Neural Codes and Convexity

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Very Important People

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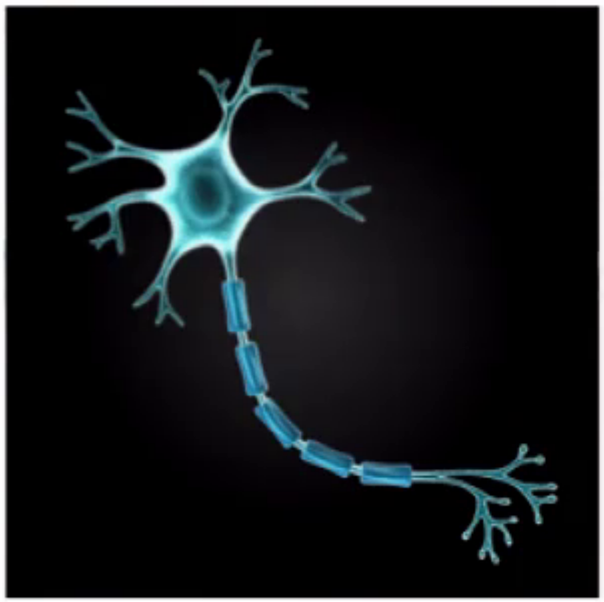
Vladimir Itskov (Penn State)
William Kronholm (Whittier College)
Yan Zhang (UC Berkeley)

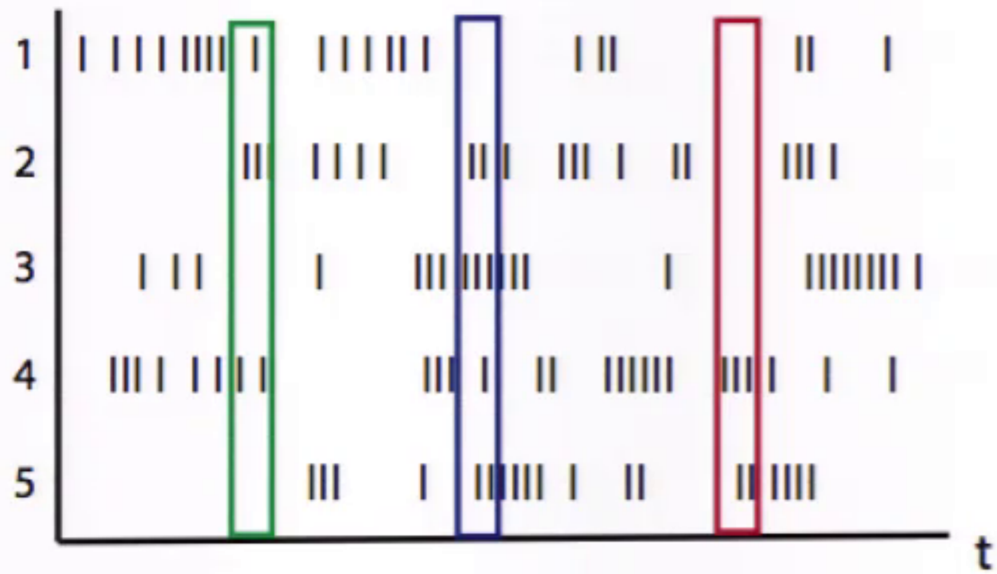
The neural code

Neurons communicate by firing signals called *action potentials* or *spikes*.
Spike times are collected in a table called a raster.



