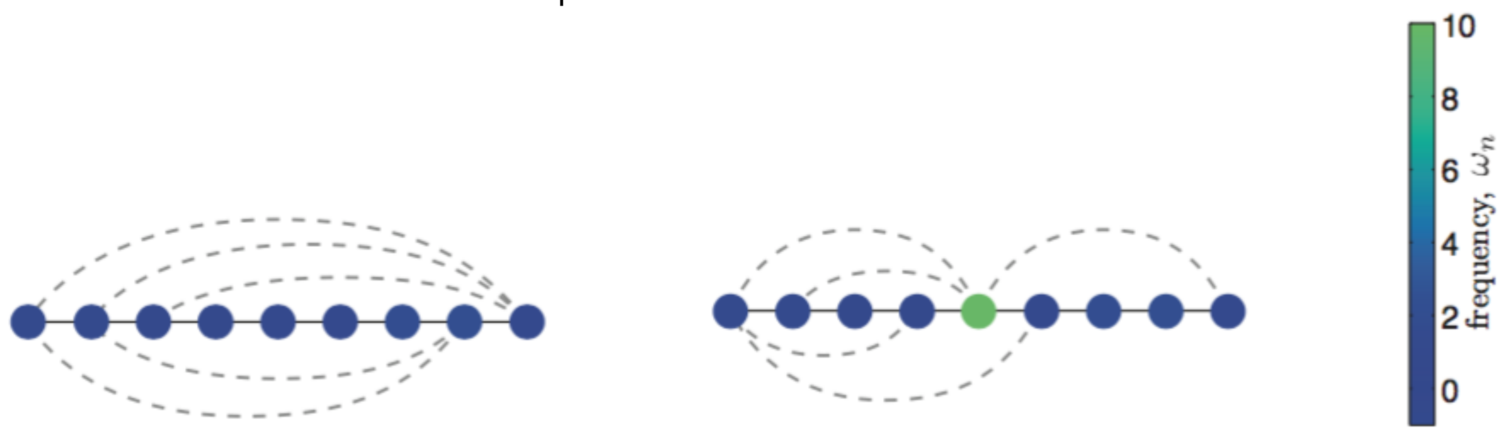
- We add edges to a chain network 
  - In one case, the oscillator frequencies are nonidentical, but similar
  - In the other, case we make oscillator 5 very fast

- We observe two regimes:
  - When the network is the limiting factor, the edge additions aim to improve the network's general (i.e., agnostic of frequencies) synchronizability
  - When oscillator heterogeneity is the limiting factor, the edge additions aim to counter balance the oscillators' frequencies

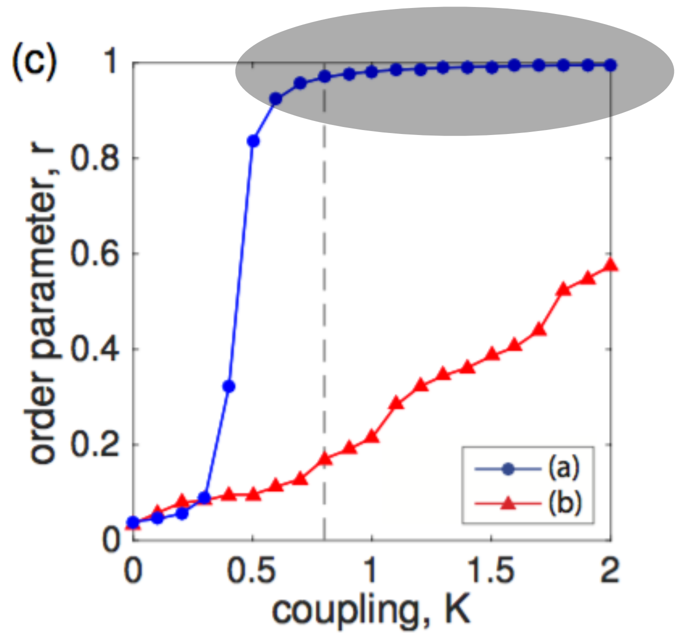
top-ranked edges:



# Connection to Dynamical Importance

-Restrepo, Ott, Hunt. *PRL* 97, 094102 (2006)

- We rank edges according to strong-synchronization regime



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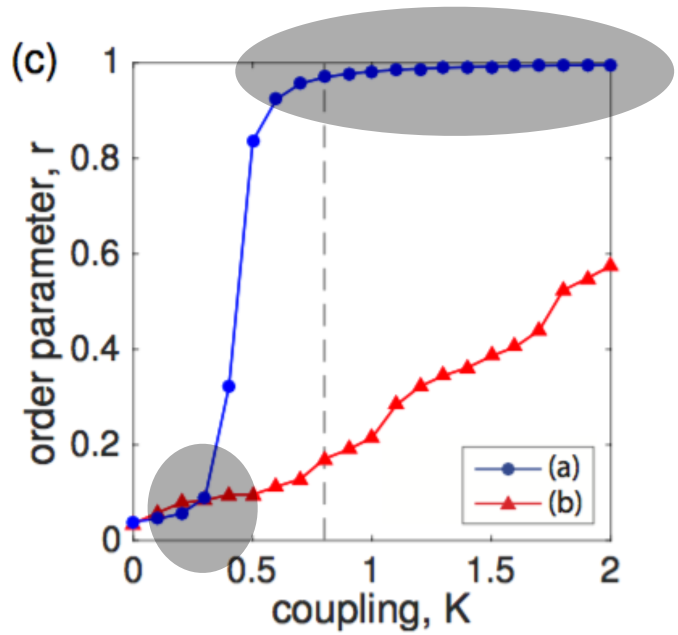


- Our work complements Dynamical Importance, $\hat{I}_k = \frac{v_k u_k}{v^T u}$, which ranks edges according their effect on the critical value where the incoherent state becomes unstable

# Algorithms for Tuning Synchronization of Phase-Oscillator Systems

- We introduce two "gradient descent" algorithms to tune synchronization
  - One that updates $Q_{pq}$ after modifications
  - One that does not update $Q_{pq}$
- We compare these algorithms to two other methods
  - Modifications made at random
  - Modifications chosen to maximize $\lambda_2$

