A combined GDM–ELLAM–MMOC (GEM) scheme with local volume conservation for advection dominated PDEs

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Joint work with J. Droniou and K. N. Le

Australian Government
Australian Research Council

Plan

Advection-reaction model

$$\begin{cases} \phi \dfrac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u} c) &= f(c) \text{ on } Q_T := \Omega \times (0, T) \\ c(\cdot, 0) &= c_{\mathrm{ini}} \qquad \text{on } \Omega \end{cases}$$

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- $\mathbf{u} \cdot \mathbf{n} = 0$ on $\partial\Omega$
- $f(c) = f(c, \boldsymbol{x}, t) : \mathbb{R} \times Q_T \to \mathbb{R}$ is Lipschitz continuous w.r.t. its first variable and $f(0, \cdot, \cdot) \in L^\infty(Q_T)$.

Gradient discretisation

Gradient discretisation: $\mathcal{D} = (X_{\mathcal{D}}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ with

- $X_{\mathcal{D}}$ finite dimensional space *(encodes the unknowns)*.
- $\Pi_{\mathcal{D}} : X_{\mathcal{D}} \to L^\infty(\Omega)$ *(reconstructs a function)*.
- $\nabla_{\mathcal{D}} : X_{\mathcal{D}} \to L^\infty(\Omega)^d$ *(reconstructs a gradient)*.

Plan

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Weak formulation and proper choice of test functions

Weak formulation between two time steps $t^{(n)}$ and $t^{(n+1)}$: for $\varphi \in C^\infty(\overline{\Omega} \times [t^{(n)}, t^{(n+1)}])$:

$$\boxed{-\int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega c(\phi\partial_t\varphi + \mathbf{u} \cdot \nabla\varphi)}$$
$$+ \int_\Omega \phi c(t^{(n+1)})\varphi(t^{(n+1)}) - \int_\Omega \phi c(t^{(n)})\varphi(t^{(n)})$$
$$= \int_{t^{(n)}}^{t^{(n+1)}} \int_\Omega f(c, \boldsymbol{x}, t) d\boldsymbol{x} dt.$$

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Choice of test function: φ that satisfy $\phi \partial_t \varphi + \mathbf{u} \cdot \nabla \varphi = 0$...

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- With $\frac{dF_t}{dt} = \frac{\mathbf{u}^{(n+1)}(F_t)}{\phi(F_t)}$, $F_0(\boldsymbol{x}) = \boldsymbol{x}$, we have

$$\varphi(\boldsymbol{x}, t) = \varphi(F_{t^{(n+1)}-t}(\boldsymbol{x}), t^{(n+1)}).$$

ELLAM scheme

- Given $\mathcal{C}$ gradient discretisation,

Find $c^{(n+1)} \in X_{\mathcal{C}}$ such that for all $z \in X_{\mathcal{C}}$,

$$\int_\Omega \phi \Pi_{\mathcal{C}} c^{(n+1)} \Pi_{\mathcal{C}} z - \int_\Omega \phi \Pi_{\mathcal{C}} c^{(n)} v_z(t^{(n)}) \\ = w \delta t^{(n+\frac{1}{2})} \int_\Omega f_n v_z(t^{(n)}) + (1-w)\delta t^{(n+\frac{1}{2})} \int_\Omega f_{n+1} \Pi_{\mathcal{C}} z,$$

where $w \in [0,1]$, $f_k := f(\Pi_{\mathcal{C}} c^{(k)}, \cdot, t^{(k)})$

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where $w \in [0,1]$, $f_k := f(\Pi_{\mathcal{C}} c^{(k)}, \cdot, t^{(k)})$ and
$v_z : \Omega \times (t^{(n)}, t^{(n+1)}] \to \mathbb{R}$ solves

$$\phi \partial_t v_z + \mathbf{u}^{(n+1)} . \nabla v_z = 0 \text{ on } (t^{(n)}, t^{(n+1)}), \quad v_z(\cdot, t^{(n+1)}) = \Pi_{\mathcal{C}} z,$$

with $\mathbf{u}^{(n+1)} \in L^2(\Omega)^d$ and $\nabla \cdot \mathbf{u}^{(n+1)} \in L^\infty(\Omega)$.

