An Evolving Introduction to Game Theory

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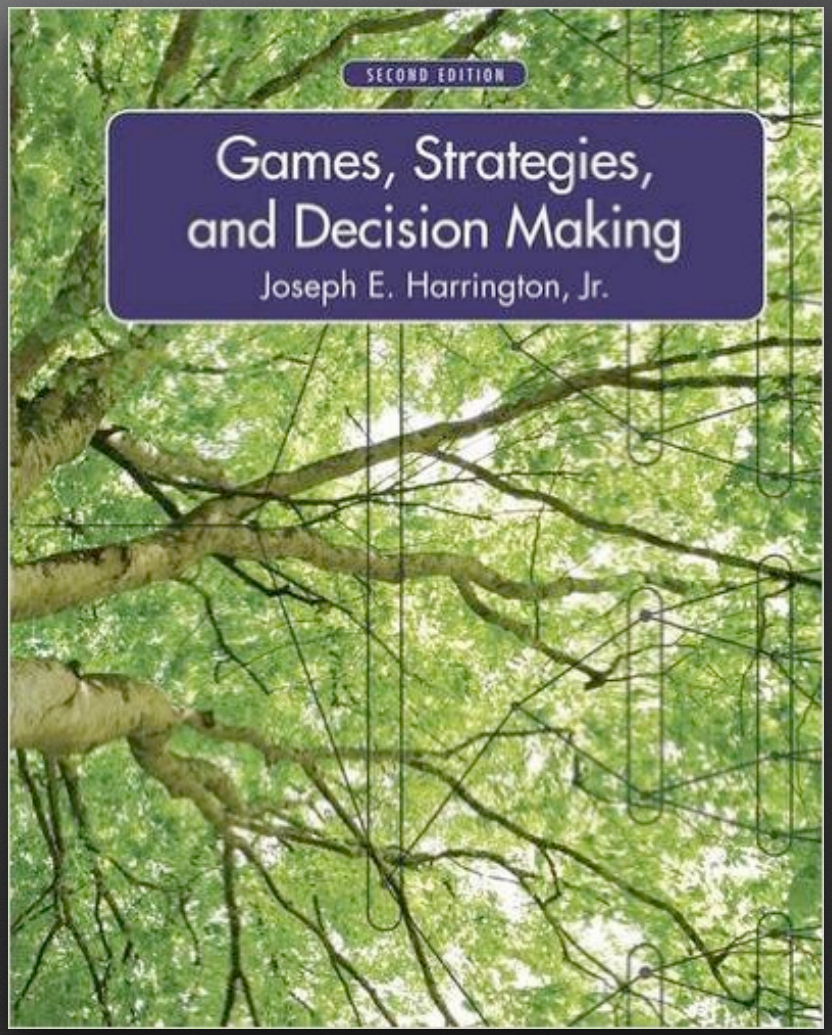
Game Theory for Sophomores +

- Cross-listed: Math & Econ
- Applied Statistics (AP-level) prerequisite
- Game theory text popular in economics
  - contains biology-based material
- + instructor-authored
  - beginning (~1 week)
  - end (~2 weeks) of the class

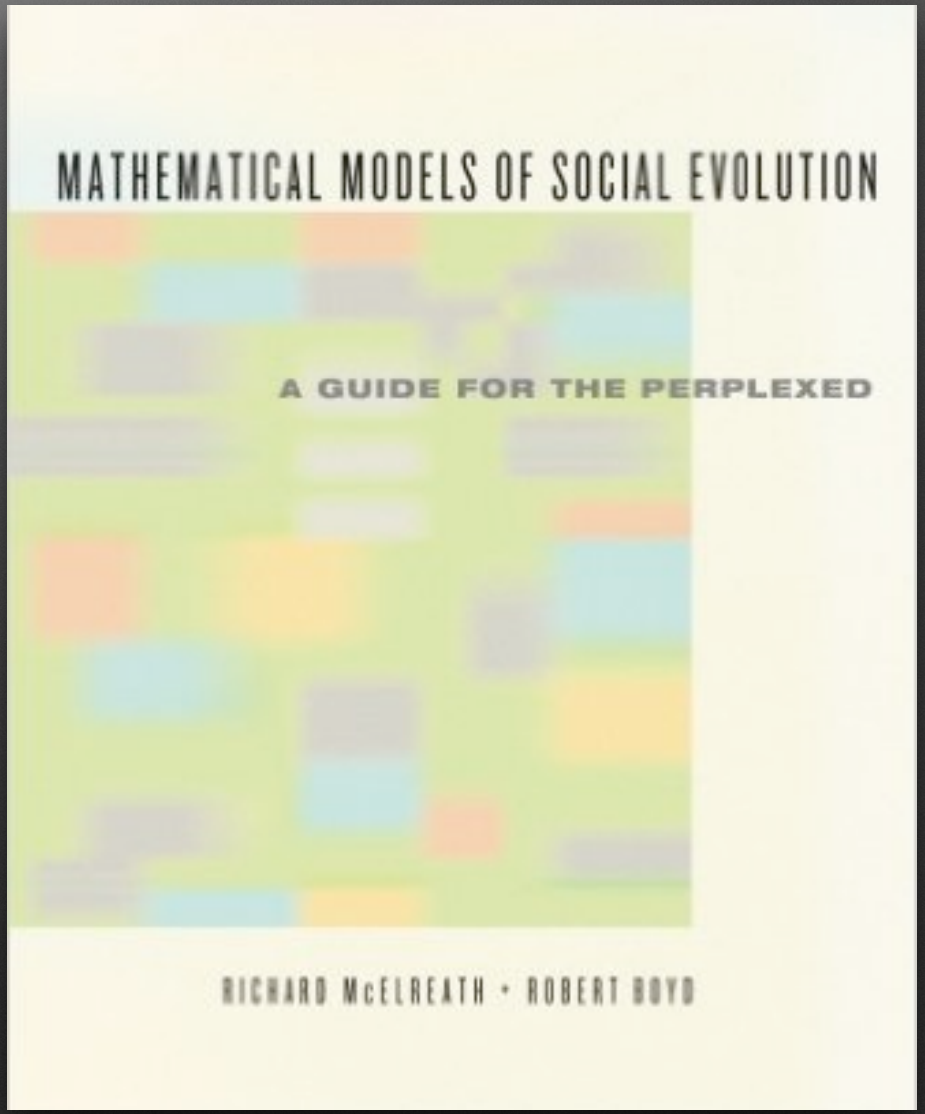


Not your usual sophomore-level math class

- Diverse audience, including as regards math background
  - Neurosci, never taken Econ
  - Math, unfazed by [simple math]
  - Econ, daunted by the math: Didn't expect so much
- Previous game theory experience: two-by-two matrices
  … need to quickly get to the more mathematical treatment
- Applications dominate theory, but theory is vibrant and central
- Material amenable to innovative pedagogy and challenging the breadth of student aptitude

Keep it Simple!