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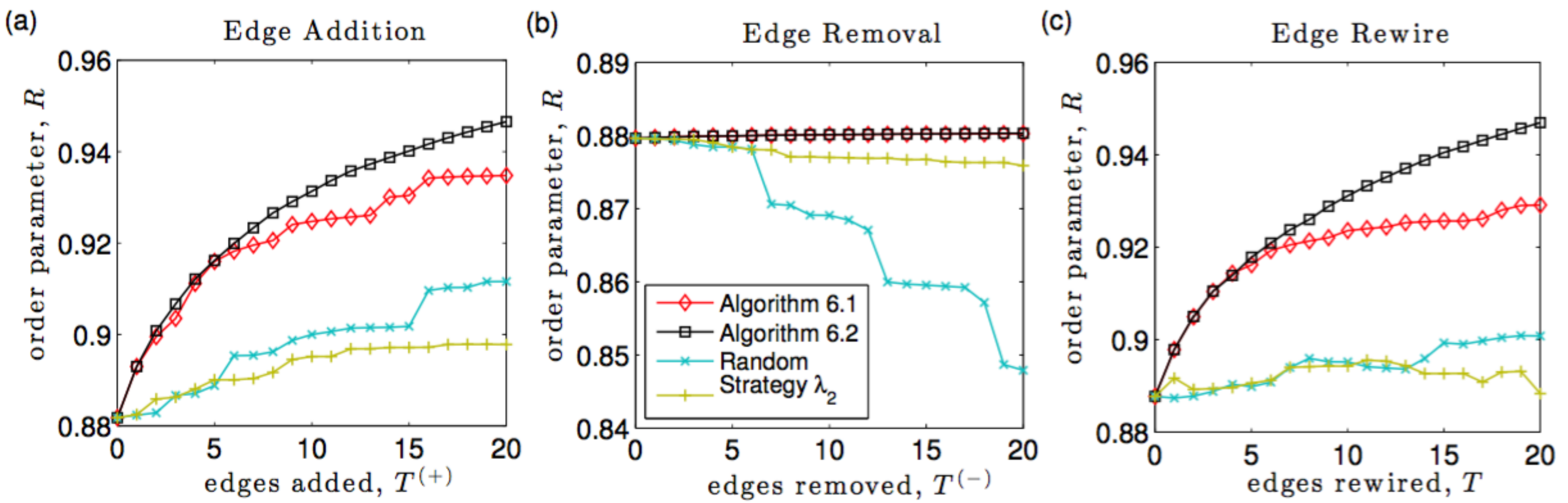
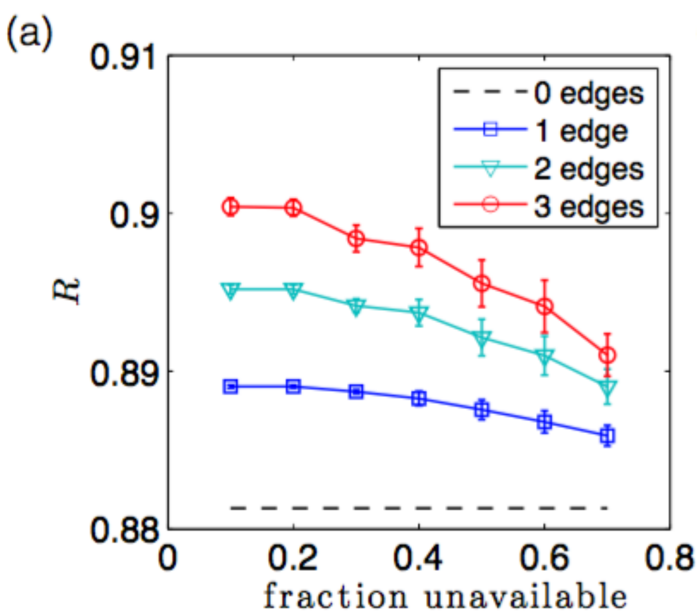


# Algorithms Performance for Non-ideal Scenarios

- We consider edge additions under 3 scenarios

  (a) A fraction of potential new edges are "off limits" for addition



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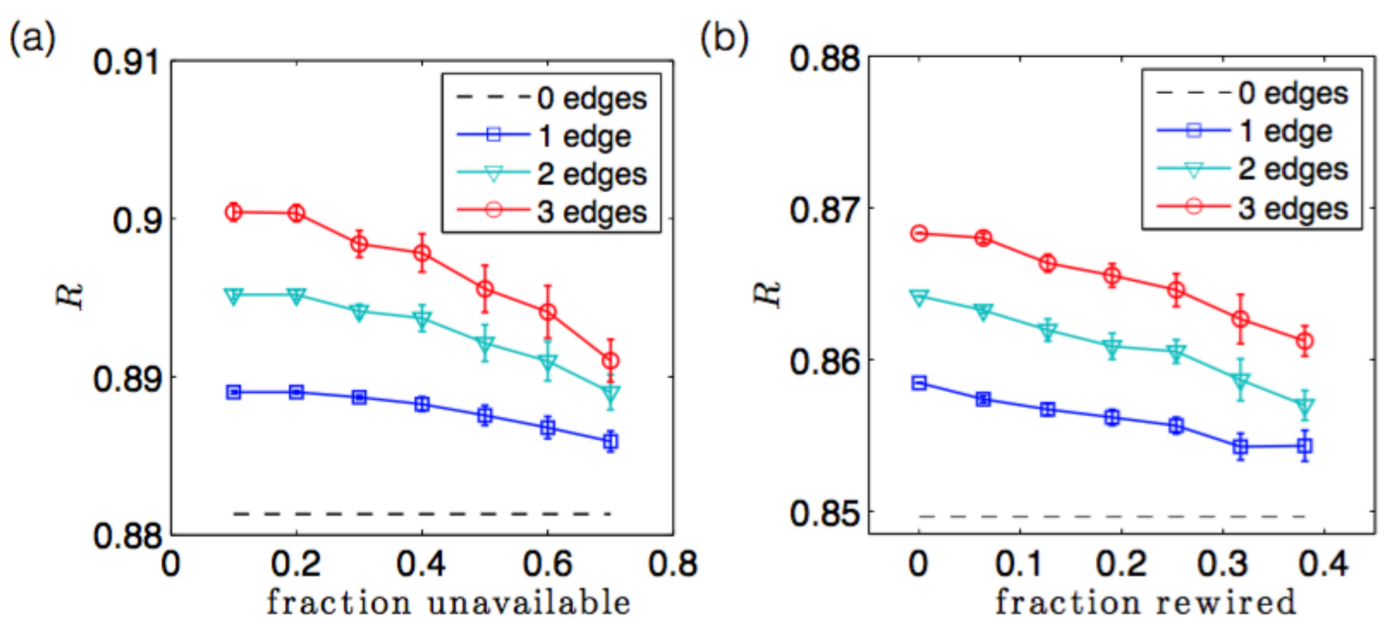