ELLAM scheme - condensed

- $f^{(n,w)}(\boldsymbol{x}) := \left(wf(\boldsymbol{x}, t^{(n)}), (1-w)f(\boldsymbol{x}, t^{(n+1)}) \right)$
- $g_F(\boldsymbol{x}) := \left(g(F_{\delta t^{(n+\frac{1}{2})}}(\boldsymbol{x})), g(\boldsymbol{x}) \right)$

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Find $c^{(n+1)} \in X_{\mathcal{C}}$ such that

$$\int_K \phi c_K^{(n+1)} d\boldsymbol{x} = \int_\Omega \phi \sum_{M \in \mathcal{M}} c_M^{(n)} \mathbb{1}_M(\boldsymbol{x}) \mathbb{1}_K(F_{\delta t^{(n+\frac{1}{2})}}(\boldsymbol{x})) d\boldsymbol{x}$$
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$$|K|_\phi c_K^{(n+1)} = \sum_{M \in \mathcal{M}} |M \cap F_{-\delta t^{(n+\frac{1}{2})}}(K)|_\phi c_M^{(n)} + \delta t^{(n+\frac{1}{2})} \int_\Omega f^{(n,w)} \cdot (\mathbb{1}_K)_F d\boldsymbol{x},$$

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where $\mathbf{e} := (1,1)$.

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- discrete mass balance error

$$e_{\mathrm{mass}} := \left| \sum_{K\in\mathcal{M}} |K|_\phi c_K^{(n+1)} - \sum_{M\in\mathcal{M}} |M|_\phi c_M^{(n)} - \sum_{K\in\mathcal{M}} \delta t^{(n+\frac{1}{2})} \int_\Omega f^{(n,w)} \cdot \mathbf{e} d\boldsymbol{x} \right|.$$

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- Sum over $K \in \mathcal{M}$.

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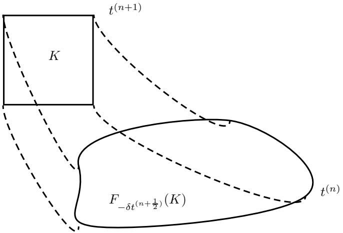
Figure: Interpretation: Piecewise constant approximations



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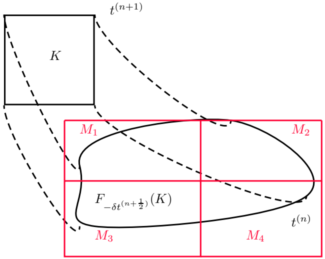

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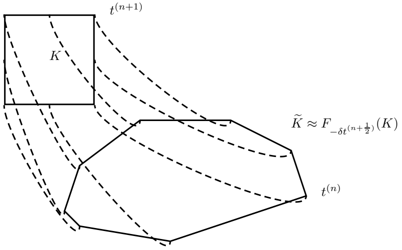
Figure: Numerical implementation: Piecewise constant approximations



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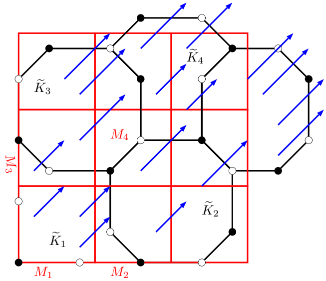

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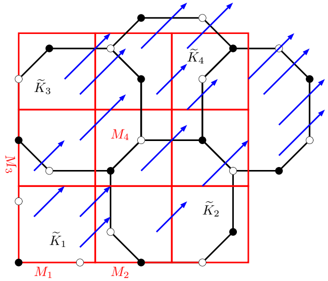

Figure: Trace back regions $\widetilde{K}_i$ (left: initial; right: illustration of possible perturbed cells after local volume adjustments).