- Use only discrete probability and discrete generations to avoid most Calculus (except geometric series)
- Need expected value, variance and covariance to convey Price equation
- Elementary presentations by McElreath & Boyd and Bowles & Gintis instrumental but needed cleaning



What is covered?

I. The elements

- Evolution: Natural Selection & Speciation
- Rationality vs Natural Selection
- Iterated Deletion of Strictly Dominated Strategies
- Pure/Mixed Nash Equilibrium
- Backward Induction & Subgame Perfection
- Imperfect Information

What is covered?
II. Evolutionary Games

- Repeated Games: Partial Optimality for Nash Equils
- Games in Institutions with Indefinite Lives
- Evolutionarily Stable Strategies & Replicator Dynamic
- Positive Assortment & Hamilton’s Rule
- Multilevel Selection and Price’s Equation

COURSE GUIDELINE

Chapter	Core	Broad Social Science	Private Information	Repeated Interaction	Biology	Simple	Advanced
1: Introduction to Strategic Reasoning	✔	✔	✔	✔	✔	✔	✔
2: Building a Model of a Strategic Situation	✔	✔	✔	✔	✔	✔	✔
3: Eliminating the Impossible: Solving a Game when Rationality Is Common Knowledge	✔	✔	✔	✔	✔	✔	✔
4: Stable Play: Nash Equilibria in Discrete Games with Two or Three Players	✔	✔	✔	✔	✔	✔	✔
5: Stable Play: Nash Equilibria in Discrete *n*-Player Games		✔					✔
6: Stable Play: Nash Equilibria in Continuous Games							✔
7: Keep 'Em Guessing: Randomized Strategies		✔	✔		✔		✔
8: Taking Turns: Sequential Games with Perfect Information	✔	✔	✔	✔	✔	✔	✔
9: Taking Turns in the Dark: Sequential Games with Imperfect Information	✔	✔	✔	✔	✔	✔	✔
10: I Know Something You Don't Know: Games with Private Information		✔	✔				
11: What You Do Tells Me Who							

in Discrete *n*-Player Games		✔					✔
6: Stable Play: Nash Equilibria in Continuous Games							✔
7: Keep 'Em Guessing: Randomized Strategies		✔	✔		✔		✔
8: Taking Turns: Sequential Games with Perfect Information	✔	✔	✔	✔	✔	✔	✔
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10: I Know Something You Don't Know: Games with Private Information		✔	✔				
11: What You Do Tells Me Who You Are: Signaling Games		✔	✔				
12: Lies and the Lying Liars That Tell Them: Cheap-Talk Games			✔				
13: Playing Forever: Repeated Interaction with Infinitely Lived Players		✔		✔	✔	✔	
14: Cooperation and Reputation: Applications of Repeated Interaction with Infinitely Lived Players		✔		✔	14.3	✔	
15: Interaction in Infinitely Lived Institutions				✔			
16: Evolutionary Game Theory and Biology: Evolutionarily Stable Strategies					✔		
17: Evolutionary Game Theory and Biology: Replicator Dynamics				✔	✔		

Supplement Main Thread with Student Presentations

- Students from wide variety of majors
- Following their interests results in great breadth of presentations
- Variation in sophistication allows strong math students a chance to exercise
- Students generally enjoy presentations of peers
- Change of pace especially valued by students less comfortable with math

Useful pedagogy: BYOD “Clickers”

- Each lecture presentation incorporates a few questions that students respond to either using clickers or phones/web browsers (PollEverywhere)
- Particularly helpful for reinforcing concepts vs techniques
- Easier to incorporate than quizzes, with faster feedback and reinforcement of ideas immediately after introduction

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I. The elements

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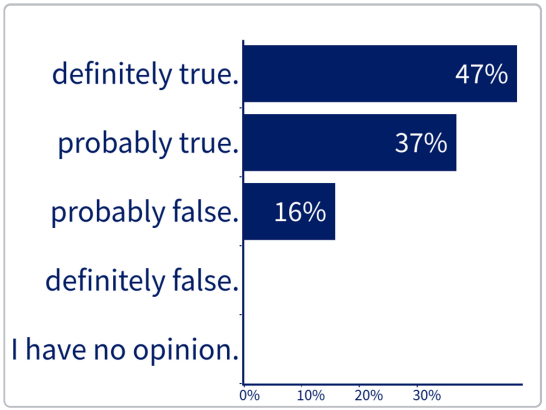
    - Rationality & Common Knowledge are simple assumptions that paper over complex behavior

Useful pedagogy: “Clickers”

Evolution is



Response options

definitely true.

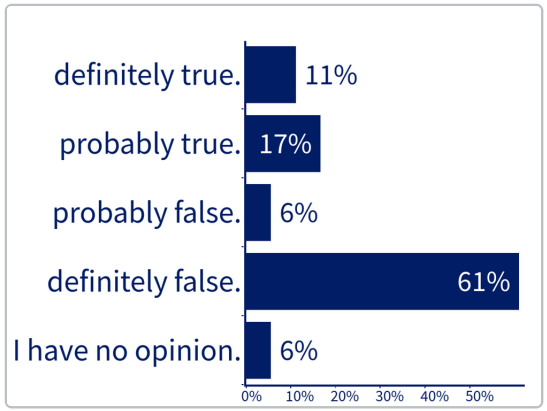
probably true.

probably false.

definitely false.