# Algorithms Performance for Non-ideal Scenarios

- We consider edge additions under 3 scenarios

  (a) A fraction of potential new edges are "off limits" for addition

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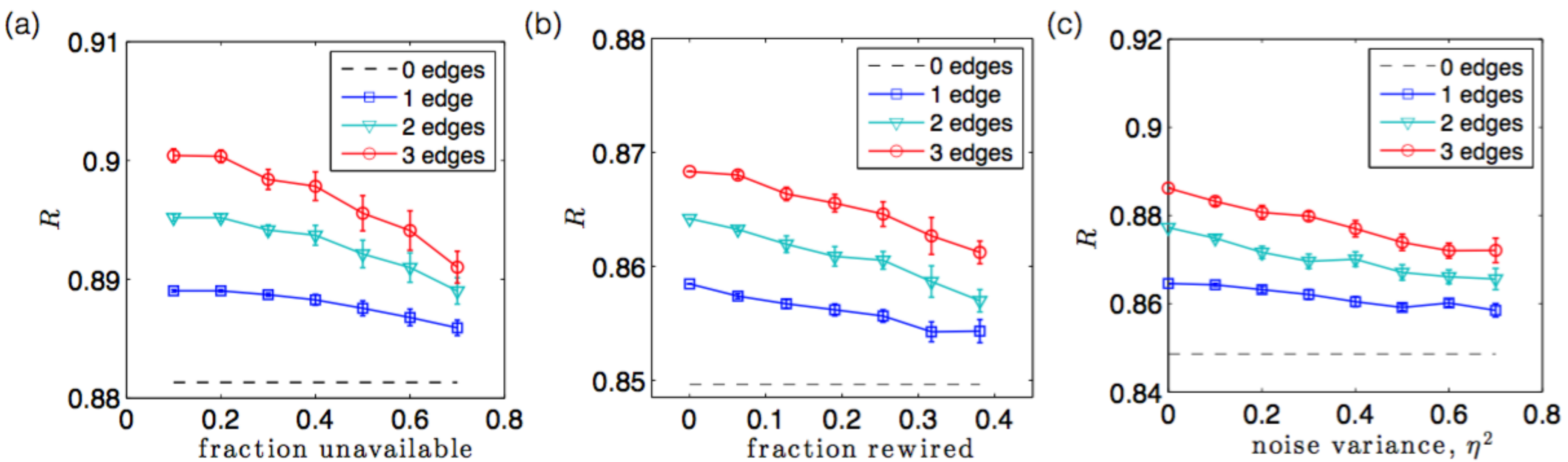
  (c) We introduce misinformation about the frequenting by adding Gaussian noise to them



# Conclusion

- The Synchrony Alignment Function (SAF) describes the variance of phases for strongly-coupled phase-locked oscillators
- We developed a first-order spectral perturbation analysis for how network modifications affect the SAF
- These perturbations rank network edges (and potential new edges) according to their importance to synchronization
  - Rankings take into account the oscillators' heterogeneous frequencies!
  - Observed network-dominated and oscillator-dominated regimes
- Based on these rankings, we develop rewiring algorithms to efficiency enhance a systems ability to synchronize

# Main Reference

- **Synchronization of heterogeneous oscillators under network modifications: Perturbation and optimization of the synchrony alignment function.** Taylor, Skardal & Sun. *SIAM Journal of Applied Dynamical Systems* 76(5), 1984-2008 (2016).

# Other SAF-Related Work

- **Optimal synchronization of complex networks.** Skardal, Taylor & Sun, *PRL* 113, 144101 (2014).

- **Erosion of synchronization in networks of coupled oscillators.** Skardal, Taylor, Sun & Arenas. *PRE* 91, 010802R (2015).

- **Optimal synchronization of directed complex networks.** Skardal, Taylor & J Sun. *Chaos* 26, 094807 (2016).

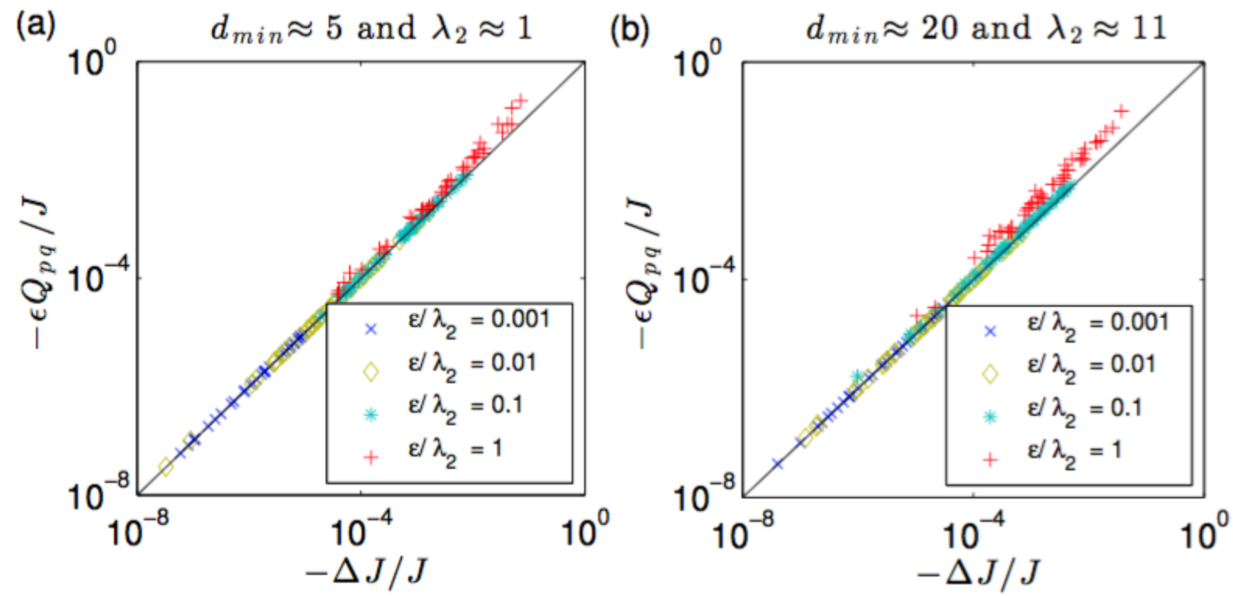
- **Collective frequency variation in network synchronization and reverse PageRank.** Skardal, Taylor, Sun & Arenas. *PRE* 93, 042314 (2016).

*dane.r.taylor@gmail.com* *https://sites.google.com/site/danetaylorresearch*

# Accuracy of First-Order Approximation

- Approximation error for edge additions

$$\Delta J = J(\omega, L(0)) - J(\omega, L(\epsilon))$$
$$\approx \epsilon Q_{pq}$$