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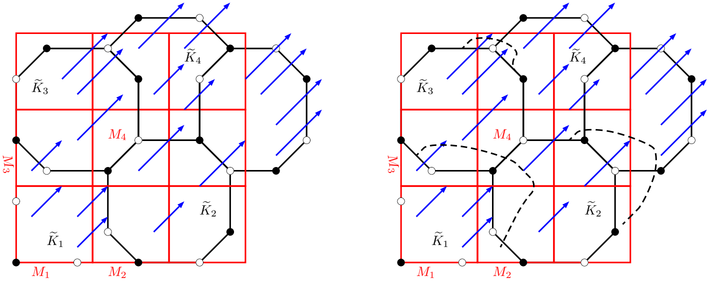

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- Note: Dotted figures are not explicitly computed

Volume adjustment algorithm

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iii) Obtain local volume conservation for K_1 by

$$|\widetilde{K}_1 \cap M_i| \rightsquigarrow |\widetilde{K}_1 \cap M_i| + \frac{|\mathbf{u}_{1,i}|}{\sum_{j=2}^{4} |\mathbf{u}_{1,j}|} e_{K_1}.$$

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$$|\widetilde{K}_2 \cap M_2| \rightsquigarrow |\widetilde{K}_2 \cap M_2| - \frac{|\mathbf{u}_{1,2}|}{\sum_{j=2}^{4} |\mathbf{u}_{1,j}|} e_{K_1}$$

$$\vdots$$

Steep back-tracked regions

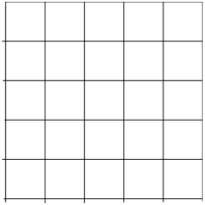

Figure: Mesh cells K

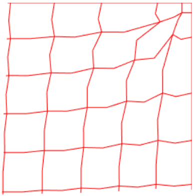

Figure: Back-tracked regions $F_{-\delta t^{(n+\frac{1}{2})}}(K)$

Plan

- characteristics-based scheme

MMOC

- characteristics-based scheme
- characteristic derivative is approximated in a different manner compared to ELLAM

Piecewise constant approximations

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$$
\begin{aligned}
|K|_\phi c_K^{(n+1)} = &\sum_{M\in\mathcal{M}} |F_{\delta t^{(n+\frac{1}{2})}}(M)\cap K|_\phi c_M^{(n)} \\
&+ \delta t^{(n+\frac{1}{2})} \int_K f^{(n,w)} \cdot \mathbf{e} d\boldsymbol{x} \\
&- \delta t^{(n+\frac{1}{2})} \int_K \left[(c_K)^{(n,w)} \nabla \cdot \mathbf{u}^{(n+1)}\right] \cdot \mathbf{e} d\boldsymbol{x}.
\end{aligned}
$$

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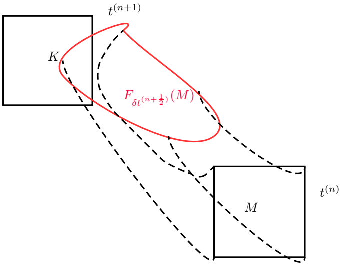
- $\delta t^{(n+\frac{1}{2})} \to 0$
- $\nabla \cdot \mathbf{u}^{(n+1)} = 0$
- c is almost constant in the non-divergence free regions

$$|K|_\phi c_K^{(n+1)} = \sum_{M\in\mathcal{M}} |F_{\delta t^{(n+\frac{1}{2})}}(M) \cap K|_\phi c_M^{(n)}$$

Figure: Interpretation: Piecewise constant approximations



Forward-tracked regions

Figure: Back-tracked regions $F_{-\delta t^{(n+\frac{1}{2})}}(K)$

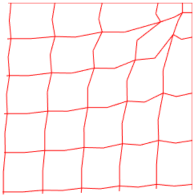
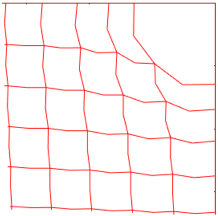


Figure: Forward-tracked regions $F_{\delta t^{(n+\frac{1}{2})}}(K)$



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ELLAM–MMOC formulation

advection-reaction equation

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$$\phi \frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u} c) = f(c).$$