The neural code

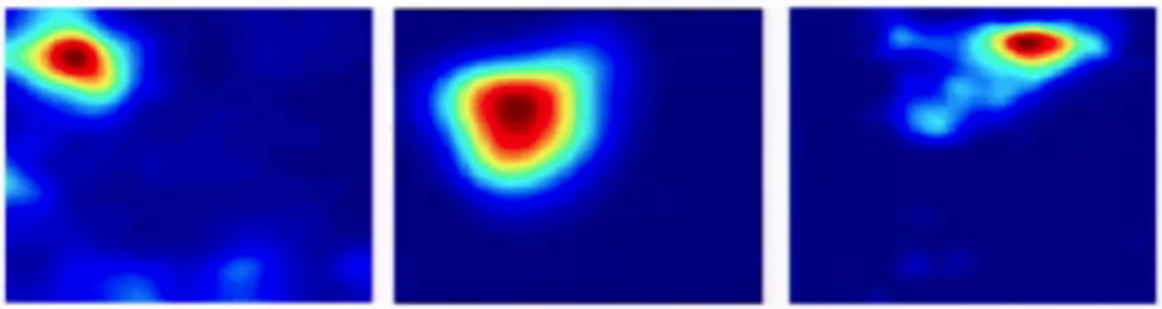
A **neural code** $\mathcal{C} \subseteq \{0,1\}^n$ is a set of firing patterns, or codewords.



11010 01111 00011

Place cells

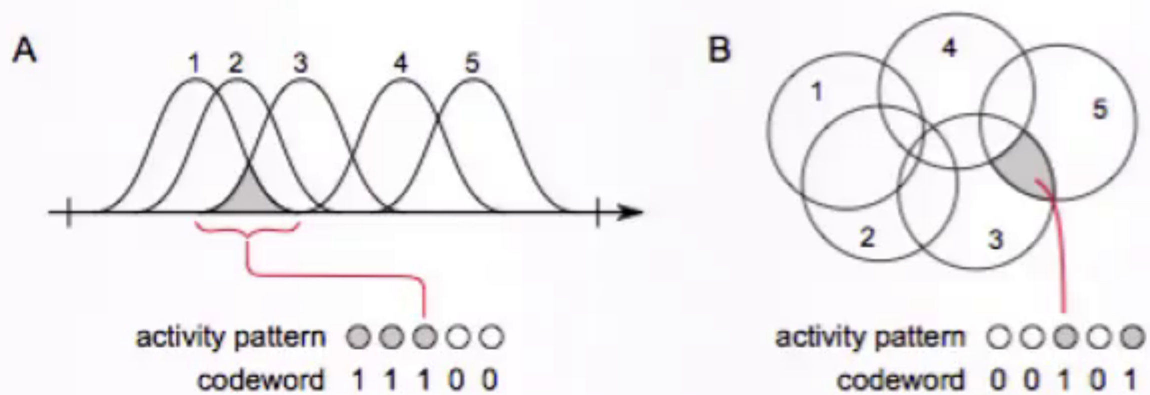
- Place cells: a type of neuron, found in the hippocampus (navigation, memory)
- Each place cell has a place field - a region to which it is sensitive.



Neuron 1 Neuron 2 Neuron 3

Receptive fields

Let $X \subset \mathbb{R}^d$ be a stimulus space. A subset $U_i \subset X$ is the **receptive field** for neuron i if that neuron has a high firing rate for stimuli in U_i.



Receptive field codes

If $\mathcal{C}$ represents the full set of regions for some collection of receptive fields $\mathcal{U}$, then $\mathcal{C} = \mathcal{C}(\mathcal{U})$ is a **receptive field code**.

Receptive field codes

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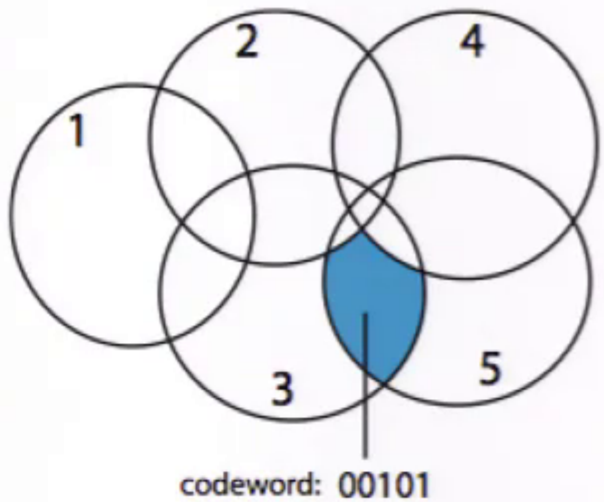
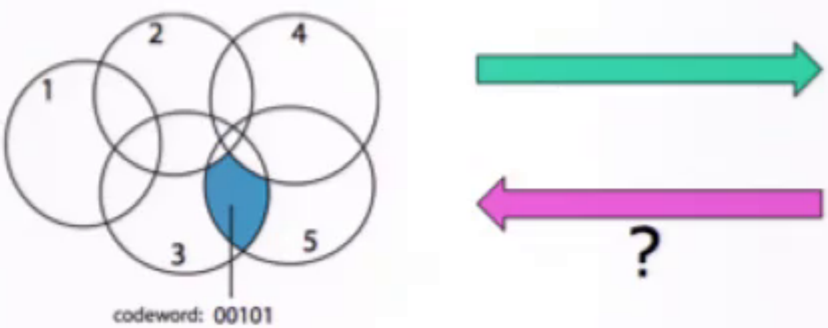
If $\mathcal{C} = \mathcal{C}(\mathcal{U})$ for some set of receptive fields $\mathcal{U}$ where each U_i is a convex open subset of $\mathbb{R}^d$, then $\mathcal{C}(\mathcal{U})$ is a **convex** receptive field code.



