Bridging Scales

M. Hairer

Imperial College London

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Guiding principles in probability

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Symmetry

If different outcomes are equivalent (from the perspective of the mechanism causing them), they should have the same probability.

Universality

In many instances, if a random outcome is a consequence of many different sources of randomness, the details of its description should not matter much. (Outcomes of successive coin tosses: de Moivre 1733, Laplace 1812, ...)

Einstein-Smoluchowski

Independently in 1905-1906 give a theory of Brownian motion, building on earlier work by Lord Rayleigh.

1. Physics: Random motion caused by collision with fluid molecules.
2. Mathematics: Probability distribution of the position of the particle is described by the heat equation.

Comes with quantitative predictions, verified experimentally by Perrin in 1908 (Nobel prize 1926). Settles the debate about existence of atoms.

Bachelier

Submits his thesis "Théorie de la spéculation" in 1900, but does not find much recognition. Besides a very short time at Sorbonne (interrupted by WWI), he obtains his first permanent position at age 57!

1. Finance: Describes mechanism for evolution of stock prices.
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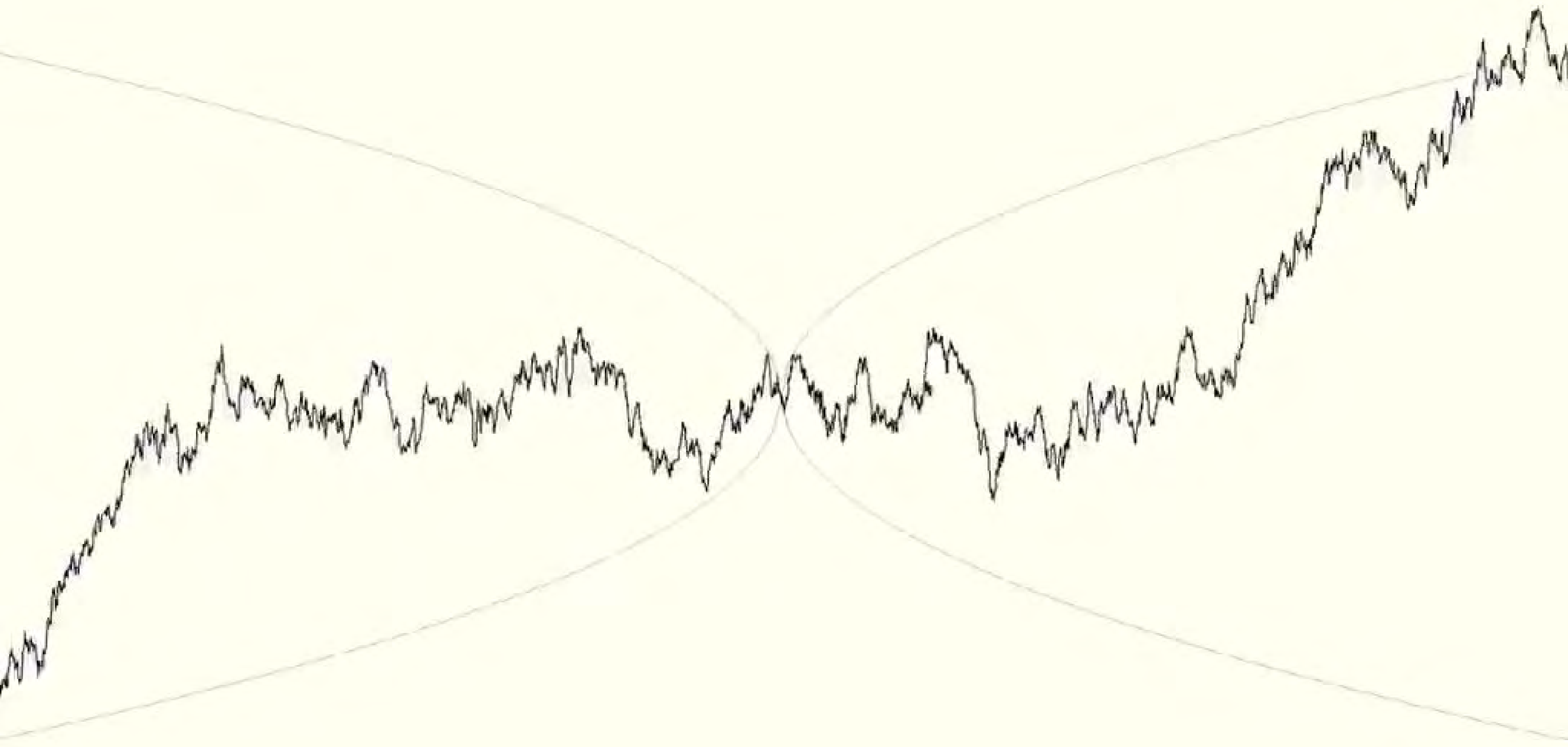
Lays foundations for the works of Black & Scholes (1973) who were awarded the 1997 "Nobel prize" in Economics