- The error vanishes as $\epsilon \to 0$

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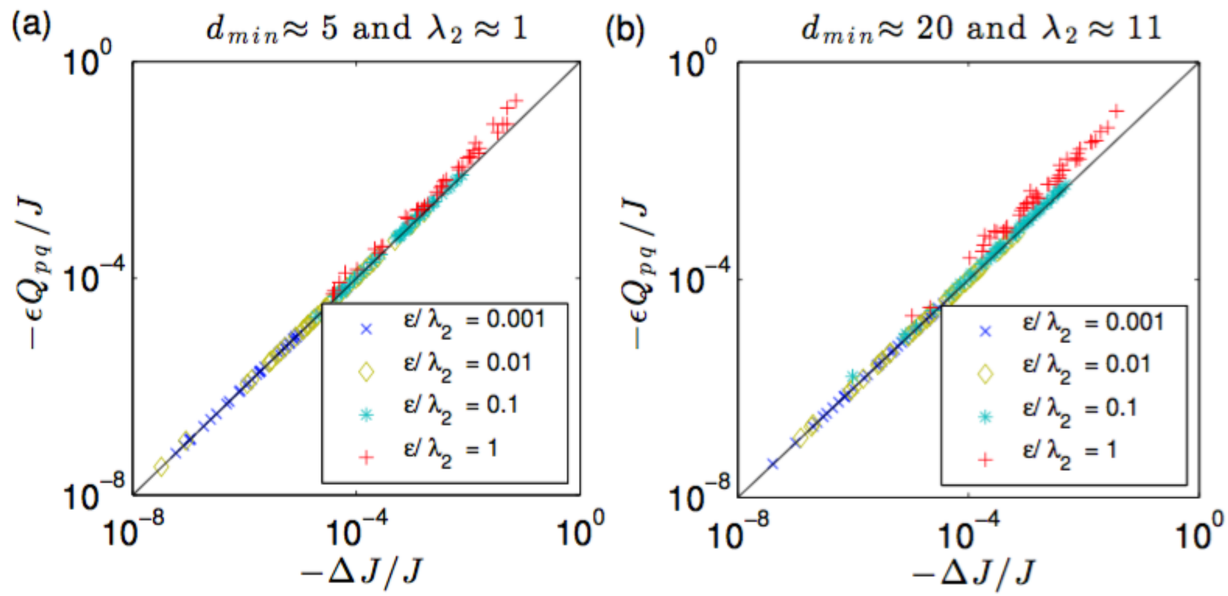
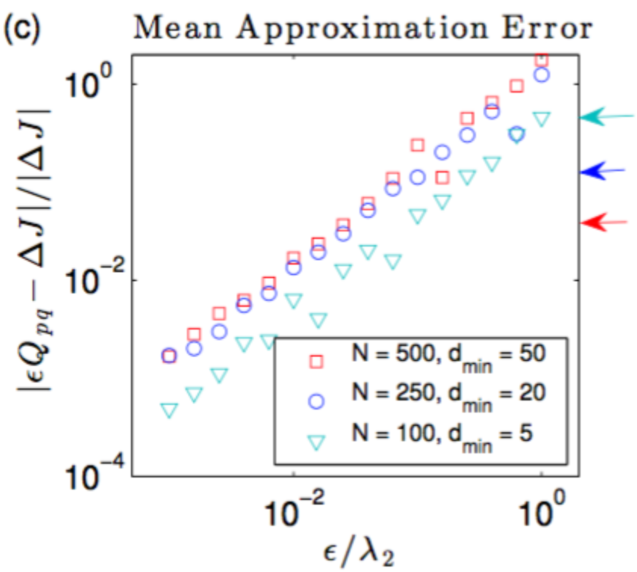


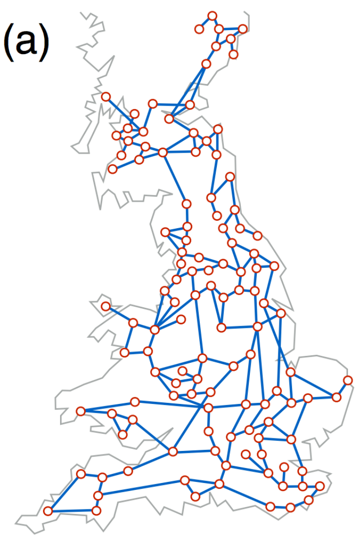

- The error vanishes as $\epsilon \to 0$

- The error also vanishes with fixed $\epsilon$ and increasing network size $N$ and number of edges

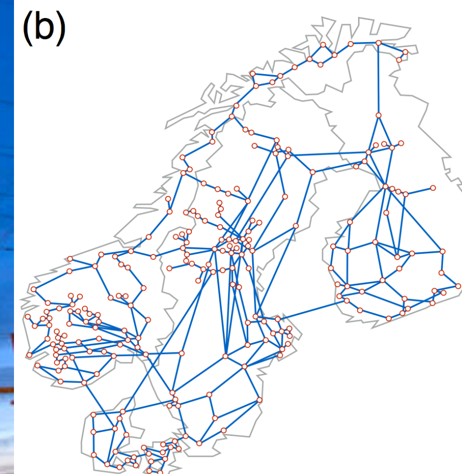
# Power-Grid Synchronization

- T Nishikawa & AE Motter (2015) Comparative analysis of existing models for power-grid synchronization. New Journal of Physics, 17(1), 015012.
- M Rohden, A Sorge, M Timme & D Witthaut (2012) Self-organized synchronization in decentralized power grids. Physical Review Letters, 109(6), 064101.
- F Dorfler & F Bullo (2012) Synchronization and transient stability in power networks and nonuniform Kuramoto oscillators. SIAM Journal on Control and Optimization, 50(3), 1616-1642.

(a)



(b)



# Spectral Perturbation Analysis

- Classical perturbation analysis of eigenvalues and eigenvectors

  [4] KE Atkinson, "An introduction to numerical analysis," John Wiley & Sons, 2008.

Theorem 4.1 (Perturbation of Simple Eigenvalues and their Eigenvectors [4]). *Let $L$ be a symmetric matrix with simple eigenvalues $\{\lambda_n\}$ and normalized eigenvectors $\{\boldsymbol{v}^{(n)}\}$. Consider a fixed symmetric perturbation matrix $\Delta L$, and let $L(\epsilon) = L + \epsilon \Delta L$. Denote the eigenvalues and eigenvectors of $L(\epsilon)$ by $\lambda_n(\epsilon)$ and $\boldsymbol{v}^{(n)}(\epsilon)$, respectively, for $n = 1, 2, \ldots, N$. It follows that*

$$\begin{aligned} \lambda_n(\epsilon) &= \lambda_n + \epsilon \lambda'(0) + \mathcal{O}(\epsilon^2), \\ \boldsymbol{v}^{(n)}(\epsilon) &= \boldsymbol{v}^{(n)} + \epsilon \boldsymbol{v}^{(n)'}(0) + \mathcal{O}(\epsilon^2), \end{aligned} \tag{4.1}$$

*and the derivatives with respect to $\epsilon$ at $\epsilon = 0$ are given by*

$$\begin{aligned} \lambda_n'(0) &= (\boldsymbol{v}^{(n)})^T \Delta L \boldsymbol{v}^{(n)} \\ \boldsymbol{v}^{(n)'}(0) &= \sum_{m \neq n} \frac{(\boldsymbol{v}^{(m)})^T \Delta L \boldsymbol{v}^{(n)}}{\lambda_n - \lambda_m} \boldsymbol{v}^{(m)}. \end{aligned} \tag{4.2}$$

# General Network-Perturbation Result

- Perturbation result for SAF: $J(\omega, L) = \frac{1}{N} \sum_{n=2}^{N} \frac{\langle \omega, v^{(n)} \rangle^2}{\lambda_n}$

THEOREM 4.2 (Perturbation of the SAF under a Network Modification). *Let $J(\boldsymbol{\omega}, L)$ denote the SAF given by Eq. (3.3) for natural frequencies $\boldsymbol{\omega}$ and symmetric network Laplacian $L$, and let $J(\boldsymbol{\omega}, L(\epsilon))$ denote the SAF for the network after it undergoes a symmetric modification $\epsilon \Delta L$. Assume the eigenvalues of $L$ and $L(\epsilon) = L + \epsilon \Delta L$ are simple, and that the original and perturbed networks are both connected. Then the first-order expansion in $\epsilon$ for the perturbed SAF is given by*

