Stochastic collocation methods with multi-fidelity models

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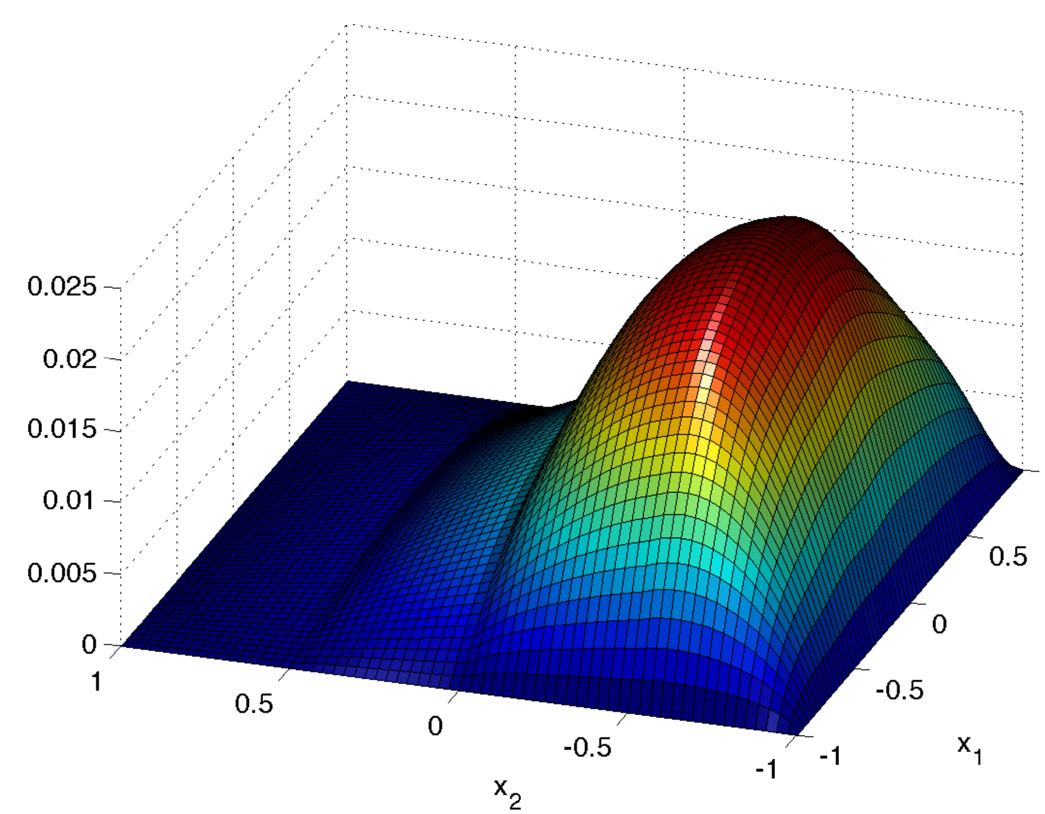


Motivation

Collocation methods are widely used in uncertainty quantification of practical problems:

- Non-intrusive: run different parameter realizations of the deterministic solver.
- Fast convergence can be achieved for smooth problems.

However,

- High cost: repetitive runs of deterministic solvers
- Extreme case: "I can only afford 10 simulations"

What can we do with this scenario?

Motivation

For many problems, there usually exists low fidelity models:

- Affordable cost: coarser meshes, simplified physics, coarse-grained model, …
- Low accuracy but resolve some important features

Wishlist:

- Efficiency from low fidelity models
- Accuracy from high fidelity models

How can we glue models together?

Setup

$$\begin{cases} u_t(x,t,Z) = \mathcal{L}(u), & \text{in } D \times (0,T] \times I_Z, \\ \mathcal{B}(u) = 0, & \text{on } \partial D \times [0,T] \times I_Z, \\ u = u_0, & \text{in } D \times \{t = 0\} \times I_Z. \end{cases}$$

u^L : low fidelity solution (cheap)

u^H : high fidelity solution (expensive)

The goal: build a surrogate of u^H in a non-instrusive way

$$v(x,t,z) = \sum_{n=1}^{m} c_n(z) u^H(x,t,z_n), \quad z_n \in I_Z$$

Q1: How to choose z_n intelligently?

Q2: How to compute c_n without extensive sampling the high-fidelity solver?

Point Selection

Key idea: Explore the parameter space by the cheap low-fi model

- Search the space by the low-fi model via greedy algorithm:

$$z_m = \arg\max_{z \in \Gamma} d(u^L(z), U^L_{m-1})$$

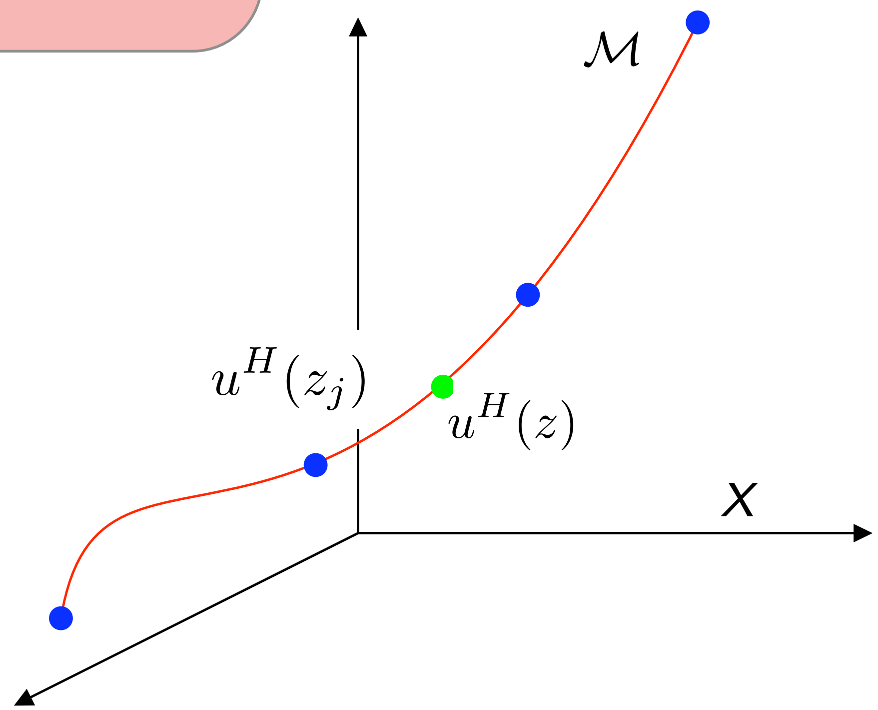
Where

Candidate set:

$$\Gamma = \{z_1, \ldots, z_M\}$$

Low-fi approximation space:

$$U^L_{m-1}(\gamma) = \{u^L(z_1), \ldots, u^L(z_{m-1})\}$$



✓ Enrich the space by finding the furtherest point away from the space spanned by the existed set

✓ The greedy choice is (almost) asymptotically optimal. (Devore, 2013)

Lifting procedure

The coefficients are possibly inferred from the low-fi model

- Construct $\{c_n\}$ as projection coefficients from low-fidelity model:

$$v(z)^L = \mathcal{P}_{U^L(\gamma)} u^L(z) = \sum_{n=1}^{m} c_n(z) u^L(z_n)$$

- Using the same coefficients, construct the approximation rule for high-fidelity model

$$v(x,t,z)^H = \sum_{n=1}^{m} c_n(z) u^H(z_n)$$

  - This is in fact a lifting operator from low-fi space to high-fi space
  - Justification: e.g., linear scaling, coarse/fine mesh

Overview of Bi-fidelity algorithm

- Run the low-fi models at each point of Γ to obtain $u^L(\Gamma), U^L(\Gamma)$
  select the most "important" m points — γ
- Run the high-fi models on those m points to get $u^H(\gamma)$

  most expensive part!
- Bi-fidelity approximation:
  - For any given $z \in I_z$, compute low-fi projection coefficients

$$v(z)^L = \mathcal{P}_{U^L(\gamma)} u^L(z) = \sum_{n=1}^{m} c_n(z) u^L(z_n)$$

  - Apply the same coefficients to $u^H(\gamma)$ to get the bi-fi approximation

$$v(z)^H = \sum_{n=1}^{m} c_n(z) u^H(z_n)$$

The approximation quality depends on how well the low-fi model approximates the functional variation of high-fi model in the parameter space

Tri-fidelity senario

- Run the low-fi model to select the most "important" m points
- Run the medium and high-fidelity models at those m points to get

  $u^H(\gamma) \quad u^{M_1}(\gamma)$

- Tri-fidelity approximation:
  - For any given $z \in I_z$, compute med-fi projection coefficients

$$v(z)^{M_1} = \mathcal{P}_{U^{M_1}(\gamma)} u^{M_1}(z) = \sum_{n=1}^{m} c_n^{M_1}(z) u^{M_1}(z_n)$$

