 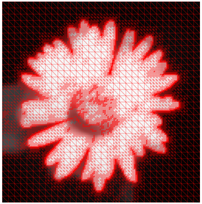 

A posteriori error control for the binary Mumford–Shah model

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The binary Mumford–Shah model

u_0

$\mathcal{O} \subset \Omega$

Binary Mumford–Shah model $\theta_1, \theta_2 \in L^1(\Omega)$

$$E[\mathcal{O}] = \int_{\mathcal{O}} \theta_1 \, \mathrm{d}x + \int_{\Omega \setminus \mathcal{O}} \theta_2 \, \mathrm{d}x + \mathrm{Per}[\mathcal{O}]$$

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- $\theta_i = \frac{1}{\nu}(c_i - u_0)^2$ $(i = 1, 2)$ for weight parameter $\nu > 0$

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- $\theta_i = \frac{1}{\nu}(c_i - u_0)^2$ $(i = 1, 2)$ for weight parameter $\nu > 0$
- $c_1 = (\int_\Omega \chi \, \mathrm{d}x)^{-1} \int_\Omega \chi u_0 \, \mathrm{d}x$, $c_2 = (\int_\Omega 1 - \chi \, \mathrm{d}x)^{-1} \int_\Omega (1 - \chi) u_0 \, \mathrm{d}x$

The binary Mumford–Shah model (cont.)

$$E[\chi, c_1, c_2] = \int_\Omega \theta_1 \chi + \theta_2 (1 - \chi)\,\mathrm{d}x + |\mathrm{D}\chi|(\Omega)$$

minimize E over the set $\chi \in BV(\Omega, \{0,1\})$ (c_1, c_2 fixed)

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Goal: derive functional a posteriori error estimates for

$$\|\chi - \chi_h\|_{L^1(\Omega)}$$

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- thresholding of the relaxed solution and cut out argument

Convex relaxation and thresholding - CEN

Idea Relax range of χ to $[0,1]$ [Nikolova, Esedoḡlu, Chan '06]

$$E_{\mathsf{CEN}}[u] = \int_\Omega u\theta_1 \,\mathrm{d}x + \int_\Omega (1-u)\,\theta_2 \,\mathrm{d}x + |\mathrm{D}u|(\Omega).$$

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Theorem

$$u^* \in \operatorname*{argmin}_{u \in BV(\Omega,[0,1])} E_{\mathsf{CEN}}[u] \quad \Rightarrow \quad [u^* > s] \in \operatorname*{argmin}_{\chi \in BV(\Omega,\{0,1\})} E[\chi]$$

for all a.e. $s \in [0,1]$,

where $[u > s]$ is the s-superlevel set of u, i.e. $\{x : u(x) > s\}$.

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- ✔ $BV(\Omega,[0,1])$ convex
- ✘ E_{CEN} convex, but not uniformly convex in u

Uniformly convex approach

$$E^{\mathsf{rel}}[u] = \int_\Omega u^2 \theta_1 \,\mathrm{d}x + \int_\Omega (1-u)^2\, \theta_2 \,\mathrm{d}x + |\mathrm{D}u|(\Omega)$$

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Theorem

[Chambolle/Darbon '08][B. SSVM '09]

Let $\theta_1, \theta_2 \in L^1(\Omega)$, $\theta_1, \theta_2 \geq 0$ a. e.. Then,

- E^{rel} has a unique minimizer on $BV(\Omega)$.
- $u = \underset{v \in BV(\Omega)}{\operatorname{argmin}} E^{\text{rel}}[v] \Rightarrow \chi_{[u>0.5]} \in \underset{\chi \in BV(\Omega, \{0,1\})}{\operatorname{argmin}} E[\chi]$.
  Moreover, $u(x) \in [0,1]$ for a.e. $x \in \Omega$.

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✔ $BV(\Omega)$ convex (even unconstrained)

✔ E_{UC} uniformly convex in u

Convex relaxation and thresholding - UC (cont.)