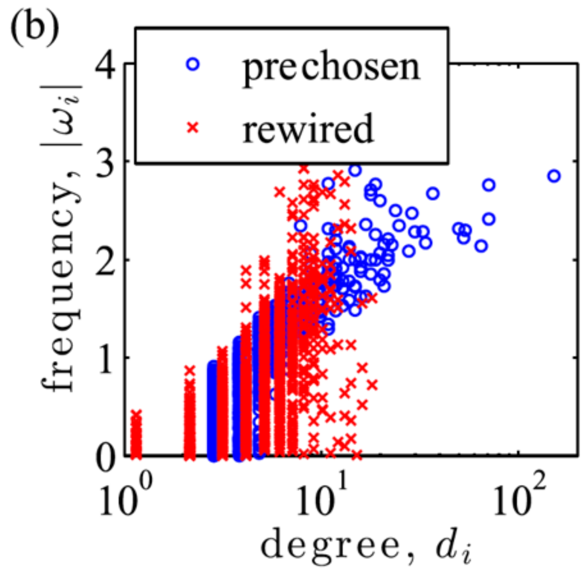
$$J(\boldsymbol{\omega}, L(\epsilon)) = J(\boldsymbol{\omega}, L) + \epsilon J'(\epsilon) + \mathcal{O}(\epsilon^2), \tag{4.3}$$

*where*

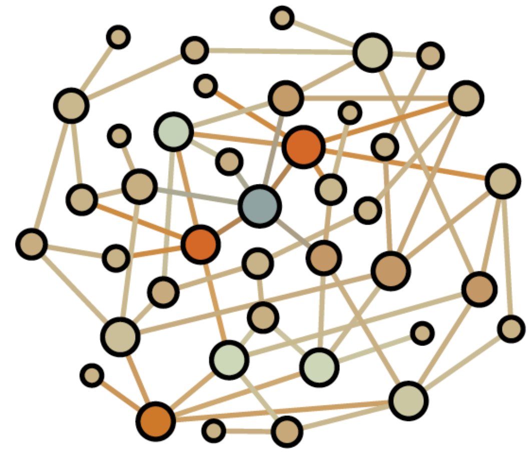
$$J'(\epsilon) = \frac{2}{N} \sum_{n=2}^{N} \left( \frac{\boldsymbol{\omega}^T \boldsymbol{v}^{(n)}}{\lambda_n^3} \right) \left( \sum_{m=2}^{N} \frac{[\boldsymbol{\omega}^T \boldsymbol{v}^{(m)}][(\boldsymbol{v}^{(m)})^T \Delta L \boldsymbol{v}^{(n)}]}{(1 - \lambda_m/\lambda_n) - \delta_{nm}} \right). \tag{4.4}$$

# Observation 2: SAF-Optimized Systems Exhibit **Correlations**

**Positive correlation between a node's frequency and degree**




**homogeneous network**



-color indicates frequency

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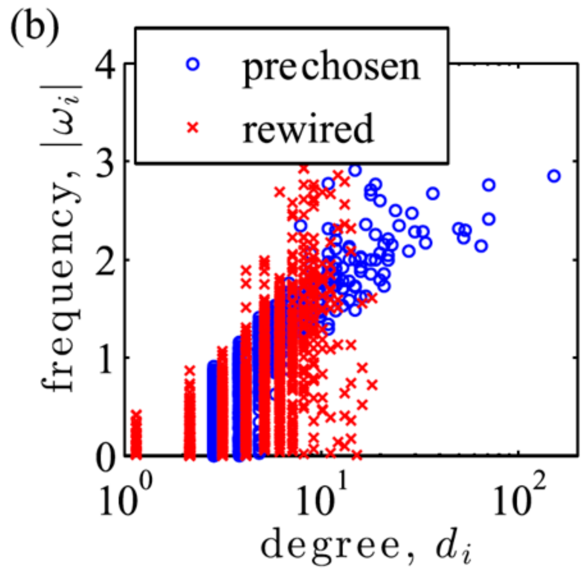


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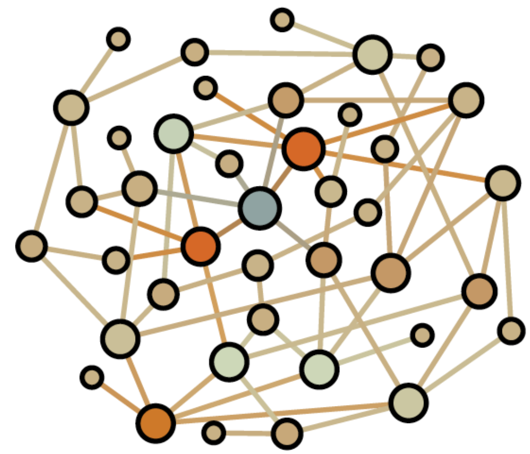
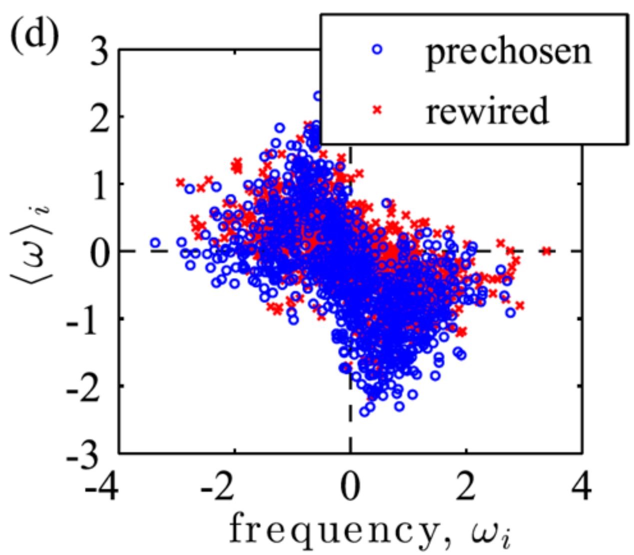
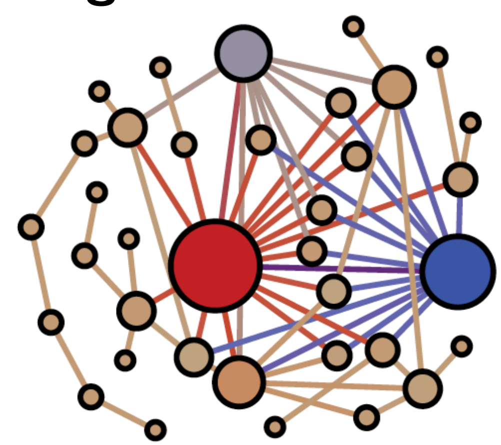


-color indicates frequency

**Negative correlation the frequencies of neighboring nodes**



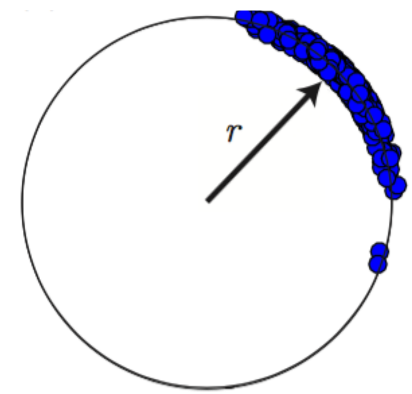

**heterogeneous network**



# Synchrony Alignment Function (SAF)

- The SAF estimates the variance of phases for phase-locked oscillators

$$J(\omega, L) \approx \frac{||\theta^*||^2}{K^2 H'^2(0)}$$




- and approximates the Kuramoto order parameter

$$r \simeq 1 - J(\omega, L)/2K^2 H'^2(0)$$

- We develop methodology to optimize the SAF, thereby optimizing $r$

# Optimization Experiment I: allocate frequencies to a network

- **Problem:** Given a network, allocate a set of oscillator frequencies with unit variance to maximize the order parameter $r = 1 - 1/2\lambda_N^2 K^2$
- **Solution:** If one can choose <u>any frequencies</u>, the optimal frequency vector is , which gives $\boldsymbol{\omega} \propto \boldsymbol{v}^N$