Big Questions

1. Given a code $\mathcal{C}$, is $\mathcal{C}$ a convex receptive field code?
2. If so, what is the smallest dimension d so $\mathcal{C}$ can be realized as $\mathcal{C}(\mathcal{U})$ for convex sets $U_i \subset \mathbb{R}^d$?



Code C:

00000	10000	01000	00100
00010	00001	11000	10100
01100	01010	00101	00011
11100	01110	01101	01011
00111	01111		

The Neural Ring

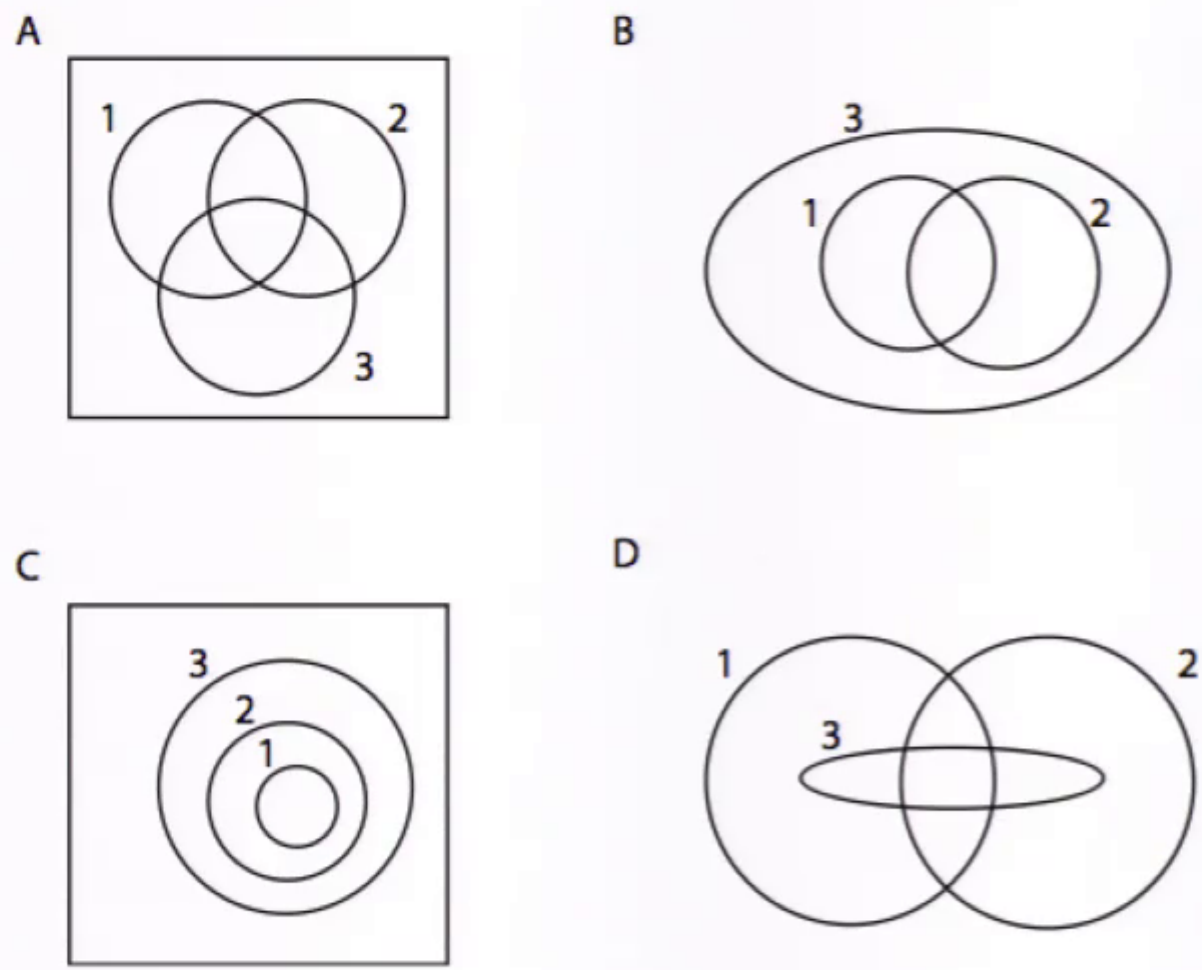
Simplicial complex of $\mathcal{C}$

- A **simplicial complex** Δ on n vertices is a collection of subsets of $\{1, ..., n\} = [n]$ such that if $\sigma \in \Delta$ and $\tau \subset \sigma$, then $\tau \in \Delta$ also.
- A code $\mathcal{C} \subset \{0, 1\}^n$ corresponds to a set of subsets of $[n]$:

$$\mathrm{supp}(\mathcal{C}) = \{\sigma \subset [n] \,|\, \sigma = \mathrm{supp}(c) \text{ for some } c \in \mathcal{C}\}.$$

- The simplicial complex of the code, denoted $\Delta(\mathcal{C})$, is the smallest simplicial complex containing $\mathrm{supp}(\mathcal{C})$.

Why go beyond the simplicial complex?



All codes realized here have the same simplicial complex.

Stanley-Reisner rings

Let Δ be a simplicial complex on n vertices and $\mathbf{k}$ a field. The **Stanely-Reisner ideal** of Δ is defined

$$I_\Delta = \langle x_{i_1} \cdots x_{i_s} \mid \{i_1, ..., i_s\} \notin \Delta \rangle$$