I have no opinion.

Creationism is



Response options

definitely true.

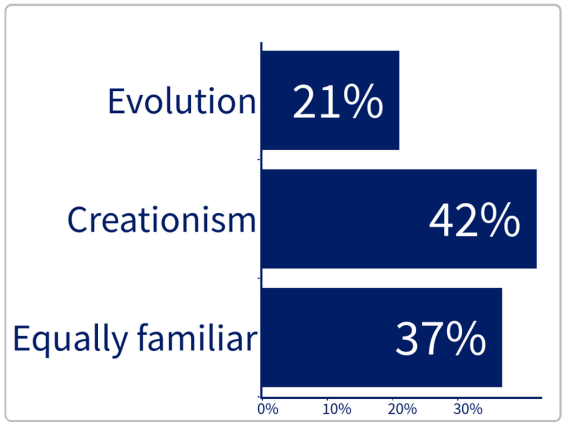
probably true.

probably false.

definitely false.

I have no opinion.

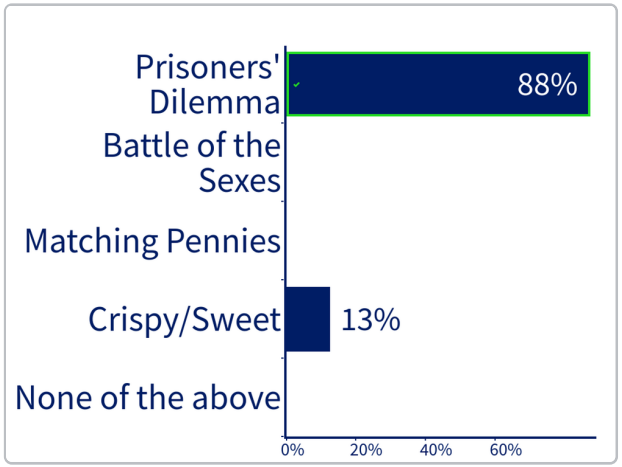
With which of these positions do you think Americans are more familiar?



Response options	Count
Evolution	4
Creationism	**8**
Equally familiar	7

Useful pedagogy: BYOD "Clickers"

Which games have unique (pure-strategy) equilibria?



Response options

Prisoners' Dilemma

Battle of the Sexes

Matching Pennies

Crispy/Sweet

None of the above

- How many children would you like to have?
- How many children are in your family?

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5: Stable Play: Nash Equilibria in Discrete *n*-Player Games		✔			

Boxed Pigs

		Large pig	
		Press lever	*Wait at dispenser*
Small pig	*Press lever*	1,5	−1,9
	Wait at dispenser	4,4	0,0

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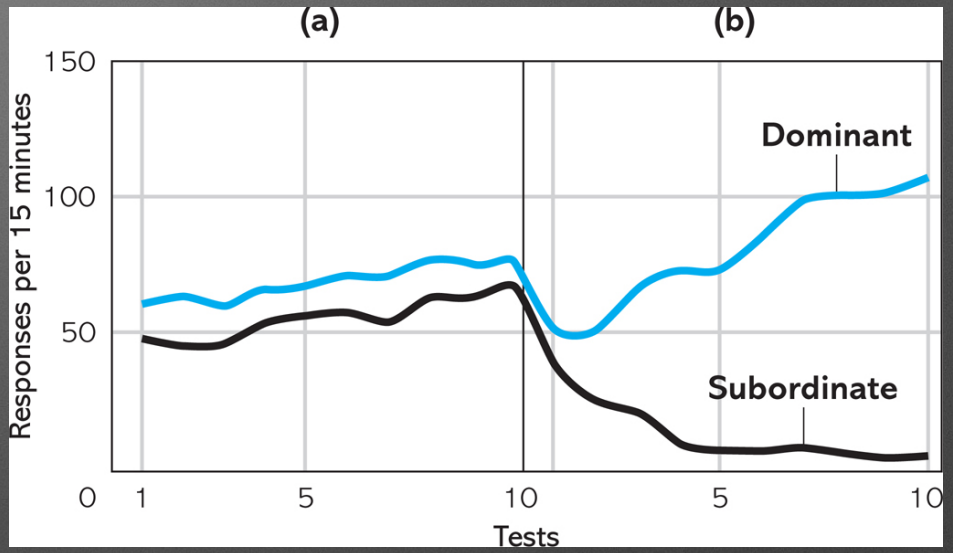

- Check out Harrington's language!
  "How does [the] outcome *emerge*?"
  - We'll see this language when considering **complexity**
  - Similar to evolution, but more often in **population** behavior

in Continuous Games					
7: Keep 'Em Guessing: Randomized Strategies		✔	✔		✔
8: Taking Turns: Sequential Games with Perfect Information	✔	✔	✔	✔	✔
9: Taking Turns in the Dark: Sequential Games with Imperfect Information	✔	✔	✔	✔	✔

Bkwds Ind'n … Really?

- Centipede game (Chain-store paradox)

- Experimental evidence in parentheses
- Sure, it ***does*** work, to a degree.
- Segue: At node 4, should assumptions change?

Mixed Strategy & Expectation

Probability p defines a mixed strategy for officer, and d defines dealer's mixed strategy.