Proof sketch: [Chambolle/Darbon '09]

- $E^{\text{rel}}[\min\{\max\{0,v\},1\}] \leq E^{\text{rel}}[v]$ for any $v \in BV(\Omega)$.

Convex relaxation and thresholding - UC (cont.)

Proof sketch: [Chambolle/Darbon '09]

- $E^{\text{rel}}[\min\{\max\{0,v\},1\}] \le E^{\text{rel}}[v]$ for any $v \in BV(\Omega)$.
- $\Psi(x,t) := t^2\theta_1(x) + (1-t)^2\theta_2(x)$, $C_\Psi := \int_\Omega \Psi(x,0)\,\mathrm{d}x$.

$$\int_\Omega \Psi(x,u)\,\mathrm{d}x = \int_\Omega \Psi(x,0)\,\mathrm{d}x + \int_0^1 \int_\Omega \partial_t \Psi(x,s)\chi_{[u>s]}\,\mathrm{d}x\,\mathrm{d}s$$

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$$\Rightarrow E^{\mathrm{rel}}[u] = C_\Psi + \int_0^1 E_s^{\mathrm{rel}}\left[\chi_{[u>s]}\right]\mathrm{d}s \geq C_\Psi + \int_0^1 E_s^{\mathrm{rel}}\left[\chi^s\right]\mathrm{d}s$$
$$= E^{\mathrm{rel}}[u^*] \geq E^{\mathrm{rel}}[u]$$

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$$F[q] = I_{\overline{B}_1}[q] = \begin{cases} 0 & \text{if } |q| \leq 1 \text{ a.e.} \\ +\infty & \text{else} \end{cases}$$

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- $\Lambda = \mathrm{div}$, $\Lambda^* = -\nabla$ holds in the sense

$$\langle \Lambda^* v, q \rangle = (v, \mathrm{div}\, q)_{L^2(\Omega)} \quad \forall v \in \mathcal{V},\ q \in \mathcal{Q}$$

Predual functional (cont.)

From general theory one knows

$$(D^{\mathsf{rel}})^*[v] = F^*[-\Lambda^* v] + G^*[v].$$

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$$= \sup_{q \in \mathcal{Q}, \|q\|_\infty \leq 1} \int_\Omega v \operatorname{div} q \, \mathrm{d}x = |\mathrm{D}v|(\Omega)$$

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$$= \sup_{q\in\mathcal{Q},\|q\|_\infty\leq 1}\int_\Omega v\,\mathrm{div}\,q\,\mathrm{d}x = |\mathrm{D}v|(\Omega)$$

$$G^*[v] = \sup_{w\in L^2(\Omega)}\left((v,w)_{L^2(\Omega)} - G[w]\right)$$

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$$\begin{aligned} F^*[-\Lambda^* v] &= \sup_{q\in\mathcal{Q}}(\langle -\Lambda^* v, q\rangle - F[q]) = \sup_{q\in\mathcal{Q}}\left(-\int_\Omega v \operatorname{div} q \,\mathrm{d}x - I_{\overline{B}_1}[q]\right) \\ &= \sup_{q\in\mathcal{Q},\|q\|_\infty\le 1} \int_\Omega v \operatorname{div} q \,\mathrm{d}x = |\mathrm{D}v|(\Omega) \\ G^*[v] &= \sup_{w\in L^2(\Omega)} \left((v,w)_{L^2(\Omega)} - G[w]\right) = \int_\Omega v^2\theta_1 + (1-v)^2\theta_2 \,\mathrm{d}x \end{aligned}$$

where last supremum is attained for $w = 2v(\theta_1+\theta_2) - 2\theta_2$.

$$F[q] = I_{\overline{B}_1}[q], \quad G[v] = \int_\Omega \frac{\frac{1}{4}v^2 + v\theta_2 - \theta_1\theta_2}{\theta_1+\theta_2}\,\mathrm{d}x$$

Predual functional (cont.)