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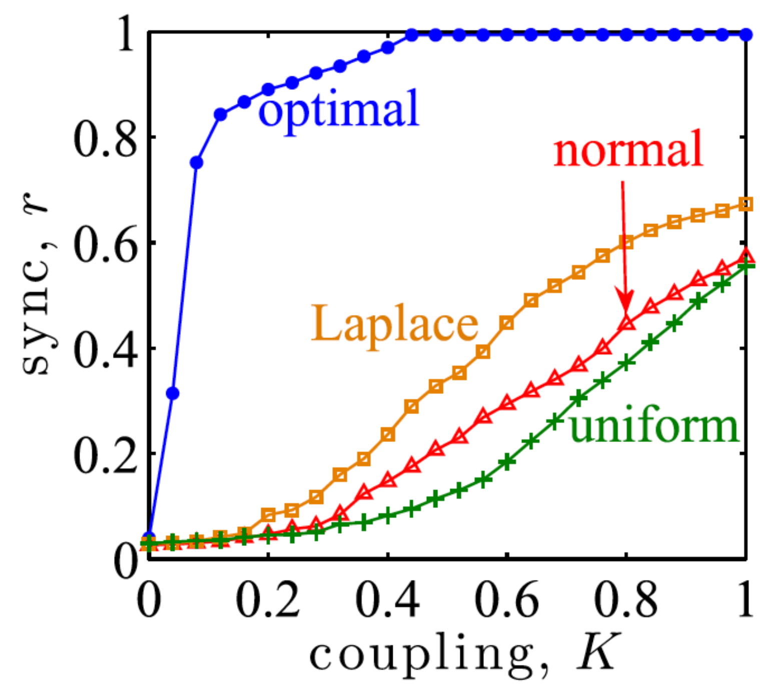
Results shown for allocating frequencies to a scale-free network with N=1000 nodes, exponent 3, and minimum degree 2



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- **Problem:** Given a network, allocate a set of oscillator frequencies with unit variance to maximize the order parameter
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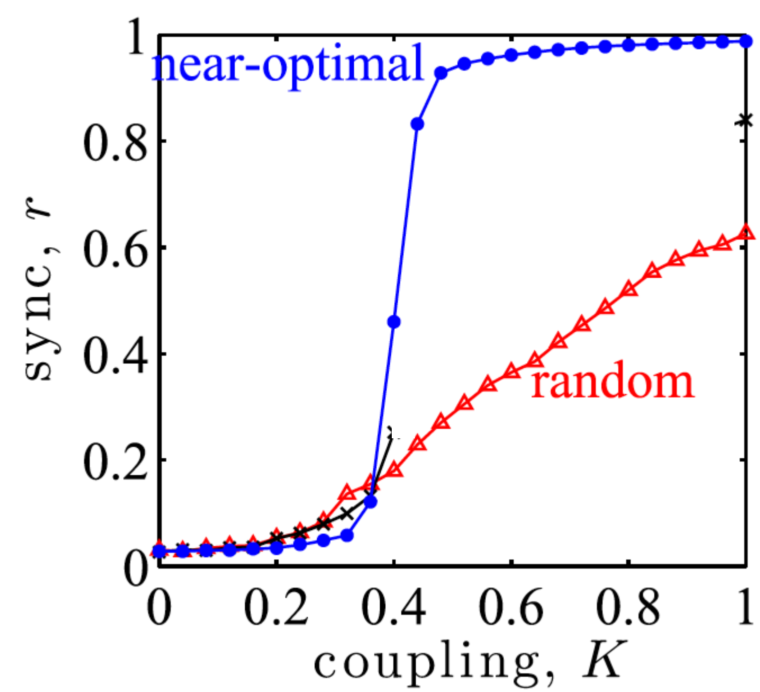
Results shown for allocating frequencies drawn from a normal distribution to a scale-free network



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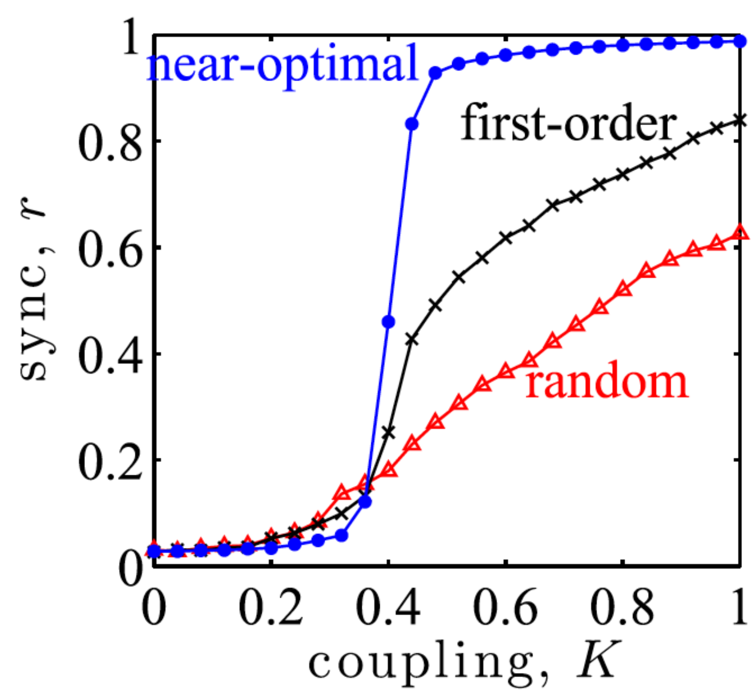
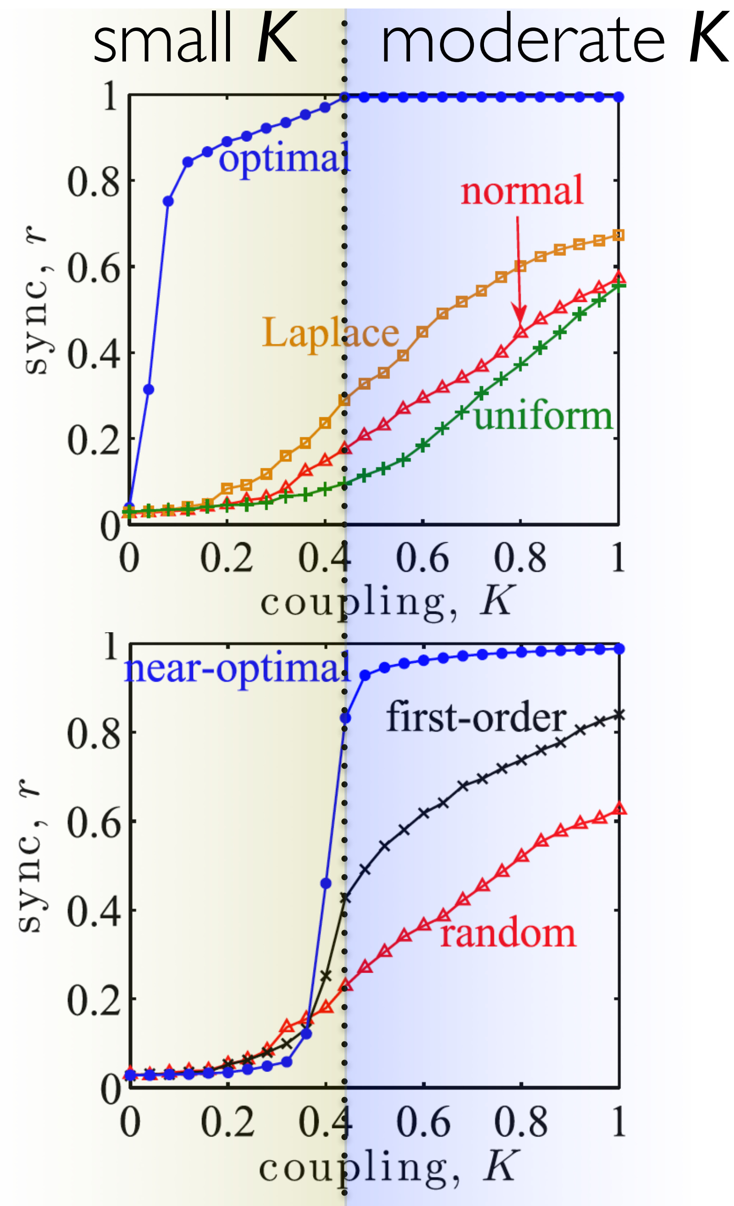
As a fast approximation, one can shuffle the frequencies to best align with $\boldsymbol{v}^N$