more accurate coefficients

  - Apply the same coefficients to $u^H(\gamma)$ to get the tri-fi approximation

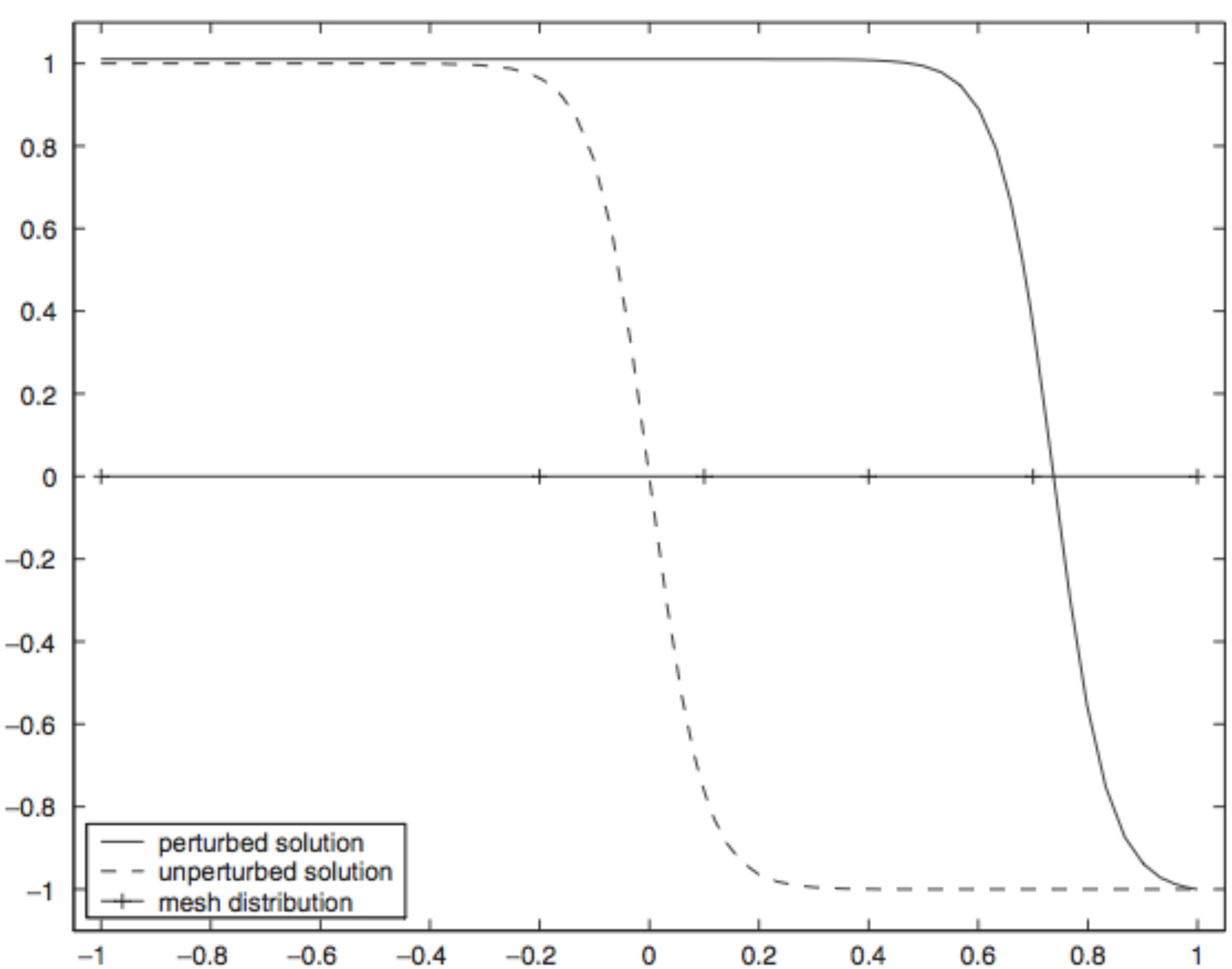
$$v^H(z) = \sum_{n=1}^{m} c_n^{M_1}(z) u^H(z_n)$$