From general theory one knows

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The Fenchel conjugates can be computed as follows:

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$$= \sup_{q\in\mathcal{Q}, \|q\|_\infty \leq 1} \int_\Omega v \,\mathrm{div}\, q \,\mathrm{d}x = |\mathrm{D}v|(\Omega)$$

$$G^*[v] = \sup_{w\in L^2(\Omega)} \left((v,w)_{L^2(\Omega)} - G[w] \right) = \int_\Omega v^2\theta_1 + (1-v)^2\theta_2 \,\mathrm{d}x$$

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$$(D^{\text{rel}})^*[v] = F^*[-\Lambda^* v] + G^*[v] = E^{\text{rel}}[v].$$

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Central insight: Minimizers p of D^{rel} and u of $(D^{\mathsf{rel}})^*$ fulfill

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&= \sup_{v\in L^2(\Omega)} \left(- \sup_{q\in H_N(\operatorname{div},\Omega)} (\langle -\Lambda^* v, q\rangle - F[q]) - G^*[v] \right) \\
&= \sup_{v\in L^2(\Omega)} (-F^*[-\Lambda^* v] - G^*[v]) \\
&= \sup_{v\in L^2(\Omega)} (-(D^{\text{rel}})^*[v]) = - \inf_{v\in L^2(\Omega)} (D^{\text{rel}})^*[v] = -(D^{\text{rel}})^*[u]\,.
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Functional a posteriori error estimates for E^{rel}

Two measures of uniform convexity for $J : X \to \mathbb{R}$

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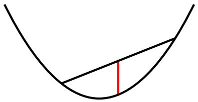

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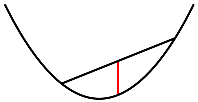
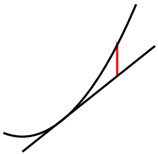


$$\langle x', x_2 - x_1\rangle + \Psi_J(x_2 - x_1) \leq J[x_2] - J[x_1] \text{ for all } x' \in \partial J[x_1]$$



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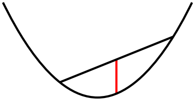

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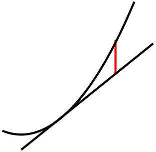

Theorem [Repin '00]: Let $u \in \operatorname{argmin}_{\tilde{v}\in\mathcal{V}} E^{\text{rel}}[\tilde{v}]$ and $q \in \mathcal{Q}$, $v \in \mathcal{V}' = \mathcal{V} = L^2(\Omega)$. Then,

$$\Phi_{G^*}(v-u) + \Phi_{F^*}(-\Lambda^*(v-u)) + \Psi_{E^{\text{rel}}}\left(\tfrac{v-u}{2}\right) \leq \tfrac{1}{2}(E^{\text{rel}}[v] + D^{\text{rel}}[q])\,.$$

Repin theorem - Proof sketch

The convexity property of Φ_{G^*} and Φ_{F^*} gives:

$$\begin{aligned}
&\Phi_{G^*}(v-u)+\Phi_{F^*}(-\Lambda^*(v-u)) \\
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&\qquad -\left(F^*\left[-\Lambda^* \tfrac{u+v}{2}\right]+G^*\left[\tfrac{u+v}{2}\right]\right)
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Adding both inequalities gives

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The claim follows using the *weak complementarity principle*

$$E^{\mathsf{rel}}[u] \geq -D^{\mathsf{rel}}[q].$$

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Theorem [Repin '00]: Let $u \in \operatorname{argmin}_{\tilde{v}\in\mathcal{V}} E^{\mathsf{rel}}[\tilde{v}]$ and $q \in \mathcal{Q}$, $v \in \mathcal{V}' = \mathcal{V} = L^2(\Omega)$. Then,

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A posteriori error estimates for the binary model

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A posteriori error estimates for the binary model