Probabilities

Drug Dealer

Officer

	Street Corner	Park
Street Corner	$p \bullet d$	$p \bullet (1\text{-}d)$
Park	$(1\text{–}p) \bullet d$	$(1\text{–}p) \bullet (1\text{–}d)$

Payoffs

Drug dealer

Officer

	Street corner	*Park*
Street corner	80,20	0,100
Park	10,90	60,40

$$V_{\text{Officer}} = p \cdot d \cdot 80 + (1-p) \cdot d \cdot 10 + p \cdot (1-d) \cdot 0 + (1-p) \cdot (1-d) \cdot 60$$

$$V_{\text{Dealer}} = p \cdot d \cdot 20 + (1-p) \cdot d \cdot 90 + p \cdot (1-d) \cdot 100 + (1-p) \cdot (1-d) \cdot 40$$

Expected Payoffs

vert axis: payoff from each pure strategy

d axis: dealer's likelihood of street

$$V_{PO}(0,d) = 60 - 50d$$

$$V_{PO}(1,d) = 80d$$

Expected Payoffs

vert axis: payoff from each pure strategy

d axis: dealer's likelihood of street

$$V_{PO}(0,d) = 60 - 50d$$

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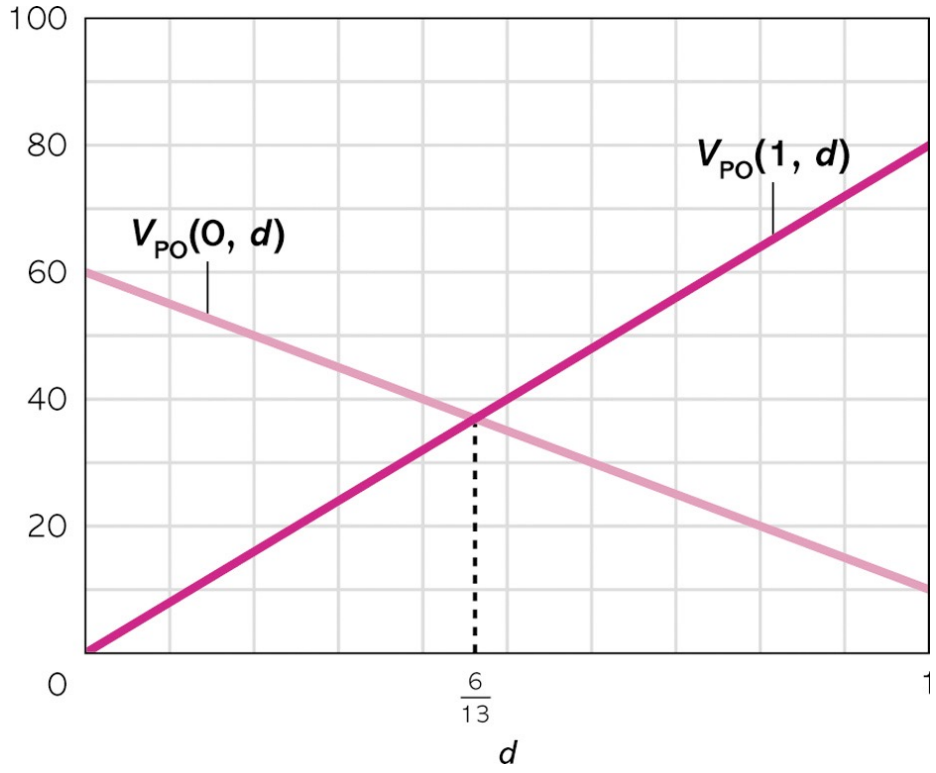
$$V_{\text{Officer}} = 60 - 60p - 50d + 130pd$$

Expected Payoffs

Text's Figure: officer's pure strategies (two)



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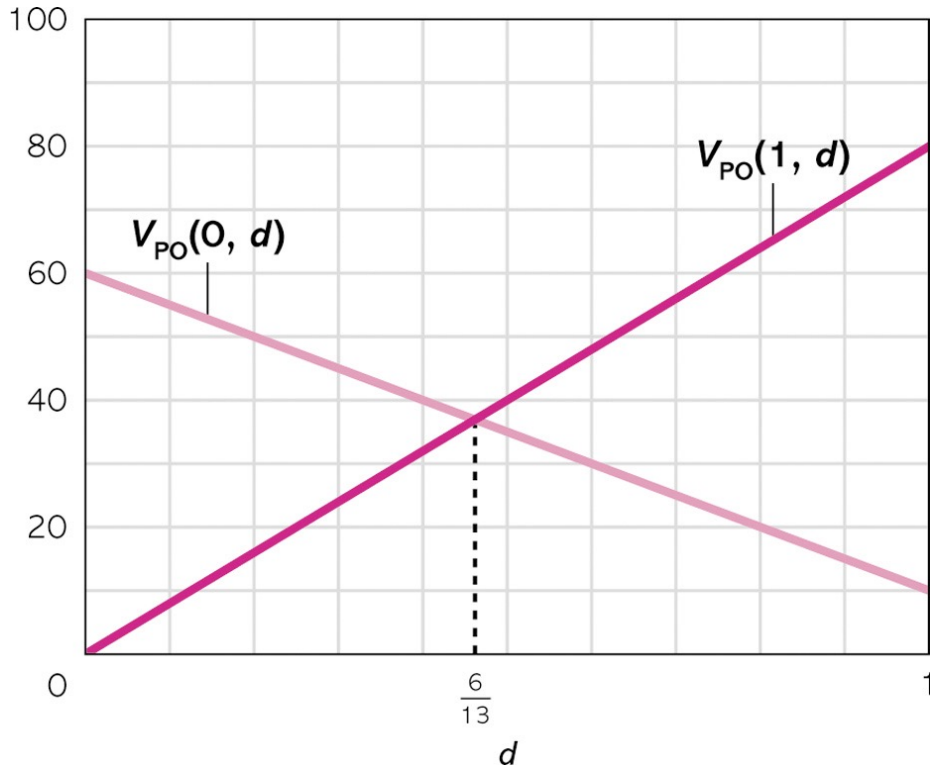