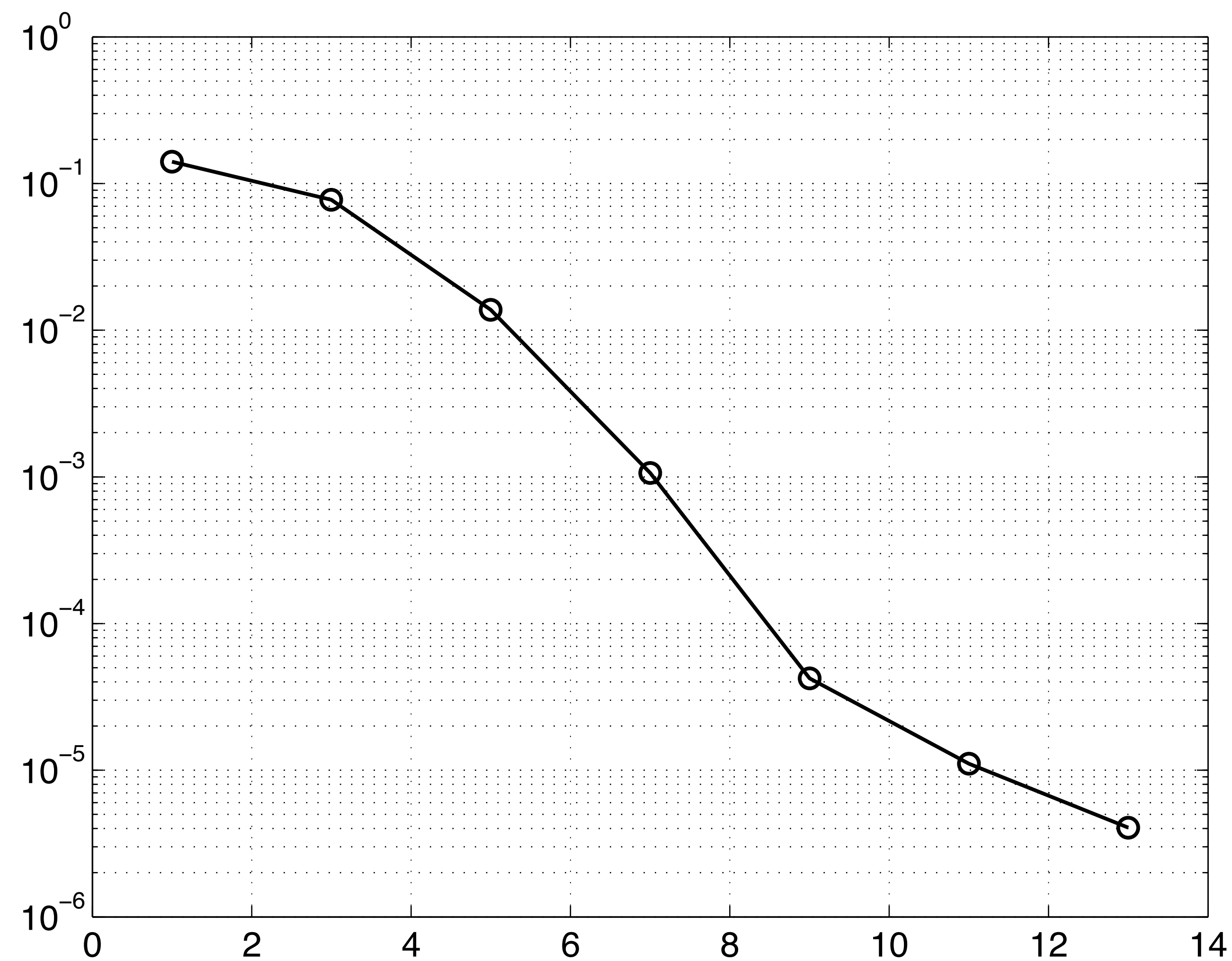
1D Burgers' equation

$$u_t + uu_x = \nu u_{xx}, \quad x \in (-1,1), \qquad \delta \sim U(0, 0.1)$$
$$u(-1) = 1 + \delta, \quad u(1) = -1. \qquad \nu = 0.5$$

low-fi: 100 points (in physical space), FD
high-fi: 1000 points, FD

mean l^2 error
ref solution: high-fi solution

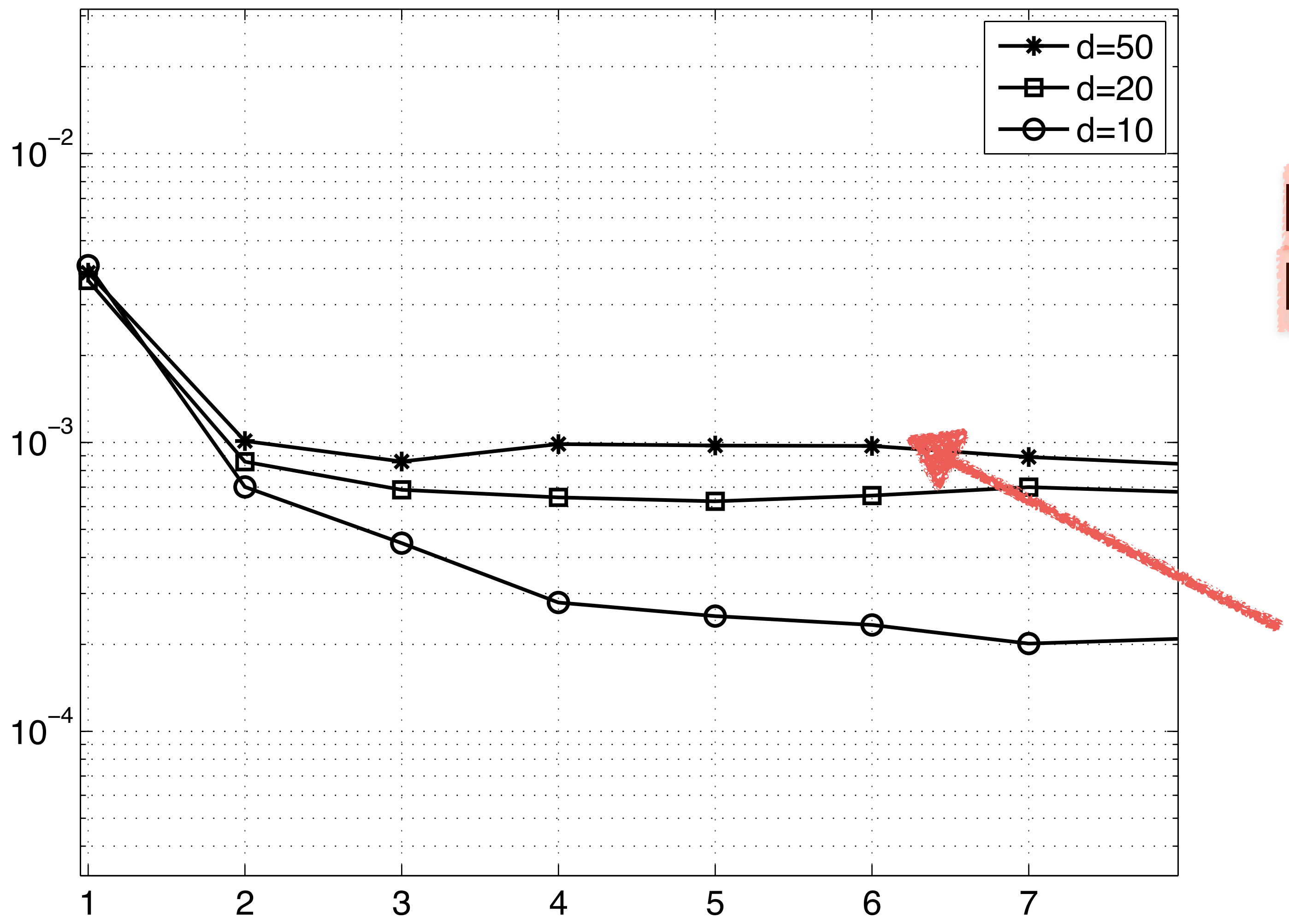




1D stochastic elliptic equation

$$\begin{cases} -(a(Z,x)u_x(Z,x))_x = 1, & (Z,x) \in I_Z \times (0,1) \\ u(Z,0) = 0, & u(Z,1) = 0. \end{cases}$$

$$a(Z,x) = 1 + \sigma \sum_{k=1}^{d} \frac{1}{k\pi} \cos(2\pi k x) Z^{(k)}, \qquad d > 1.$$