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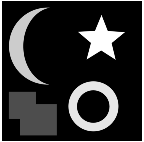
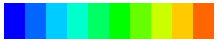
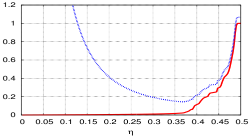


u_0

u

$\chi_{[u>0.5]}$

0  1



$\mathrm{a}[v, \eta]$ and $\mathrm{err}_\chi[v, q, \eta]$

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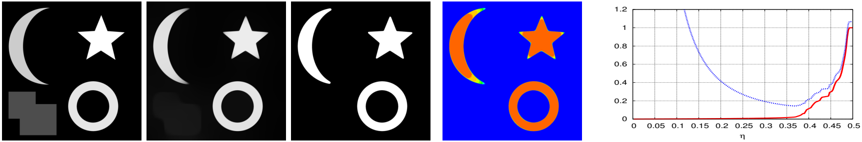

u_0 u $\chi_{[u>0.5]}$ 0 1 $\mathrm{a}[v,\eta]$ and $\mathrm{err}_\chi[v,q,\eta]$

Theorem [A posteriori error estimate for E]: Let $\chi \in BV(\Omega, \{0,1\})$ be the minimizer of the binary Mumford–Shah functional obtained from E^{rel}. Then for all $v \in \mathcal{V} = L^2(\Omega)$ and $q \in \mathcal{Q} = H_N(\mathrm{div}, \Omega)$ we have

$$\left\|\chi - \chi_{[v>\frac{1}{2}]}\right\|_{L^1(\Omega)} \leq \inf_{\eta \in (0,\frac{1}{2})} \left\{ \mathrm{err}_\chi[v,q,\eta] = \left(\mathrm{a}[v,\eta] + \frac{1}{\eta^2} \mathrm{err}_u^2[v,q] \right) \right\}.$$

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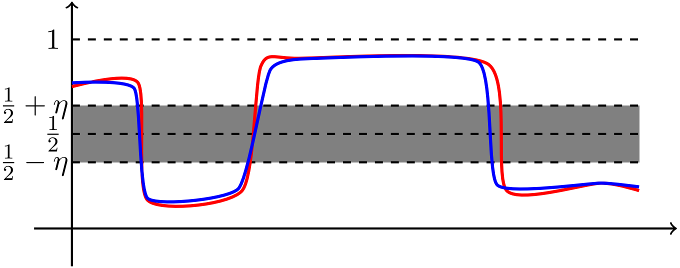
Proof: $\left[u > \frac{1}{2}\right] \Delta \left[v > \frac{1}{2}\right] \subseteq \left[\frac{1}{2} - \eta \leq v \leq \frac{1}{2} + \eta\right] \cup \left[|u - v| > \eta\right]$



u minimizer of E^{rel} $\qquad$ $v \in \mathcal{V}$

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$$\left\|\chi - \chi_{[v>\frac{1}{2}]}\right\|_{L^1(\Omega)} \leq \inf_{\eta \in (0,\frac{1}{2})} \left\{ \operatorname{err}_\chi[v,q,\eta] = \left(\mathrm{a}[v,\eta] + \tfrac{1}{\eta^2} \operatorname{err}_u^2[v,q] \right) \right\} .$$

Proof: $\left[u > \frac{1}{2}\right] \Delta \left[v > \frac{1}{2}\right] \subseteq \left[\frac{1}{2} - \eta \leq v \leq \frac{1}{2} + \eta\right] \cup [|u - v| > \eta]$

Using the estimate for $\|u - v\|_{L^2(\Omega)}^2$, we get

$$\mathcal{L}^n([|u-v| > \eta]) \leq \int_{\{|u-v|>\eta\}} \tfrac{1}{\eta^2} |u-v|^2 \, \mathrm{d}x \leq \tfrac{1}{\eta^2} \operatorname{err}_u^2[v,q] \, .$$

The above holds for any $\eta \in (0, \frac{1}{2})$, so also for $\inf_{\eta \in (0,\frac{1}{2})}$.