Then, the **Stanley-Reisner ring** is given by

$$\mathbf{k}[x_1, ..., x_n]/I_\Delta.$$

This ring encodes all information about the simplicial complex. We attempt to generalize this idea for codes which are not necessarily simplicial complexes.

The Neural Ring

Given a code $\mathcal{C} \subset \{0,1\}^n$, define the ideal $I_\mathcal{C}$ of $\mathbb{F}_2[x_1, ..., x_n]$ as follows:

$$I_\mathcal{C} = \{f \in \mathbb{F}_2[x_1, ..., x_n] \mid f(c) = 0 \text{ for all } c \in \mathcal{C}\}.$$

The *neural ring* $R_\mathcal{C}$ is defined

$$R_\mathcal{C} = \mathbb{F}_2[x_1, ..., x_n]/I_\mathcal{C}.$$

$R_\mathcal{C}$ is exactly the ring of functions $f : \mathcal{C} \to \{0,1\}$.

The neural ideal: alternative generators

For each $v \in \{0,1\}^n$, we consider the 'indicator' polynomial

$$\rho_v = \prod_{v_i=1} x_i \prod_{v_i=0} (1 - x_i).$$

Then we define the **neural ideal:**

$$J_{\mathcal{C}} = \langle \{\rho_v \mid v \notin \mathcal{C}\} \rangle.$$

In fact: $I_{\mathcal{C}} = J_{\mathcal{C}} + \langle x_1(1-x_1), \ldots, x_n(1-x_n) \rangle$.

Pseudo-monomials

The polynomials ρ_v which generate $J_{\mathcal{C}}$ are an example of **pseudo-monomials**: polynomials of the form

$$x_\sigma \prod_{j \in \tau} (1 - x_j)$$

for $\sigma \cap \tau = \emptyset$.