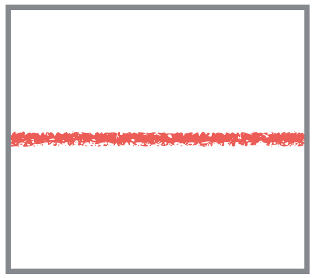
low-fi: 16 points, Cheb Collocation
high-fi: 128 points, Cheb Collocation

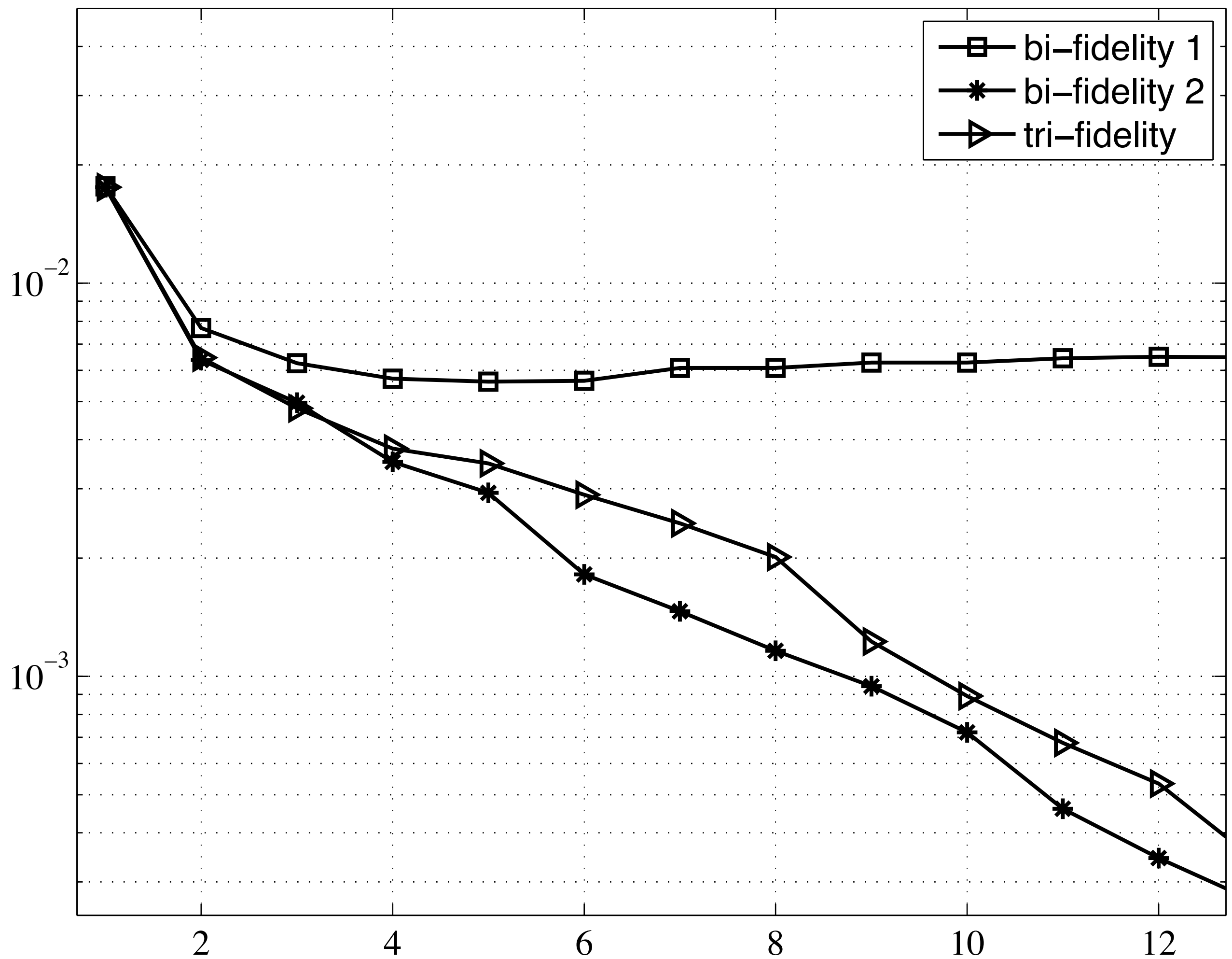
low fidelity: too coarse to resolve the feature in the high-D random space