# Optimization Experiment I: allocate frequencies to a network

MAIN INSIGHT

For small *K*, one must choose particular frequencies to achieve strong synchronization.




For moderate *K*, strong synchronization can be achieved just by rearranging the oscillators.

**observed for our experiments

# Optimization Experiment II: design network for given frequencies

- **Problem:** Given a set of oscillator frequencies, design a network with a fixed number of edges to maximize the order parameter $J(\boldsymbol{\omega}, L)$
- **Solution:** We use an accept/reject algorithm to iteratively rewire an initial network so as to always decrease

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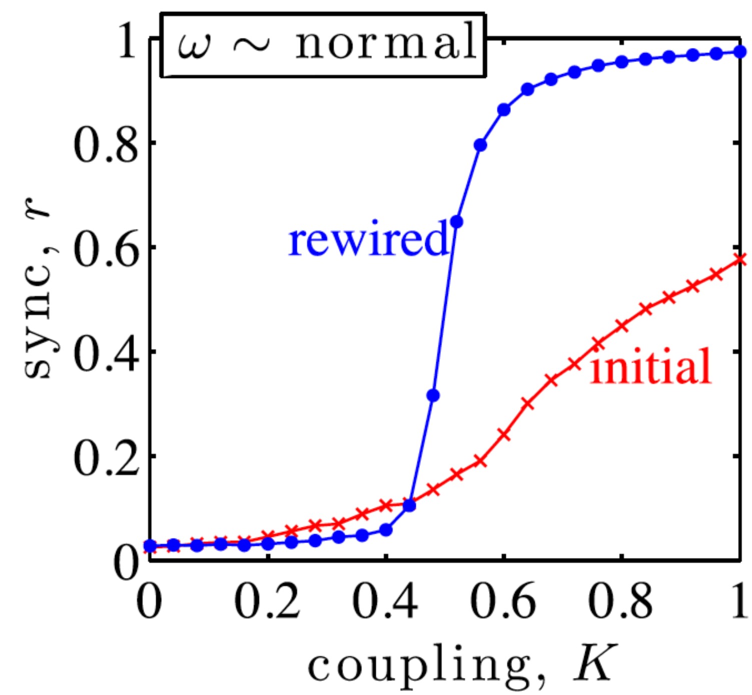
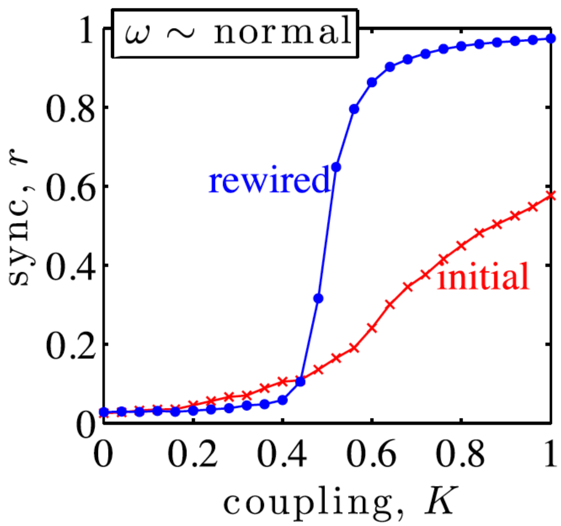
Results shown for frequencies drawn from a normal distribution.

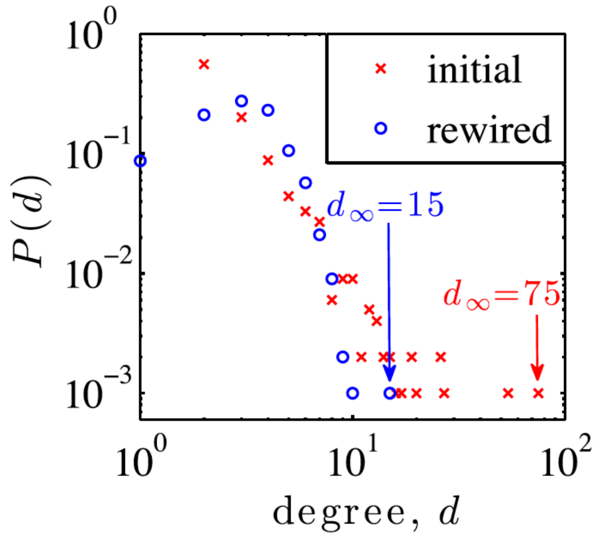
The initial network is scale free with N=1000 nodes, exponent 3, and minimum degree 2



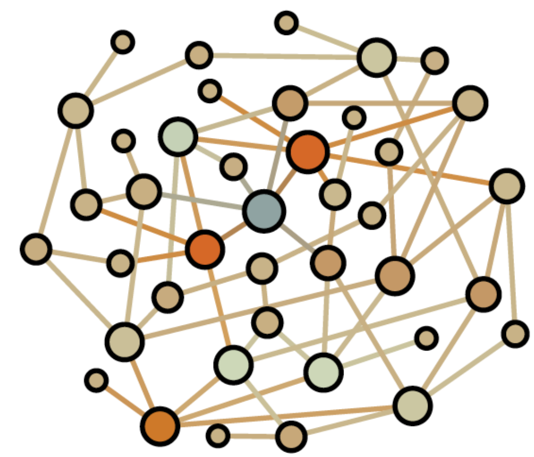
# Optimization Requires Network Heterogeneity to Match Oscillator Heterogeneity