- $c = \alpha c + (1 - \alpha) c$

ELLAM–MMOC formulation

advection-reaction equation

$$\phi\frac{\partial(\alpha c)}{\partial t} + \nabla\cdot((\alpha c)\mathbf{u}) + \phi\frac{\partial((1-\alpha)c)}{\partial t} + \nabla\cdot(((1-\alpha)c)\mathbf{u}) = \alpha f + (1-\alpha)f.$$

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Piecewise constant approximations

- For each cell $K \in \mathcal{M}$, take $\Pi_{\mathcal{C}} z_K = \mathbb{1}_K$.
- Write $\Pi_{\mathcal{C}} c^{(k)} = \sum_{K\in\mathcal{M}} c_K^{(k)} \mathbb{1}_K$.
- Choose α piecewise constant, 1 for ELLAM, 0 for MMOC.

$$
\begin{aligned}
c_K^{(n+1)}|K|_\phi - &\sum_{M\in\mathcal{M}_{\mathrm{ELLAM}}} c_M^{(n)} |M \cap F_{-\delta t^{(n+\frac{1}{2})}}(K)|_\phi \\
- &\sum_{M\in\mathcal{M}_{\mathrm{MMOC}}} c_M^{(n)} |F_{\delta t^{(n+\frac{1}{2})}}(M) \cap K|_\phi \\
=&\delta t^{(n+\frac{1}{2})} \int_\Omega \alpha f^{(n,w)} \cdot (\mathbb{1}_K)_F \\
&+ \delta t^{(n+\frac{1}{2})} \int_\Omega \left[(1-\alpha) f^{(n,w)} \cdot \mathbf{e}\right] \mathbb{1}_K \\
&- \delta t^{(n+\frac{1}{2})} \int_\Omega \left[(1-\alpha)\nabla\cdot \mathbf{u}^{(n+1)} (\Pi_{\mathcal{C}} c)^{(n,w)} \cdot \mathbf{e}\right] \mathbb{1}_K .
\end{aligned}
$$

Obtaining mass balance for ELLAM–MMOC

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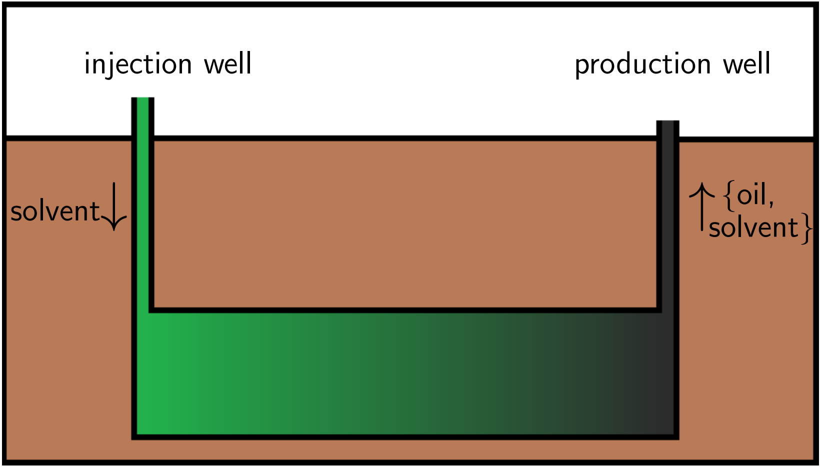
Obtaining mass balance for ELLAM–MMOC

- $\delta t^{(n+\frac{1}{2})} \to 0$
- $\nabla \cdot \mathbf{u}^{(n+1)} = 0$
- $(1-\alpha)c$ is almost constant in the non-divergence free regions

Plan

Enhanced oil recovery



Model for enhanced oil recovery

$$\begin{cases} \nabla \cdot \mathbf{u} & = q^+ - q^- := q \\ \mathbf{u} & = -\dfrac{\mathbf{K}}{\mu(c)} \nabla p \end{cases}$$

$$\phi \frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u}c - \mathbf{D}(\mathbf{x}, \mathbf{u}) \nabla c) + q^- c = q^+$$

Unknowns

- $p(\mathbf{x}, t)$ - pressure of the mixture
- $\mathbf{u}(\mathbf{x}, t)$ - Darcy velocity
- $c(\mathbf{x}, t)$ - concentration of the injected solvent