Key Idea: if $\mathcal{C} = \mathcal{C}(\mathcal{U})$, then any interesting relationships amongst the U_i are encoded by the pseudo-monomials in $J_{\mathcal{C}}$.

Pseudo-monomials

Theorem (Curto, Itskov, Veliz-Cuba,Y.)

If $\mathcal{C} = \mathcal{C}(\mathcal{U})$ *for some set of receptive fields* $\mathcal{U} = U_1, ..., U_n$ *with* $U_i \subseteq X \subseteq \mathbb{R}^d$, *then*

$$x_\sigma \prod_{j \in \tau} (1 - x_j) \in J_{\mathcal{C}}$$

if and only if

$$\bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \in \tau} U_j.$$

Canonical form

For practical purposes, we want a condensed set of generators for $J_{\mathcal{C}}$.

The **canonical form** of $J_{\mathcal{C}}$ is given by

$$CF(J_{\mathcal{C}}) = \{f \mid f \text{ is a minimal pseudo-monomial of } J_{\mathcal{C}}\}$$

- Minimal here means that f is not a multiple of another pseudo-monomial in $J_{\mathcal{C}}$.
- The pseudo-monomials in $CF(J_{\mathcal{C}})$ correspond to a minimal set of information about the relationships amongst the sets U_i.

Canonical form

The relationships in the canonical form come in 3 types:

Type 1:	x_σ	$\bigcap_{i\in\sigma} U_i = \emptyset$
Type 2:	$x_\sigma \prod_{j\in\tau}(1-x_j)$	$\bigcap_{i\in\sigma} U_i \subseteq \bigcup_{j\in\tau} U_j$
Type 3:	$\prod_{j\in\tau}(1-x_j)$	$X \subseteq \bigcup_{j\in\tau} U_j$

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Type 3: $\prod_{j\in\tau}(1-x_j)$ $$X \subseteq \bigcup_{j\in\tau} U_j$$

Minimality can also be interpreted:

Example

If $x_1x_2x_3 \in CF(J_C)$, then $U_1 \cap U_2 \cap U_3 = \emptyset$...but also, $U_1 \cap U_2 \neq \emptyset$.

Helly's Theorem

Theorem (Helly)

Let $U_1, ..., U_n$ be convex sets in $\mathbb{R}^d$, with $n > d$. If every $d+1$ of the sets have nonempty intersection, then there is a point common to all the sets.

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Example

The code $\{1110, 1101, 1011, 0111\}$ cannot be realized in $\mathbb{R}^2$.
Note $x_1x_2x_3x_4 \in CF(J_\mathcal{C})$.

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Note $x_1x_2x_3x_4 \in CF(J_\mathcal{C})$.

This only uses simplicial complex information.

Lemma (Curto, Itskov, Veliz-Cuba, Y.)

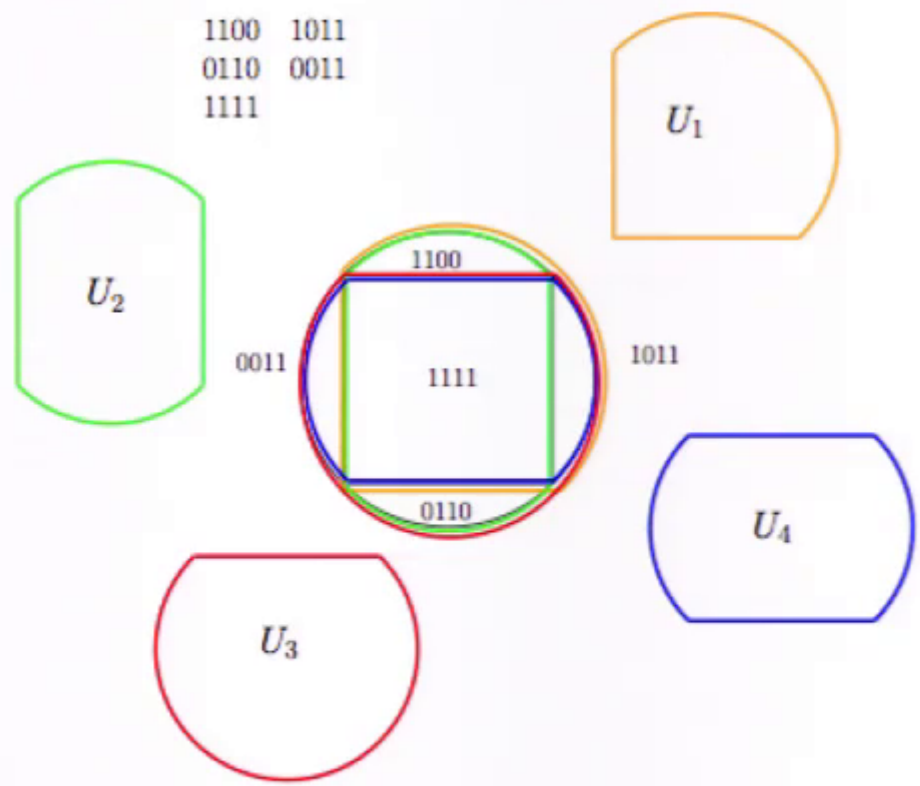
If $x_\sigma \in CF(J_\mathcal{C})$, then $\mathcal{C}$ cannot be realized in $\mathbb{R}^d$ for $d < |\sigma| - 1$.