Two discretizations and primal-dual algorithm

$\mathscr{T}$ adaptive simplicial mesh
$\mathcal{V}_h^0$ space of piecewise constant finite element functions on $\mathscr{T}$
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for $\theta_{i,h} = \mathcal{I}_h(\frac{1}{\nu}(c_i - u_0)^2)$ for $i = 1, 2$, $\mathcal{I}_h$ Lagrange interpolation

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(FE) discretization $\mathcal{V}_h = \mathcal{V}_h^1$ with L^2-scalar product and $\mathcal{Q}_h = (\mathcal{V}_h^1)^2$ with lumped mass scalar product,
$-\Lambda_h^* : \mathcal{V}_h \to \mathcal{Q}_h$ implicitly defined via (cf. [Bartels '14])

$$\int_\Omega \mathcal{I}_h(-\Lambda_h^* v_h \cdot q_h)\,\mathrm{d}x = \int_\Omega v_h \mathcal{P}_h \operatorname{div} q_h\,\mathrm{d}x \quad \forall q_h \in \mathcal{Q}_h, v_h \in \mathcal{V}_h$$

$\mathcal{P}_h : L^2(\Omega) \to \mathcal{V}_h$ denotes L^2-projection

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$\Lambda_h : \mathcal{Q}_h \to \mathcal{V}_h$ given by

$$\int_\Omega \mathcal{I}_h(\Lambda_h q_h v_h)\,\mathrm{d}x = -\int_\Omega q_h \cdot \nabla v_h\,\mathrm{d}x \quad \forall q_h \in \mathcal{Q}_h, v_h \in \mathcal{V}_h$$

$\Rightarrow -\Lambda_h^* v_h = \nabla v_h$
To evaluate err_u^2 an L^2-projection of the dual solution is performed

The primal-dual algorithm

Primal-dual algorithm

[Chambolle, Pock '11]

while $\|u_h^{k+1} - u_h^k\|_\infty > \mathit{THRESHOLD}$ **do**

$$p_h^{k+1} = (\mathrm{Id} + \sigma\partial F_h)^{-1}(p_h^k - \sigma\Lambda_h^* \bar{u}_h^k);$$
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- The operator norm of Λ can be estimated as follows
  - $\|\Lambda_h\|^2 \leq 48(3 + 2\sqrt{2})h_{\mathrm{min}}^{-2}$ for (FE') and $n = 2$
  - $\|\Lambda_h\|^2 \leq 96(3 + 2\sqrt{2})h_{\mathrm{min}}^{-2}$ for (FE)

Numerical results - "Flower"

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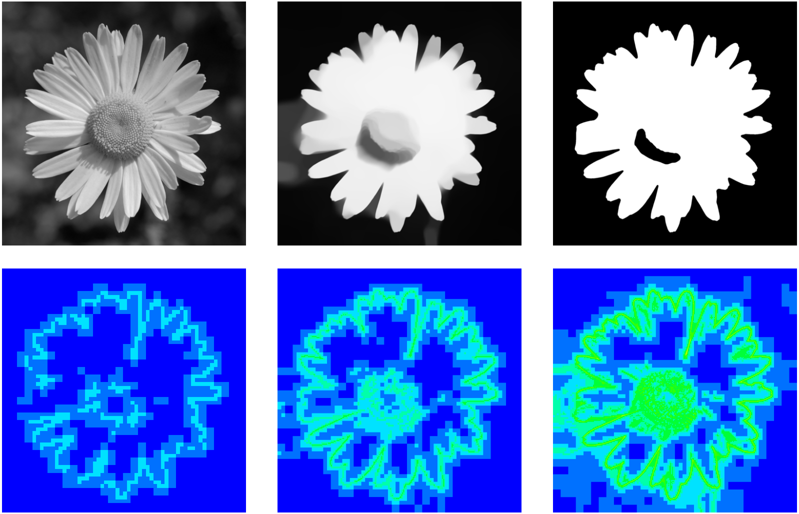
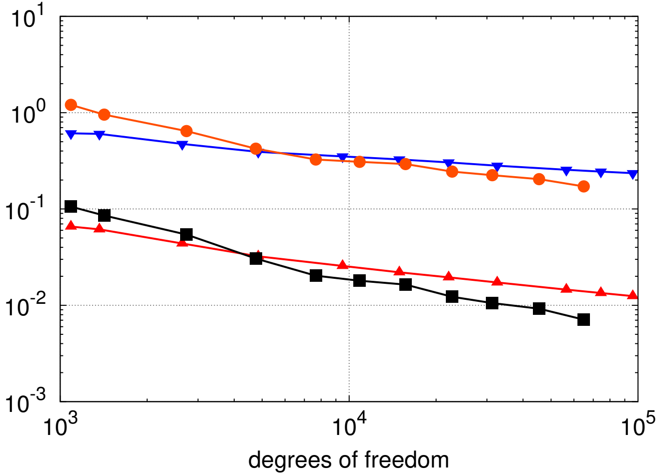
u_0 u_h $\chi_{[u_h > 0.5]}$