2D stochastic elliptic equation

$$\begin{cases} -\nabla \cdot (a(x,y,Z)\nabla u) = 0, & (x,y) \in (-1,1)^2 \\ u(-1,y,Z) = -1, \quad u(1,y,Z) = 1, & u_y(x,-1,Z) = 0, \quad u_y(x,1,Z) = 0. \end{cases}$$

$$a(x,y,Z) = 1 + \sum_{k=1}^{d} \sqrt{\lambda_k}\psi_k(x,y)Z^{(k)} \qquad d = 17$$





low-fi: FE with 80 elements, 1D model
med-fi: FE with 32 elements, 2D model
high-fi: FE with 12800 elements, 2D model

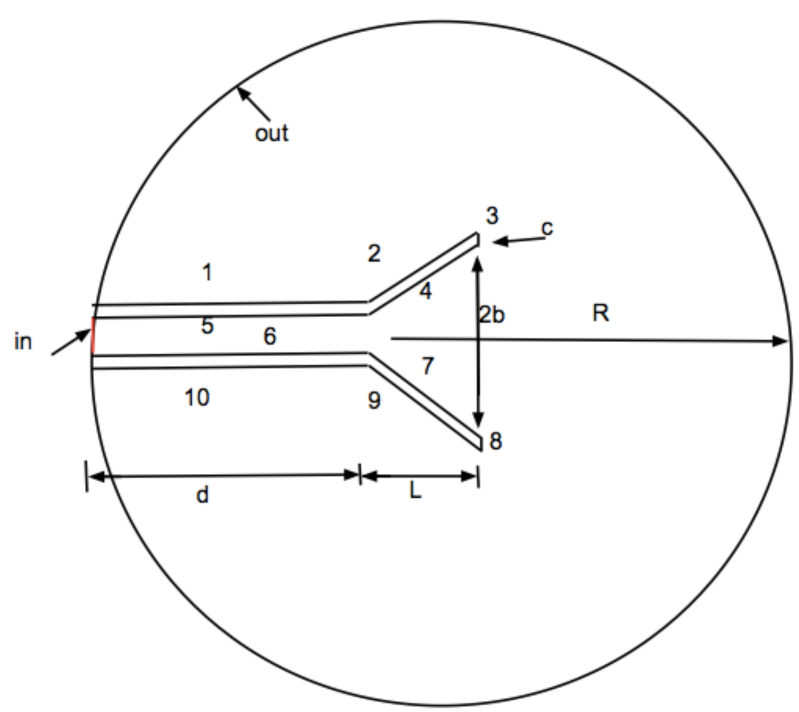
low-fi: commit modeling error

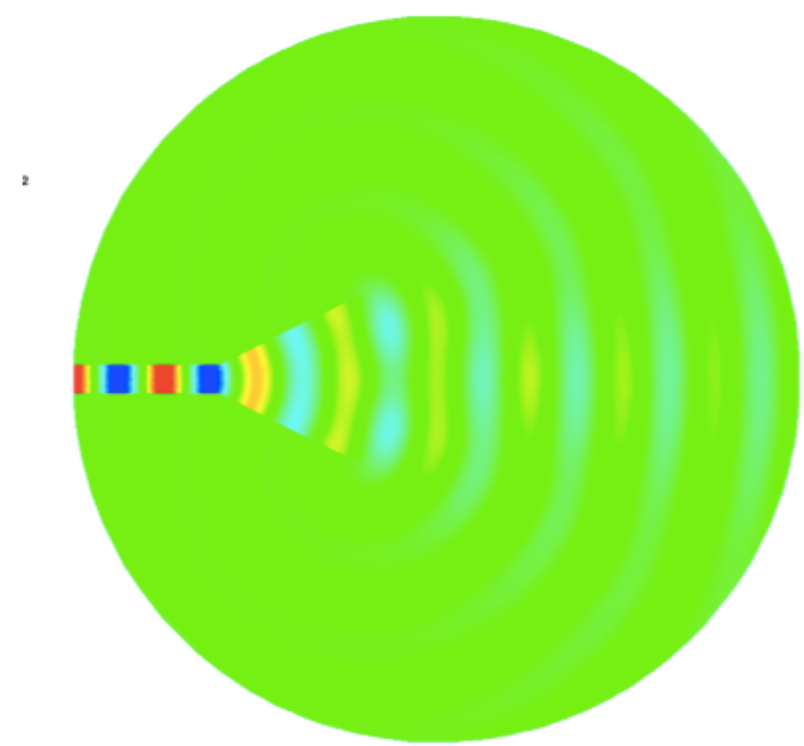
2D acoustic horn

Time-harmonic Helmholtz equation:

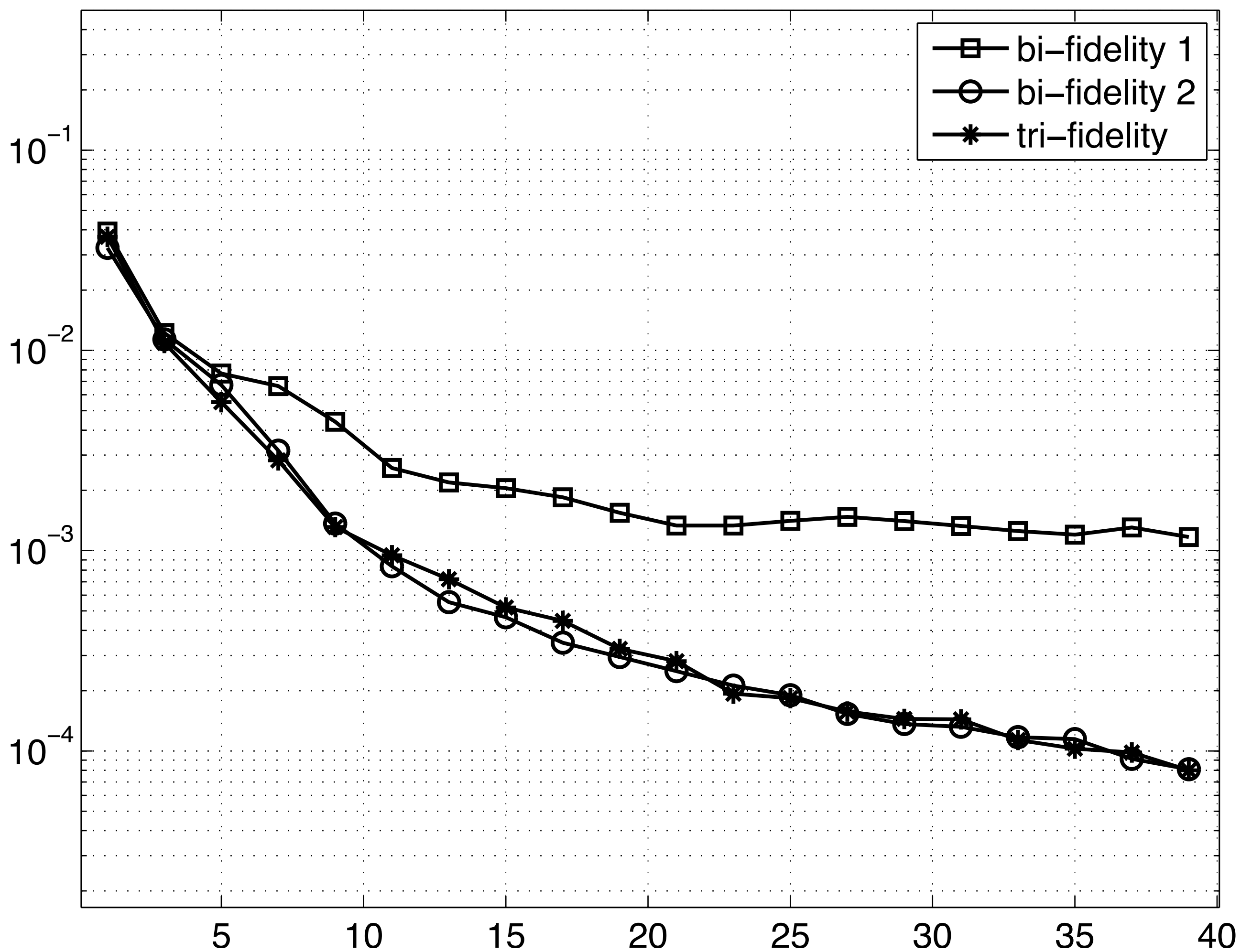
$$\begin{cases} \triangle u + 4u = 0, & \\ (2i + 1/25)u + \frac{\partial u}{\partial n} = 0, & \Gamma_{out} \\ 2iu + \frac{\partial u}{\partial n} = 4i, & \Gamma_{in} \\ \frac{\partial u}{\partial n} = 0, & \Gamma_j, j = 3, 8 \\ i\mu_j u + \frac{\partial u}{\partial n} = 0, & \text{on other boundaries,} \end{cases}$$

$$\mu = (\mu_1, \mu_2, \mu_4, \mu_5, \mu_6, \mu_7, \mu_9, \mu_{10}) \in [0, 1]^8$$





2D acoustic horn



low-fi: P2 FE with 196 elements

med-fi: P2 FE with 2061 elements

high-fi: P2 FE with 22810 elements

tri-fi is preferable due to less off-line time

fast computation of statistical moments

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$$\widetilde{\mathbb{E}}[f;\Theta] := \sum_{i=1}^{m} w_i f(z_i) \approx \mathbb{E}[f]$$

- Monte Carlo, Quadrature: requires many bi-fi constructions

Question: can we approximate hi-fi mean more efficiently?

Bi-fidelity algorithm for statistical mean

- Run the low-fi models at each point of Γ to obtain $u^L(\Gamma), U^L(\Gamma)$ compute its mean and select the most "important" m points — γ
- Run the high-fi models on those m points to get $u^H(\gamma)$

  most expensive part!
- Bi-fidelity approximation:
  - Project low-fidelity mean on the low-fi approximation space