## homogeneous frequencies

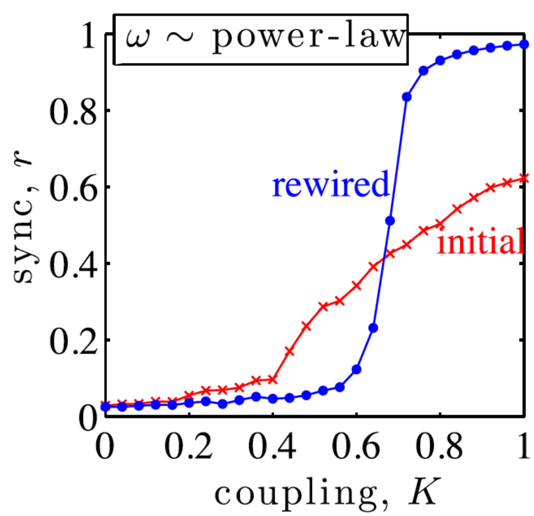


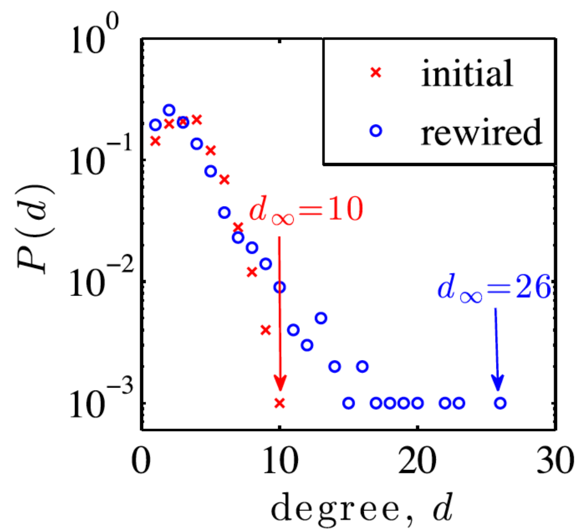




## homogeneous network

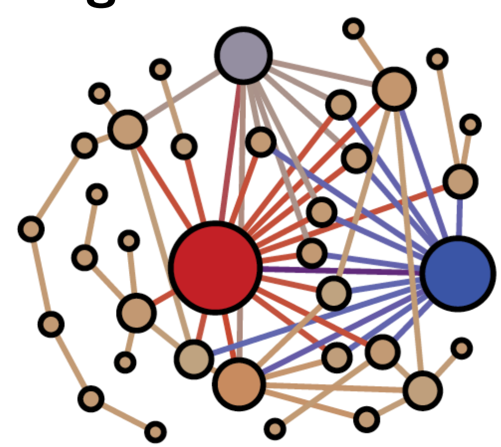


## heterogeneous frequencies



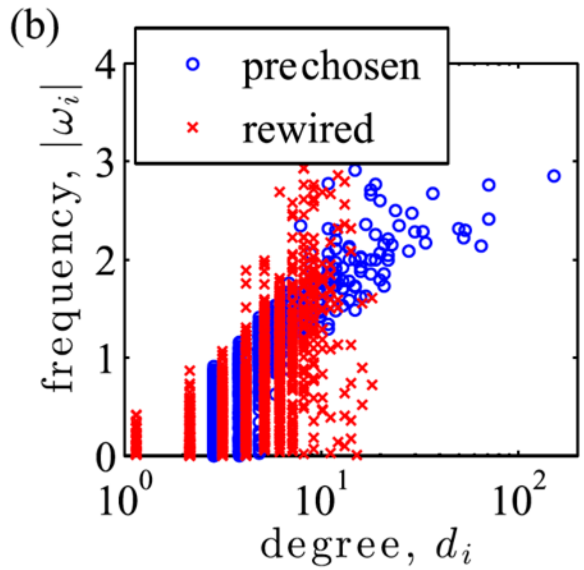

## heterogeneous network



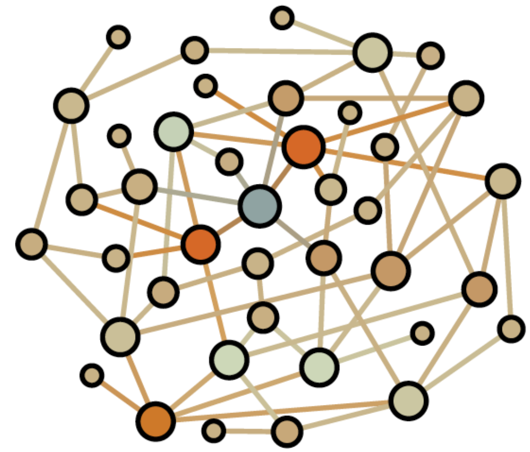
color~frequency

# Further Observations for Synchrony-Optimized Systems

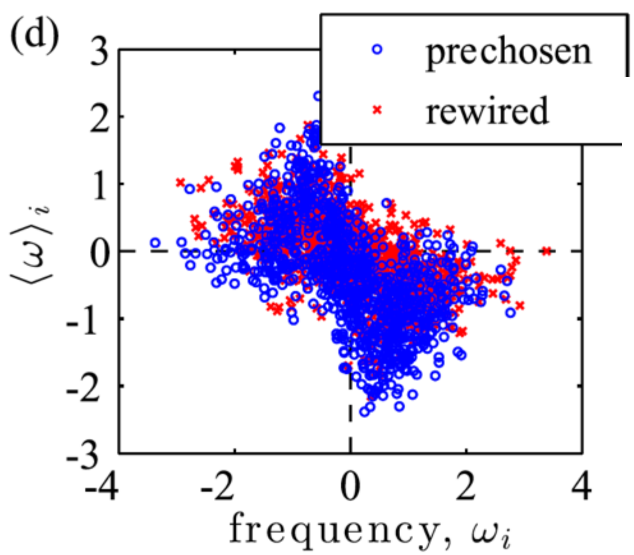
**Positive correlation between a node's frequency and degre**



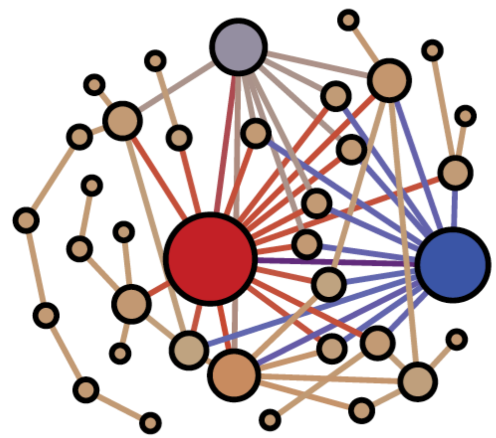

**homogeneous network**



**Negative correlation**
**frequencies of neighbo**
**nodes**




**heterogeneous network**



color~frequency

# Beyond Maximizing the Order Parameter

- **One can tune the order parameter of a system by aligning the oscillator frequencies with other eigenvectors**
  - For a given scale-free network with N=1000 nodes, we consider setting the frequencies as $\boldsymbol{\omega} \propto \boldsymbol{v}^{100}, \boldsymbol{v}^{200}, \ldots, \boldsymbol{v}^{1000}$