Using the canonical form

If there are no Type 1 relations at all, then $\Delta(C)$ is a simplex.

Lemma (MRC)

If $\Delta(\mathcal{C})$ is a disjoint union of simplices, then $\mathcal{C}$ is convex in $\mathbb{R}^d$ for $d \leq 2$.



The nerve lemma

- Given a cover $\mathcal{U}$ of a set X, the **nerve** $N(\mathcal{U})$ is the simplicial complex given by $\sigma \in N(\mathcal{U}) \Leftrightarrow \bigcap_{i \in \sigma} U_i \neq \emptyset$.
- If $\mathcal{C} = \mathcal{C}(\mathcal{U})$, then $\Delta(\mathcal{C}) = N(\mathcal{U})$.

Lemma (Nerve Lemma)

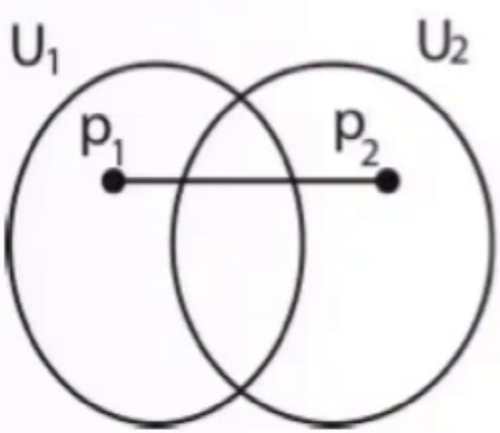
If $\mathcal{U} = U_1, \ldots, U_n$ is a finite cover of X with all intersections of U_is contractible, then $N(\mathcal{U})$ and X are homotopy equivalent.

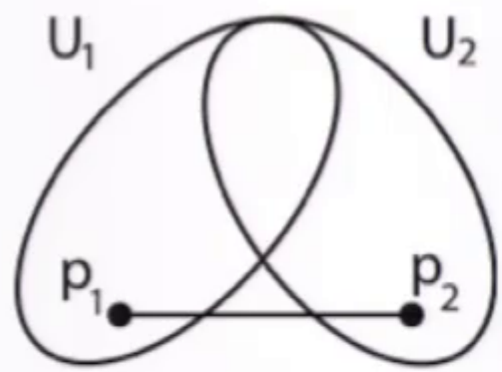
Local obstructions

Consider the code

$$\mathcal{C} = \{000, 100, 010, 110, 101, 011\}.$$

As neuron 3 fires with 1 and 2, but never with both, and never alone, this cannot be realized with open convex sets.





Local obstructions

Given $\mathcal{C}(\mathcal{U})$, a pair σ, τ with $\sigma \cap \tau = \emptyset$ forms a **local obstruction** if

$$\bigcap_{i \in \sigma} U_i \subseteq \bigcup_{i \in \tau} U_i$$

but the nerve of the cover of the intersection by sets in τ is not contractible.