mesh after 2^{nd}, 5^{th}, 10^{th} iteration

Numerical results - "Flower" (cont.)



err_u^2 estimator for $\|u - u_h\|_{L^2(\Omega)}^2$ ((FE) (black), (FE') (red))

err_χ estimator for $\|\chi - \chi_{[u_h>0.5]}\|_{L^1(\Omega)}$ ((FE) (orange), (FE') (blue))

Numerical results - “Cameraman”

u_0 u_h $\chi_{[u_h>0.5]}$

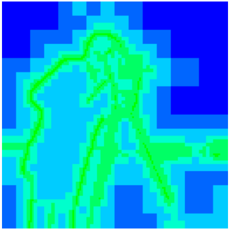
Image source: MATLAB

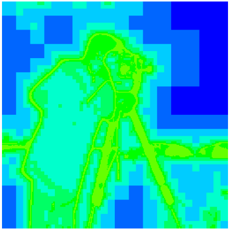
Numerical results - “Cameraman”

u_0

u_h

$\chi_{[u_h>0.5]}$

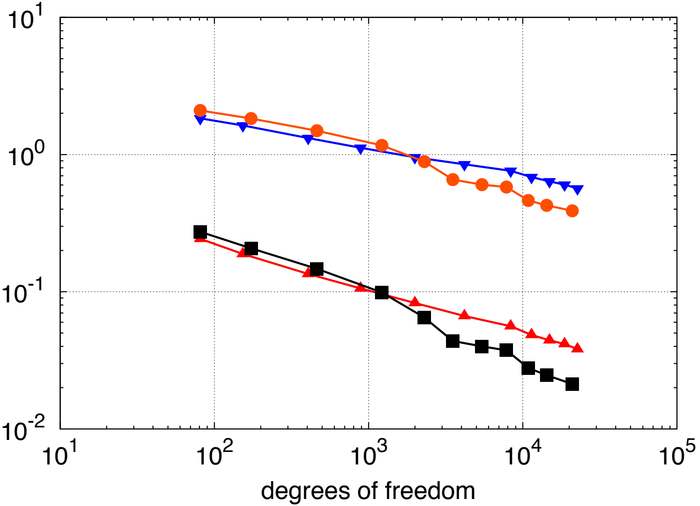




mesh after 5^{th}, 10^{th} iteration

Image source: MATLAB

Numerical results - "Cameraman" (cont.)



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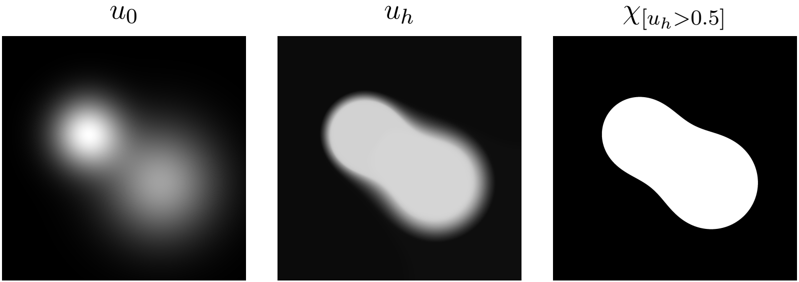
Numerical results - "Gaussians" using (FE')