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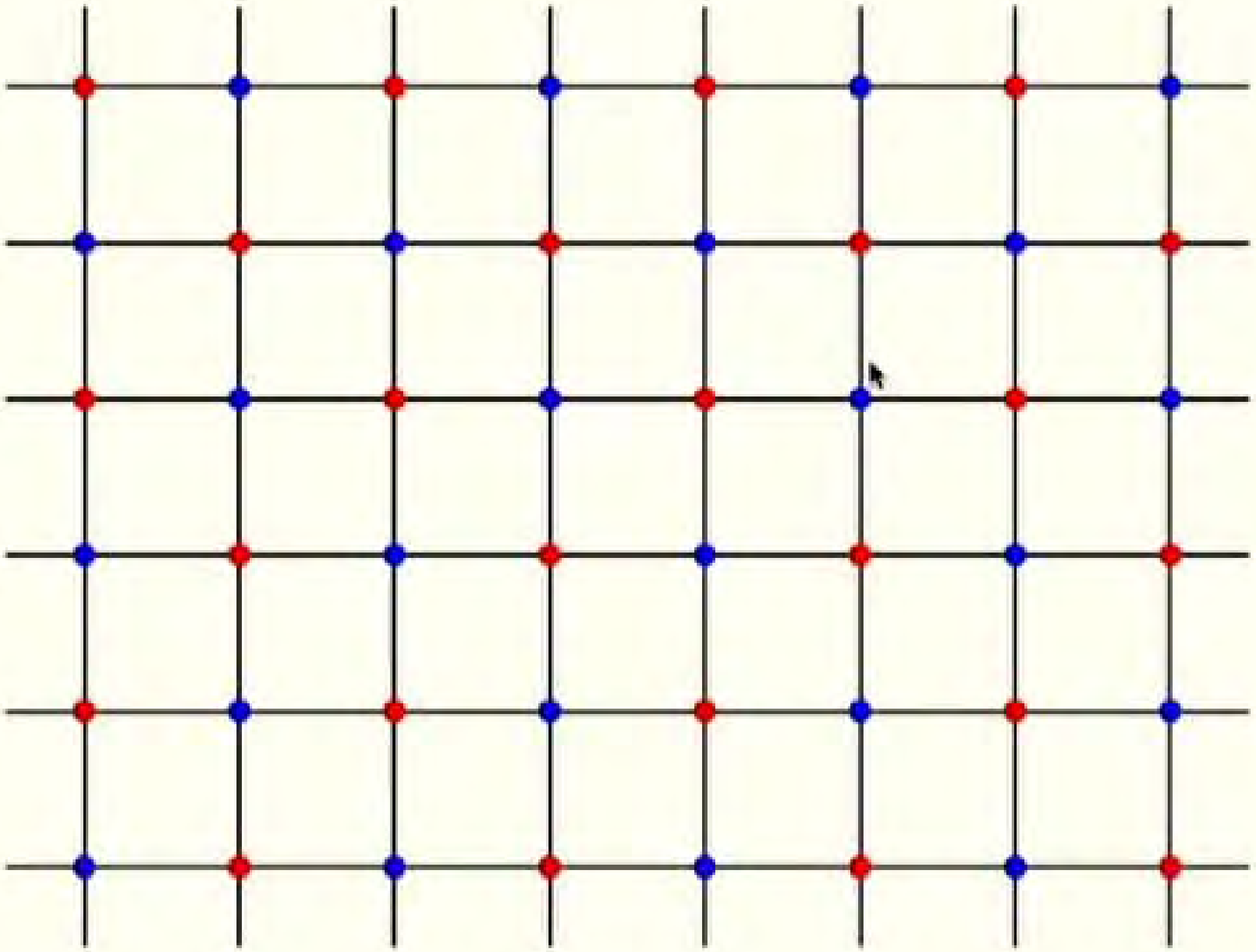
What about two dimensions?