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$$V_{PO}(0,d) = 60 - 50d$$

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Now, all of the officer's mixed strategies

$$V_{\text{Officer}} = 60 - 60p - 50d + 130pd$$

Expected Payoffs

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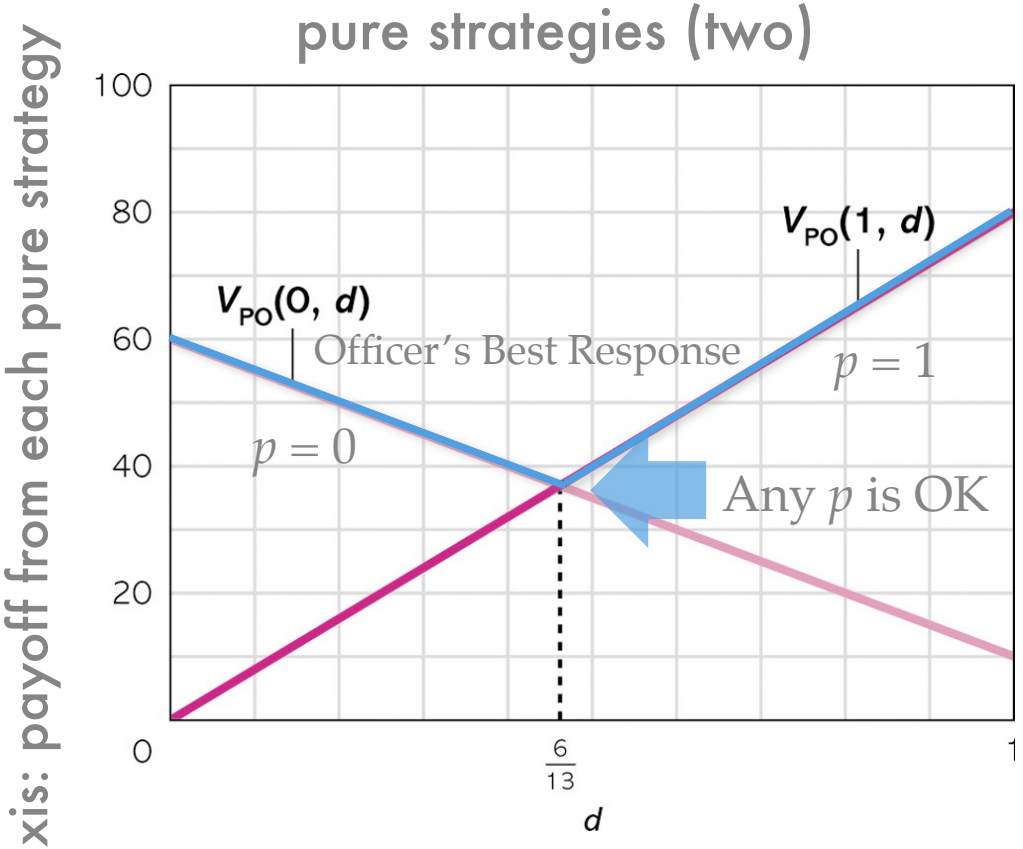

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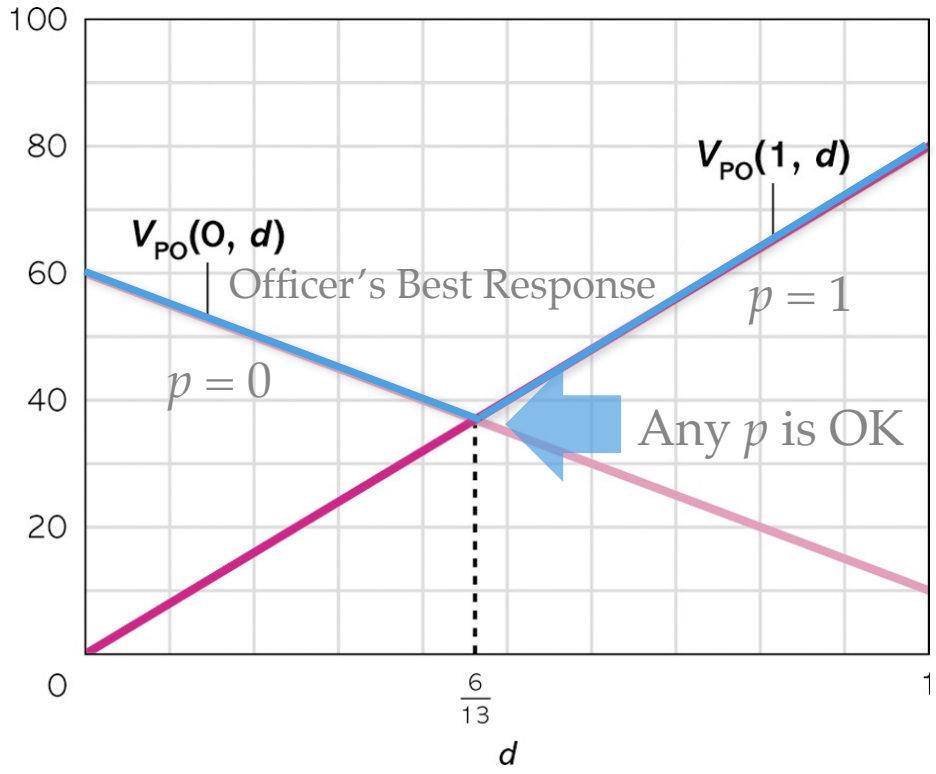