Parameters

- $\mathbf{K}(\mathbf{x})$ - permeability tensor
- $\phi(\mathbf{x})$ - porosity

Source Terms

- q^+ - injection well
- q^- - production well

Model for enhanced oil recovery

Diffusion Tensor

$$\mathbf{D}(\mathbf{x},\mathbf{u}) = \phi(\mathbf{x})\left[d_m\mathbf{I} + d_l|\mathbf{u}|\mathcal{P}(\mathbf{u}) + d_t|\mathbf{u}|\left(\mathbf{I} - \mathcal{P}(\mathbf{u})\right)\right]$$

- d_m - molecular diffusion coefficient
- d_l - longitudinal dispersion coefficient
- d_t - transverse dispersion coefficient
- $\mathcal{P}(\mathbf{u})$ - the projection matrix along the direction of $\mathbf{u}$

Viscosity

$$\mu(c) = \mu(0)\left[(1-c) + M^{1/4}c\right]^{-4}$$

- $M = \mu(0)/\mu(1)$ - mobility ratio of the two fluids

Model for enhanced oil recovery

No-flow Boundary Conditions

$$\mathbf{u} \cdot \mathbf{n} = 0, \qquad \text{on } \partial\Omega \times [0, T]$$
$$(\mathbf{D}\nabla c) \cdot \mathbf{n} = 0, \qquad \text{on } \partial\Omega \times [0, T]$$

Features

Pressure Equation

$$\begin{cases} \nabla \cdot \mathbf{u} &= q \\ \mathbf{u} &= -\dfrac{\mathbf{K}}{\mu(c)}\nabla p \end{cases} \qquad \text{in } Q_T := \Omega \times [0, T].$$

- anisotropic diffusion equation

Features

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- anisotropic diffusion equation

Concentration Equation

$$\phi\frac{\partial c}{\partial t} + \nabla \cdot (\mathbf{u}c - \mathbf{D}(\mathbf{x}, \mathbf{u})\nabla c) + q^{-}c = q^{+} \qquad \text{in } Q_T.$$

- advection–diffusion–reaction equation
- mostly advection dominated

Plan

Time-stepping: decouples the system

$0 = t^{(0)} < t^{(1)} < \cdots < t^{(N)} = T$ time steps.

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(II) Reconstruction of H_{div} Darcy velocity

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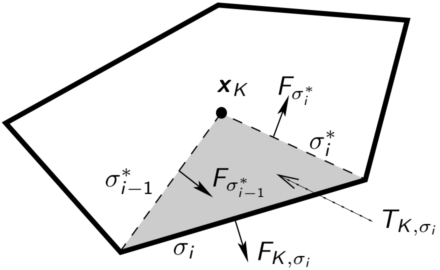
Figure: Triangulation of a cell



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(III) *Concentration equation*: find approximation $c^{(n+1)}$ of c at $t^{(n+1)}$ using $p^{(n+1)}$ and $\mathbf{u}^{(n+1)}$ for the characteristics (ELLAM–MMOC).

Choices for α

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- (αc): ELLAM

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- $((1-\alpha)c)$: MMOC

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  - C_+ injection well
  - C_- production well

Plan

Set up

Injection well: $(1000, 1000)$
flow rate: 30 ft^2/day

Porosity: $\phi = 0.1$

Permeability: $\mathbf{K} = 80\text{D}$

Diffusion-dispersion:

$\phi d_m = 0$ ft^2/day
$\phi d_l = 5$ ft^2/day
$\phi d_t = 0.5$ ft^2/day

Viscosity:

Oil viscosity: 1 cp
Mobility ratio: 41

Initial condition: $c(0) = 0$

Time step: $\delta t = 36$ days

Production well: $(0, 0)$
flow rate: 30 ft^2/day

Mesh Types

Figure: Cartesian Mesh

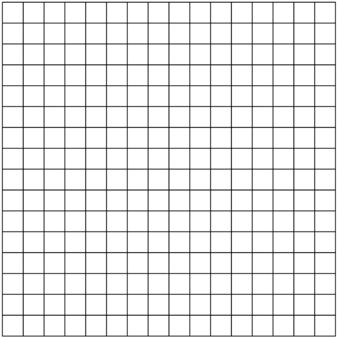
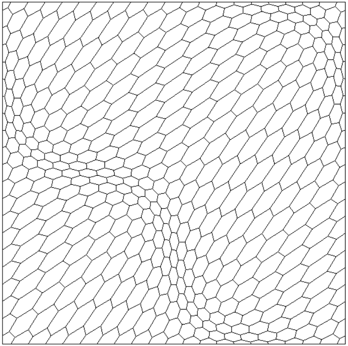
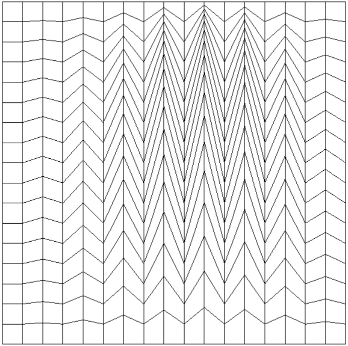


Figure: Hexahedral Mesh



Mesh Types

Figure: Kershaw Mesh



Cartesian mesh

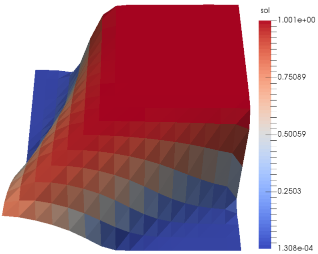


Figure: HMM–ELLAM

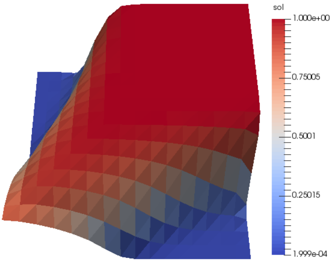


Figure: HMM–MMOC

Cartesian mesh

Figure: HMM–GEM, 1 point per edge

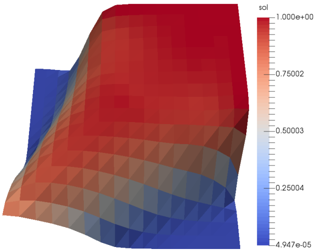
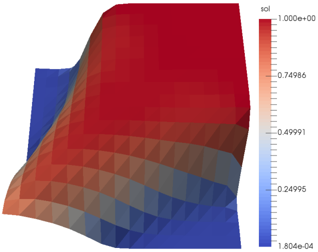


Figure: HMM–GEM, 3 points per edge



Cartesian mesh

Table: Comparison between HMM–ELLAM, HMM–MMOC and HMM–GEM schemes, Cartesian mesh

	points per edge	overshoot	$e^{(N)}_{\mathrm{mass}}$	recovery
HMM–ELLAM	1	1.11%	0.19%	70.09%
HMM–ELLAM	3	0.18%	0.21%	69.76%
HMM–MMOC	1	$<$ 0.01%	5.60%	71.97%
HMM–MMOC	3	$<$ 0.01%	2.80%	69.94%
HMM–GEM	1	$<$ 0.01%	2.35%	68.44%
HMM–GEM	3	$<$ 0.01%	0.85%	69.14%

Hexahedral mesh

Figure: HMM–ELLAM

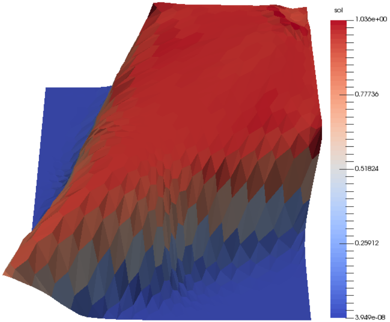
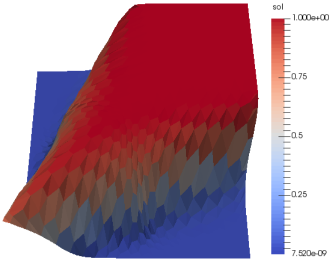


Figure: HMM–MMOC



Hexahedral mesh

Figure: HMM–ELLAM (with local volume adjustment)