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but the nerve of the cover of the intersection by sets in τ is not contractible.

Lemma (MRC, Giusti-Itskov)

If $\mathcal{C}$ is a convex code, then $\mathcal{C}$ is locally convex (has no local obstructions).

Links

The **link** of σ in Δ is given by

$$Lk_\sigma(\Delta) = \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \Delta\}$$

Collect the sets with non-contractible links in Δ:

$$M(\Delta) = \{\sigma \in \Delta \mid Lk_\sigma(\Delta) \text{ is non-contractible}\}.$$

Theorem (MRC)

$\mathcal{C}$ is locally convex if and only if $M(\Delta(\mathcal{C})) \subset \mathcal{C}$.

The codewords in $M(\Delta(\mathcal{C}))$ are **mandatory**.

Max intersection complete

Lemma (MRC)

Mandatory codewords $M(\Delta(\mathcal{C}))$ are always intersections of maximal facets/codewords.

We say $\mathcal{C}$ is **max intersection complete** if $\mathcal{C}$ contains all intersections of maximal codewords.

Theorem (MRC)

If $\mathcal{C}$ is a max intersection complete code, then $\mathcal{C}$ is locally convex.

Open questions

- Dimension - we know very little.
- **Conjecture 1:** $\mathcal{C}$ is convex if and only if $\mathcal{C}$ is locally convex.
- **Conjecture 2:** If $\mathcal{C}$ is max. intersection complete, then $\mathcal{C}$ is convex.
  Conjecture 1 $\Rightarrow$ Conjecture 2.
  The converse to Conjecture 2 does not hold.

Open questions

Conjecture 2: If $\mathcal{C}$ is max. intersection complete, then $\mathcal{C}$ is convex.

Theorem (Tancer)

if $\mathcal{C} = \Delta(\mathcal{C})$, *then* $\mathcal{C}$ *is convex.*

Open questions

Conjecture 2: If $\mathcal{C}$ is max. intersection complete, then $\mathcal{C}$ is convex.

Theorem (Tancer)

if $\mathcal{C} = \Delta(\mathcal{C})$, then $\mathcal{C}$ is convex.

Theorem (Giusti-Kronholm)

If $\mathcal{C}$ is intersection complete (contains all possible intersections of codewords) then $\mathcal{C}$ is convex.

Theorem (MRC)

If $n \leq 4$, then $\mathcal{C}$ is convex if and only if $\mathcal{C}$ is max intersection complete.

Algebraic signatures

	Algebraic signature of $J_{\mathcal{C}}$	Property of $\mathcal{C}$
A	$\exists\, x_\sigma(1-x_i)(1-x_j) \in \mathrm{CF}(J_{\mathcal{C}})$ s.t. $x_\sigma x_i x_j \in J_{\mathcal{C}}$	non-convex
B	$\exists\, x_\sigma \prod_{i\in\tau}(1-x_i) \in \mathrm{CF}(J_{\mathcal{C}})$ s.t. $G_{\mathcal{C}}(\sigma,\tau)$ is disconnected	non-convex
C	$\exists\, x_\sigma \prod_{i\in\tau}(1-x_i) \in \mathrm{CF}(J_{\mathcal{C}})$ s.t. $x_\sigma x_\tau \in \mathrm{CF}(J_{\mathcal{C}})$	non-convex
D	$\forall\, x_\sigma \prod_{i\in\tau}(1-x_i) \in \mathrm{CF}(J_{\mathcal{C}})$, $x_\sigma x_\tau \notin J_{\mathcal{C}}$	locally convex
E	$\forall\, x_\sigma \prod_{i\in\tau}(1-x_i) \in \mathrm{CF}(J_{\mathcal{C}})$, $\lvert\tau\rvert \leq 1$	convex ($\cap$-complete)

Table 1: Algebraic signatures of convexity. $G_{\mathcal{C}}(\sigma,\tau)$ is the simple graph on vertex set τ with edges $\{(ij) \in \tau\times\tau \mid x_\sigma x_i x_j \notin J_{\mathcal{C}}\}$.