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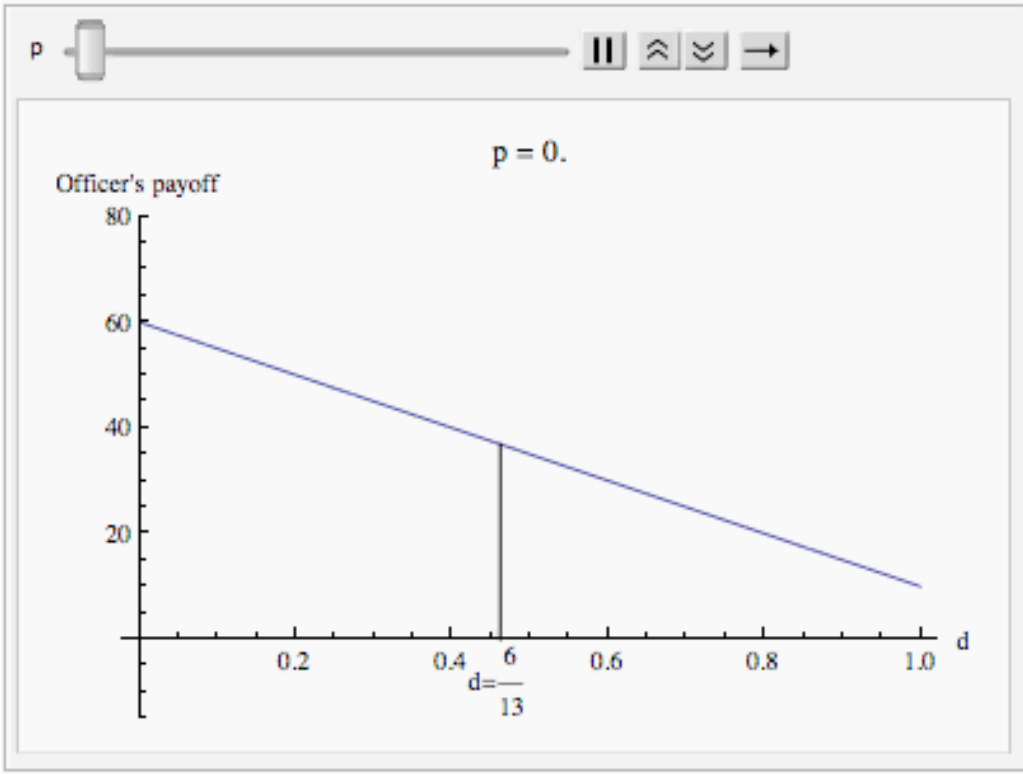
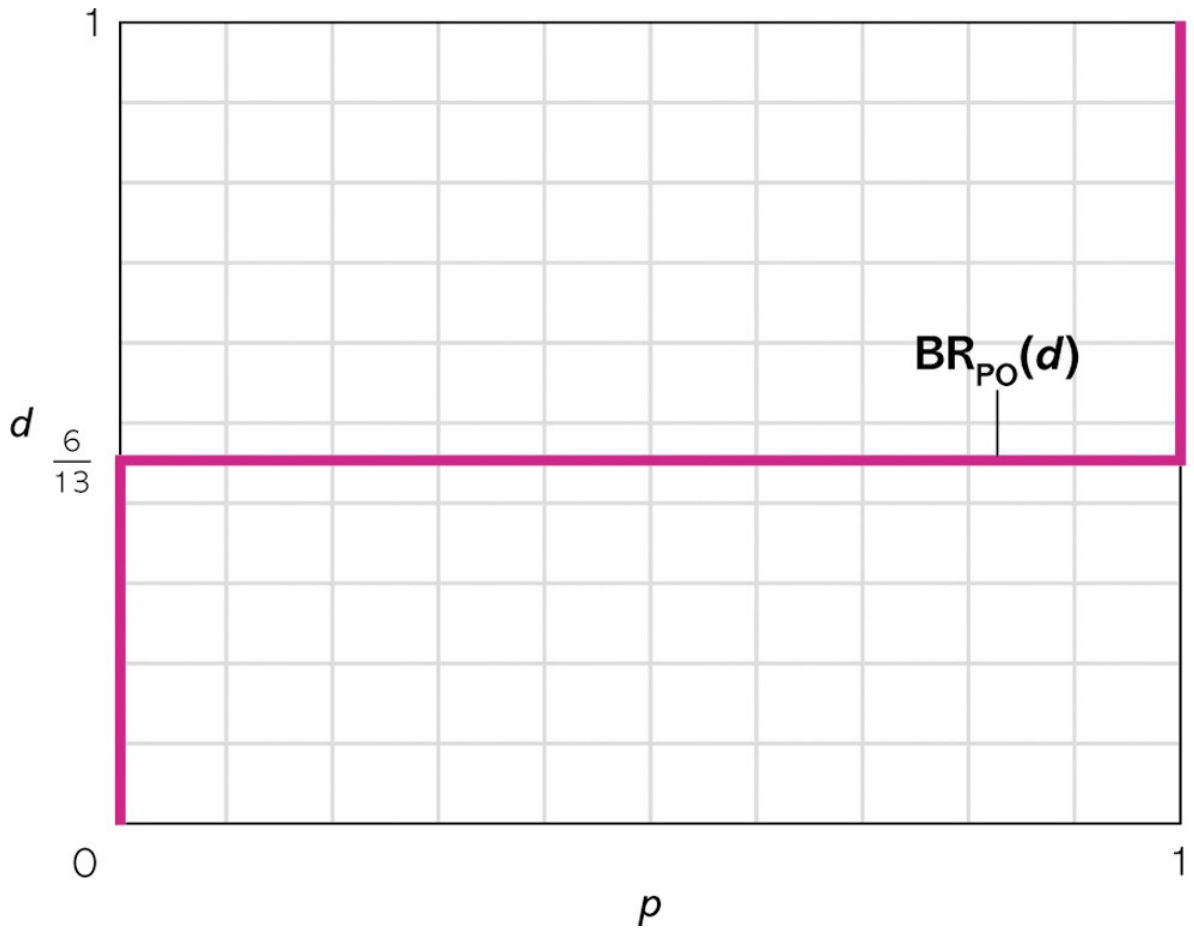


$$V_{\text{Officer}} = 60 - 60p - 50d + 130pd$$

Best Reply Strategy



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14: Cooperation and Reputation: Applications of Repeated Interaction with Infinitely Lived Players		✔		✔	14.3
15: Interaction in Infinitely Lived Institutions				✔	
16: Evolutionary Game Theory and Biology: Evolutionarily Stable Strategies					✔
17: Evolutionary Game Theory and Biology: Replicator Dynamics				✔	✔

Example: Vampire Bats

- Vampire bats share food (blood) with unrelated adults with whom they roost
- Bats starve to death after 60 hours of not eating
- Primarily among females, though males rarely will share with juveniles
- Probability of sharing is correlated with history of reciprocal sharing

Vampire Bat Stage Game

TABLE 14.2 Payoffs of Vampire Bats

Bat	Sharing	No Sharing
Fed bat	8	10
Hungry bat	4	−1

Bats discount the future at factor δ, succeed in feeding on a day with probability s