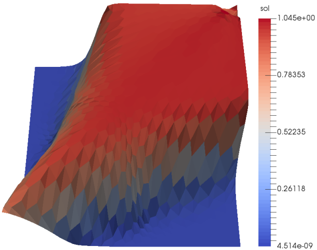
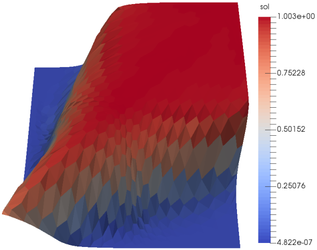


Figure: HMM–GEM



Hexahedral mesh

Table: Comparison between HMM–ELLAM, HMM–MMOC and HMM–GEM scheme, hexahedral mesh, $\Delta t = 18$ days

	points per edge	overshoot	e_{mass}	recovery
HMM–ELLAM (no adjustment)	$\lceil \log_2(m_{K\mathrm{reg}}) \rceil$	3.65%	0.62%	62.50%
HMM–ELLAM (adjusted)	$2\lceil \log_2(m_{K\mathrm{reg}}) \rceil + 1$	4.47%	0.19%	63.41%
HMM–MMOC	$\lceil \log_2(m_{K\mathrm{reg}}) \rceil$	$< 0.01\%$	1.82%	61.43%
HMM–GEM	$2\lceil \log_2(m_{K\mathrm{reg}}) \rceil + 1$	0.26%	0.70%	64.02%

Forward-tracked regions

Figure: Back-tracked regions $F_{-\delta t^{(n+\frac{1}{2})}}(K)$

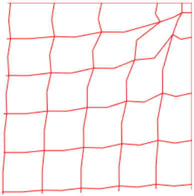
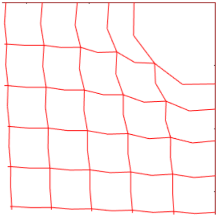


Figure: Forward-tracked regions $F_{\delta t^{(n+\frac{1}{2})}}(K)$



Kershaw mesh

Figure: HMM–ELLAM

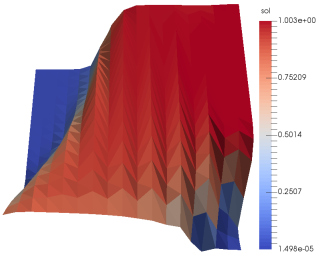
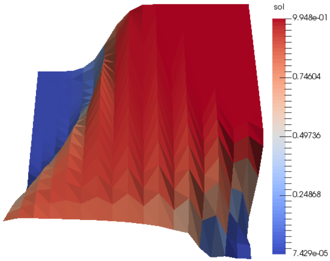
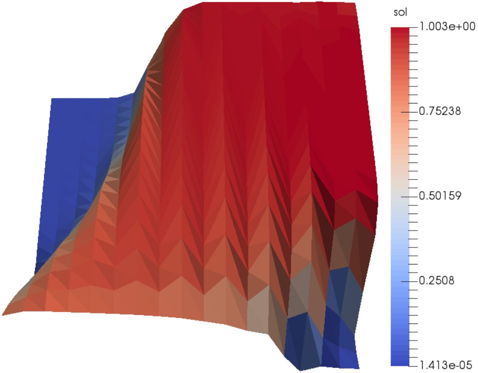


Figure: HMM–MMOC



Kershaw mesh

Figure: HMM–GEM



Kershaw mesh

Table: Comparison between HMM–ELLAM, HMM–MMOC and HMM–GEM scheme, Kershaw mesh

	points per edge	overshoot	e_{mass}	recovery
HMM–ELLAM	$\lceil \log_2(m_{K\mathrm{reg}}) \rceil$	0.28%	0.38%	72.63%
HMM–MMOC	$\lceil \log_2(m_{K\mathrm{reg}}) \rceil$	0%	4.28%	73.21%
HMM–GEM	$\lceil \log_2(m_{K\mathrm{reg}}) \rceil$	0.32%	0.13%	72.36%

Conclusion

- Mass balance analysis for characteristics based schemes
  - ELLAM
  - MMOC
  - ELLAM–MMOC

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Main papers:

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Thank you.