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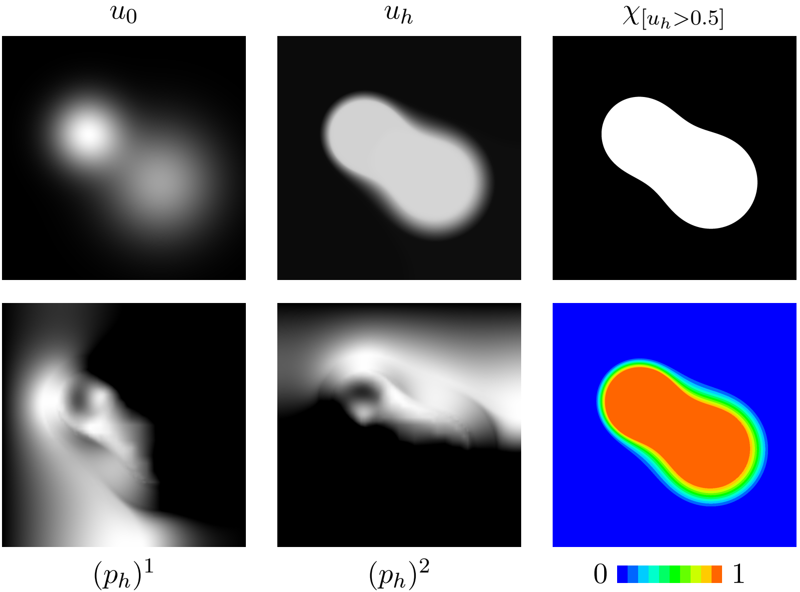

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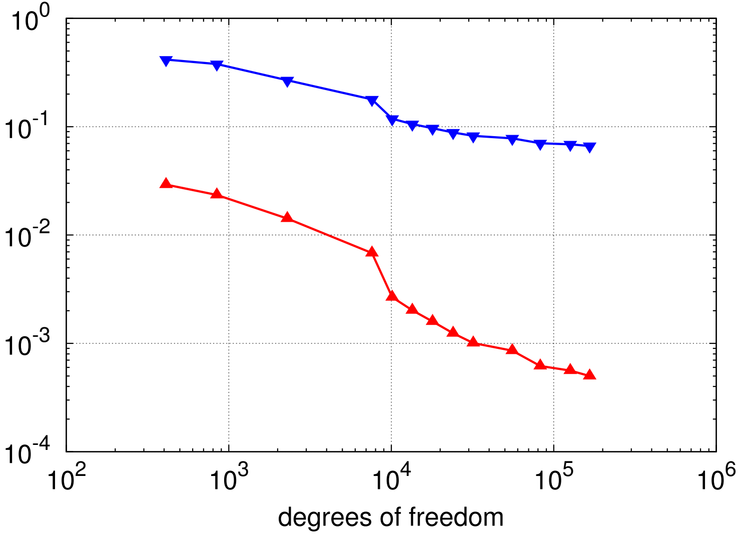

err_u^2 estimator for $\|u - u_h\|^2_{L^2(\Omega)}$ err_χ estimator for $\|\chi - \chi_{[u_h > 0.5]}\|_{L^1(\Omega)}$

Conclusion and outlook

Summary

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- Develop/use minimization algorithms that are tailored to the adaptive structure

References

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 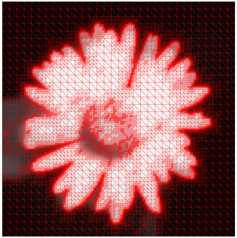 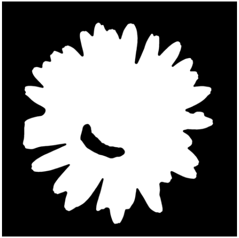

Thank you for your attention!

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