This bat has fed

This bat has not fed

$$V = s(10s + 8(1-s)) + (1-s)(4s - 1(1-s)) + \delta V$$

The other bat has fed

The other bat has not fed

Vampire Bat Stage Game

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$$V = s(10s + 8(1-s)) + (1-s)(4s - 1(1-s)) + \delta V$$

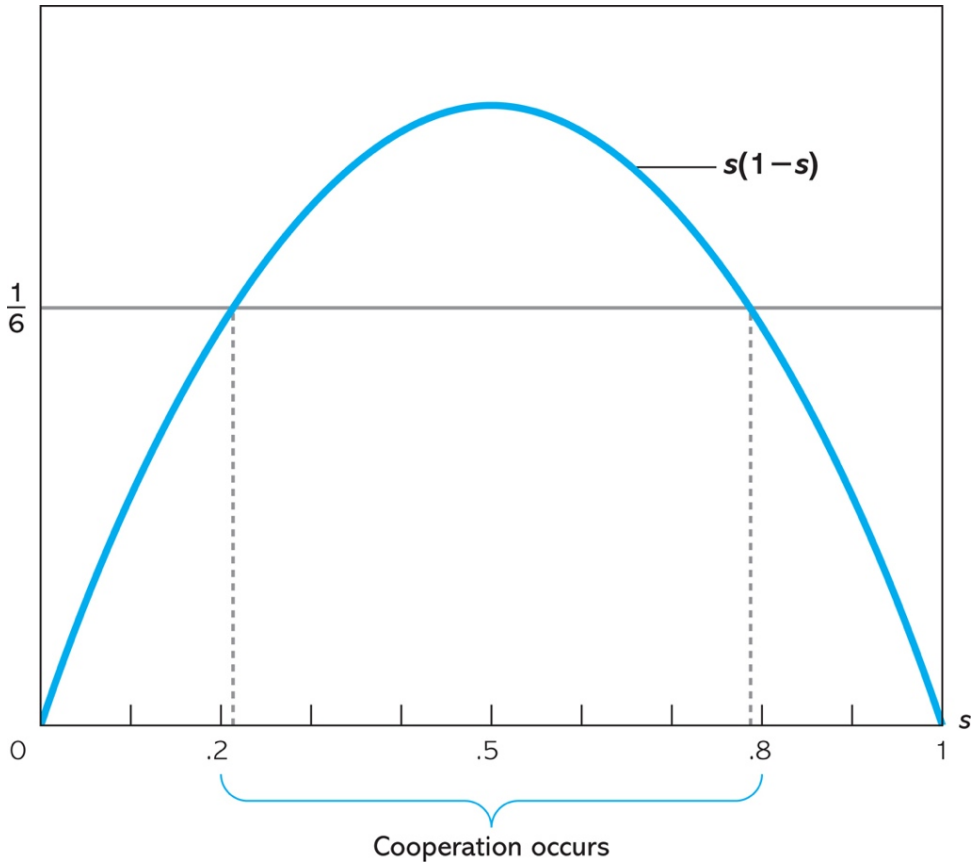
$$V = \frac{-3s^2 + 14s - 1}{1-\delta}$$

Deviation: $10s - 1(1-s) = 11s - 1$

$$8 + \delta\frac{-3s^2 + 14s - 1}{1-\delta} \geq 10 + \delta\frac{11s - 1}{1-\delta}$$

Cooperation condition

$$s(1-s) \geq \frac{2(1-\delta)}{3\delta}$$



Evolutionary game theory

Game Theory without the "rational" model

TABLE 16.1 PARALLEL CONCEPTS IN RATIONAL AND EVOLUTIONARY GAME THEORY

Rational Game Theory	Evolutionary Game Theory
Set of players	Population from which the set of players is drawn
Payoff: measure of well-being	Fitness: measure of reproductive success
Strategy is chosen by a player	Strategy is inherited by a player and "chosen" by natural selection
Equilibrium: no player can do better	Equilibrium: no small mutation in the population can survive

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Rationality

Evolutionary game theory

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Rationality

Natural Selection

Rock-Paper-Scissor Lizards

Lizards whose territorial strategies mimic RPS

		Lizard 2		
		Blue	***Orange***	***Yellow***
Lizard 1	***Blue***	0,0	−1,1	1,−1
	Orange	1,−1	0,0	−1,1
	Yellow	−1,1	1,−1	0,0

a.k.a. *The Fashion Game*

RPS-Lizards & ESS

- There is no pure strategy Nash Eq (Why?), and so no pure strategy ESS, either
- The mixed strategy Nash eq p has $F(p, p) = F(q, p)$ for any other strategy q (as it must by indifference)
- However, $F(p, q) = 0$ for any strategy q, and any strategy has $F(q, q) = 0$, as well. Thus $F(p, q) = F(q, q)$, and p cannot be a mild (or strong) ESS.

Some Games Have No Evolutionary Stable Strat!

Mathematicians like problems that are "well-posed:"

1) Solution exists
2) Solution is unique
3) Solution changes only a little when the conditions of the problem change a little

… (Evolutionary) Game Theory problems are rarely "well-posed"—*because that's how life is!*