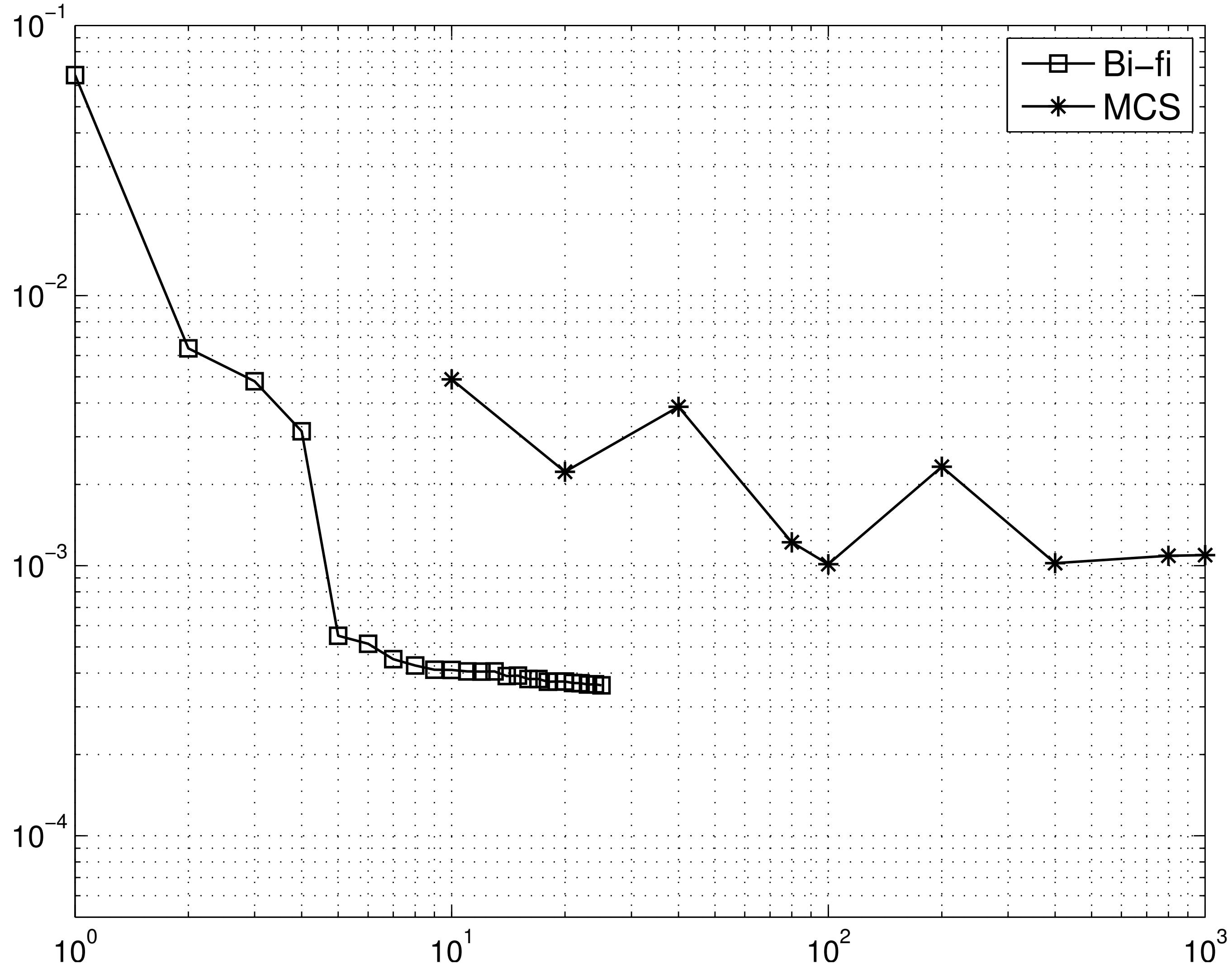
$$\mathcal{P}_{U^L(\gamma)}\mu^L = \sum_{n=1}^{m} c_n u^L(z_n) \approx \mu^L$$

  - Apply the same coefficients to $u^H(\gamma)$ to get the bi-fi mean

$$\mu^B = \mathcal{P}_{U^H(\gamma)}\mu^H = \sum_{n=1}^{m} c_n u^H(z_n)$$

instead of lifting low-fi samples, we are lifting low-fi mean!

2D acoustic horn



Summary

- ✓ Collocation methods for high-fidelity samples and mean
- ✓ Non-intrusive and implementation is straightforward
- ✓ Fast convergence if the low fidelity model can mimic the parametric dependence of the high fidelity model
- ✓ The number of high-fi simulations required is limited,e.g.,O(10)
- ✓ The discretization/models can be very different.

Next step: mismatching models

Reference

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Thank you!

Error Analysis

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Error bound

$$\left\|u^H(z) - v^H(z)\right\|^H \leq C d_{m/2}(u^H(\Gamma)) + \epsilon \left\|P_{U^H(\gamma)} u^H(z)\right\|^H + \left\|\sqrt{\mathbf{G}^H}(\mathbf{G}^L)^{-1}\mathbf{Q}\mathbf{f}^L\right\|$$

- (red) small if the solution is in the low-dimensional manifold
- (yellow) depends on how well the low-fidelity model approximates the functional variation in the parameter space

$$\epsilon = \epsilon_1 + \epsilon_2 + \epsilon_1\epsilon_2$$

$$\left\|\left(\sqrt{\mathbf{G}^H}\right)^{-}(\mathbf{f}^H - \mathbf{f}^L)\right\| \leq \epsilon_1 \left\|\left(\sqrt{\mathbf{G}^H}\right)^{-}\mathbf{f}^H\right\|$$

Coordinates behave similarly in different space

$$\left\|\sqrt{\mathbf{G}^H}\left(\mathbf{G}^L\right)^{-1}\sqrt{\mathbf{G}^H} - \mathbf{I}\right\| \leq \epsilon_2$$

Gramian matrices behave similarly

- (green) non-invertibility of high-fidelity Gramian matrix