Two Population ESSes

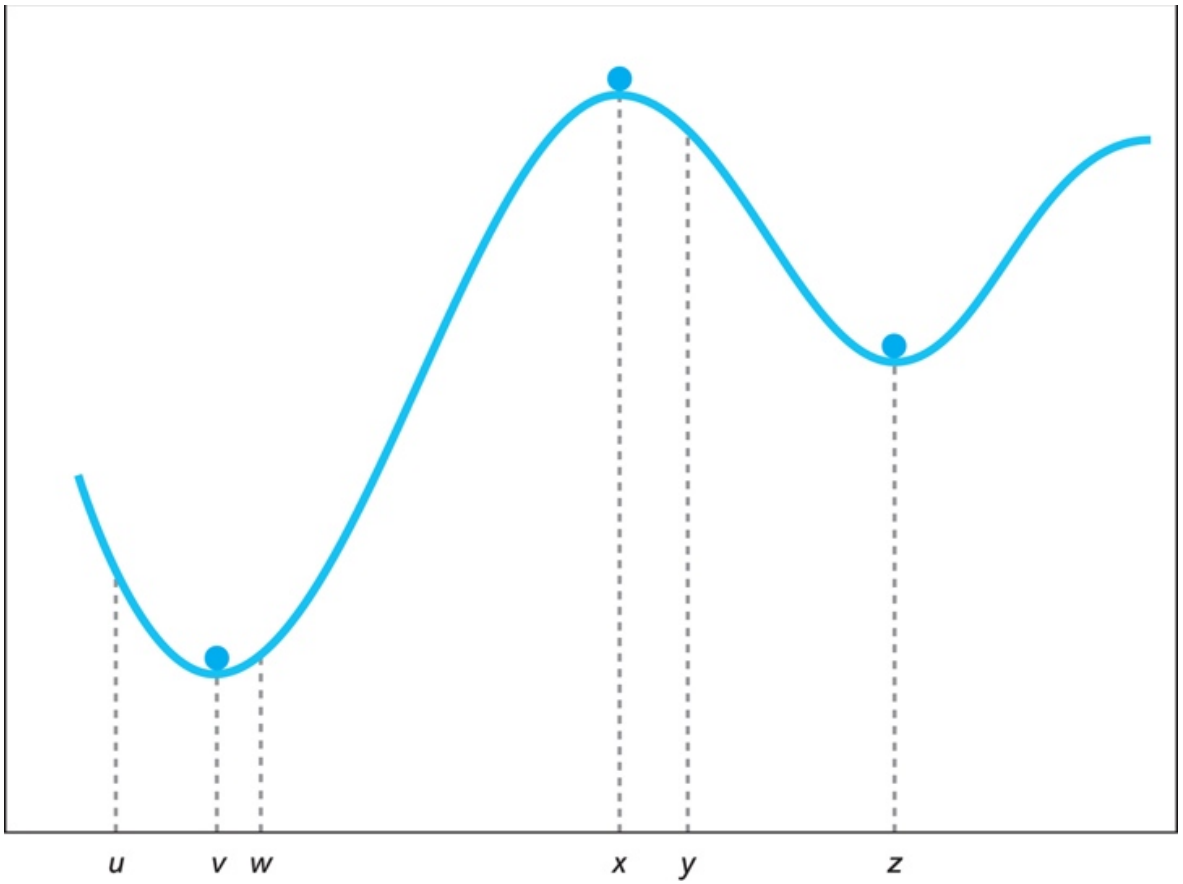
- ESS can make sense even analyzing a non-symmetric game. Consider BoS ("Battle of the Sexes" or "Boxing or the Symphony"):

		High Brow: Boxing	High Brow: Symphony
Pugilist	**Boxing**	3, 2	0, 0
	Symphony	0, 0	2, 3

2 populations: pugilists and high brows. Can we still have an ESS?

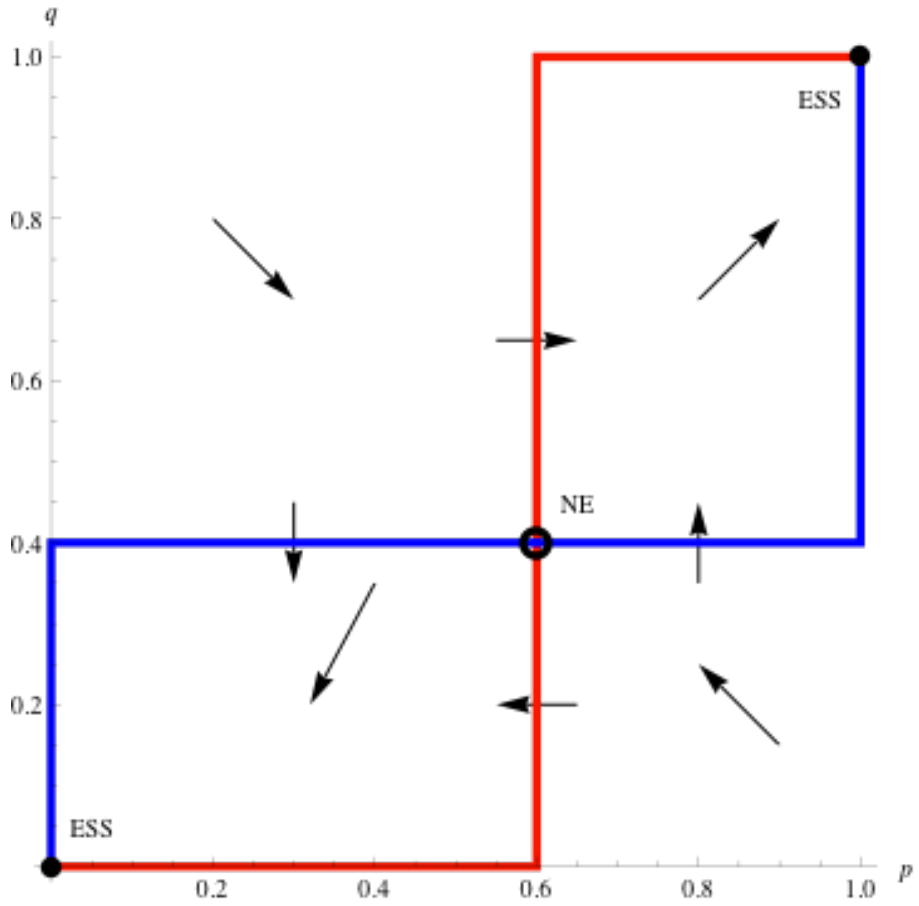
Rest points, stability, attractors

- Top of the hill versus the bottom
- nudge away



Best Response for BoS

- Best-response curves for Pugilist and High-Brow:



- Best response predicts dynamics to ESS

Not your usual sophomore-level math class

- Diverse audience, including as regards math background
  - Neurosci, never taken Econ
  - Math, unfazed by [simple math]
  - Econ, daunted by the math: Didn't expect so much
- Previous game theory experience: two-by-two matrices
  … need to quickly get to the more mathematical treatment
- Applications dominate theory, but theory is vibrant and central
- Material amenable to innovative pedagogy and challenging the breadth of student aptitude