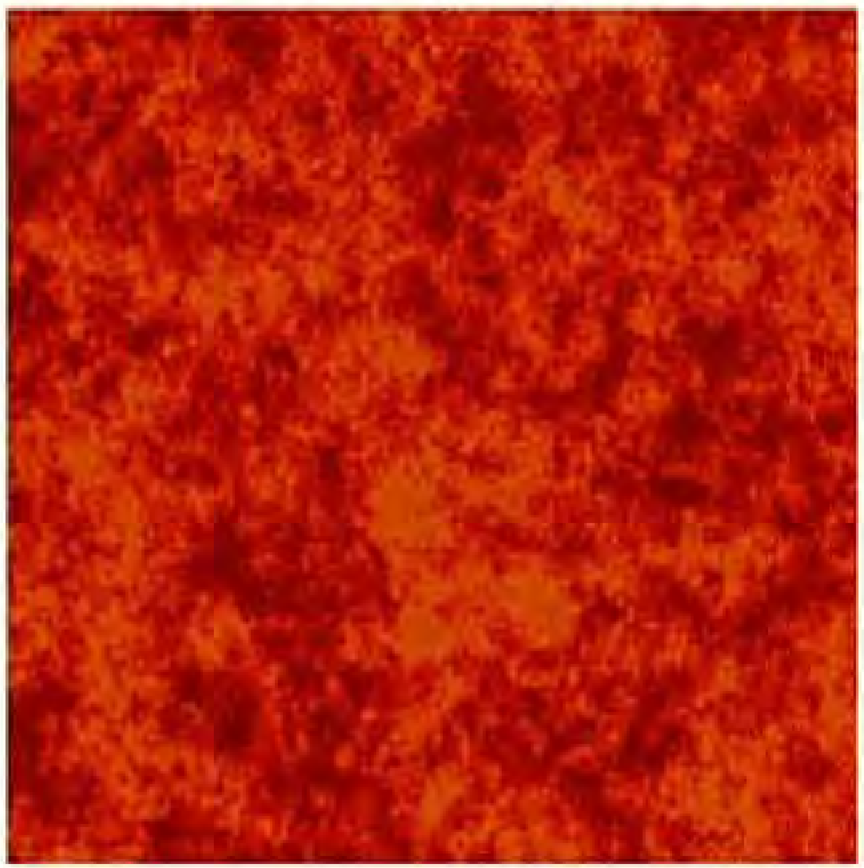
Simple dimensional analogue of random walk:

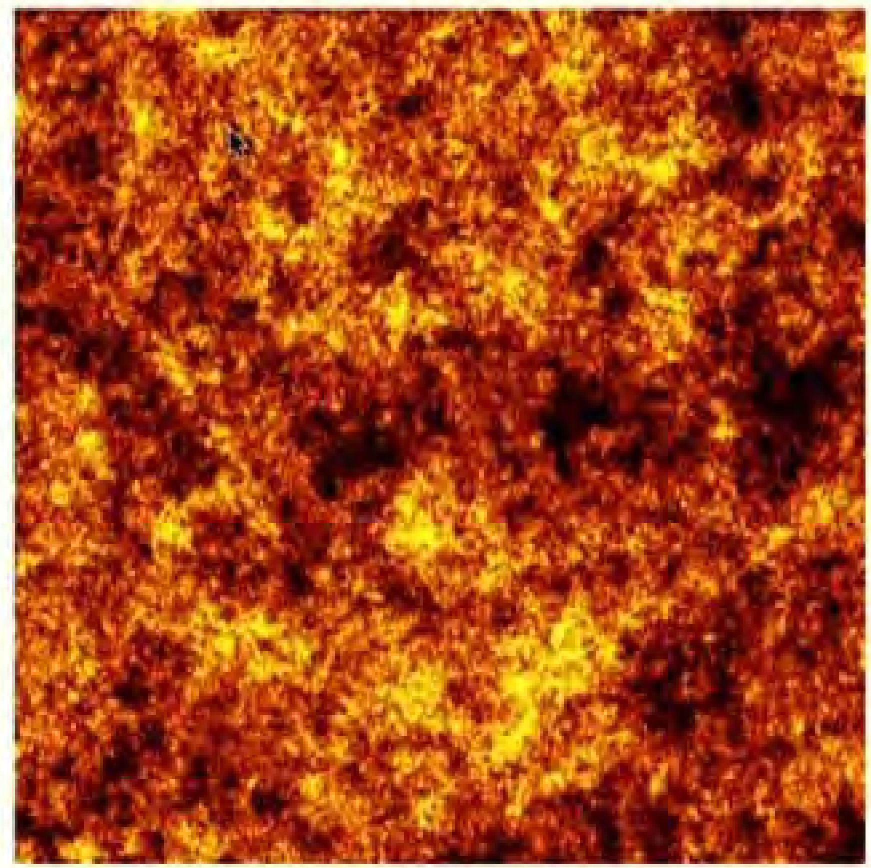


Random function $h\colon Grid \to \mathbf{Z}$ such that $|h(x) - h(y)| = 1$ for $x \sim y$. What does h look like at very large scales?

Free field

Large scale behaviour should be described by "free field", Gaussian generalised function with $\mathbf{E}h(x)h(y) = -\log|x-y|$. No proof yet! (But for similar models, see Borodin, Johansson, Kenyon, Okounkov, Peled, Toninelli, etc.)

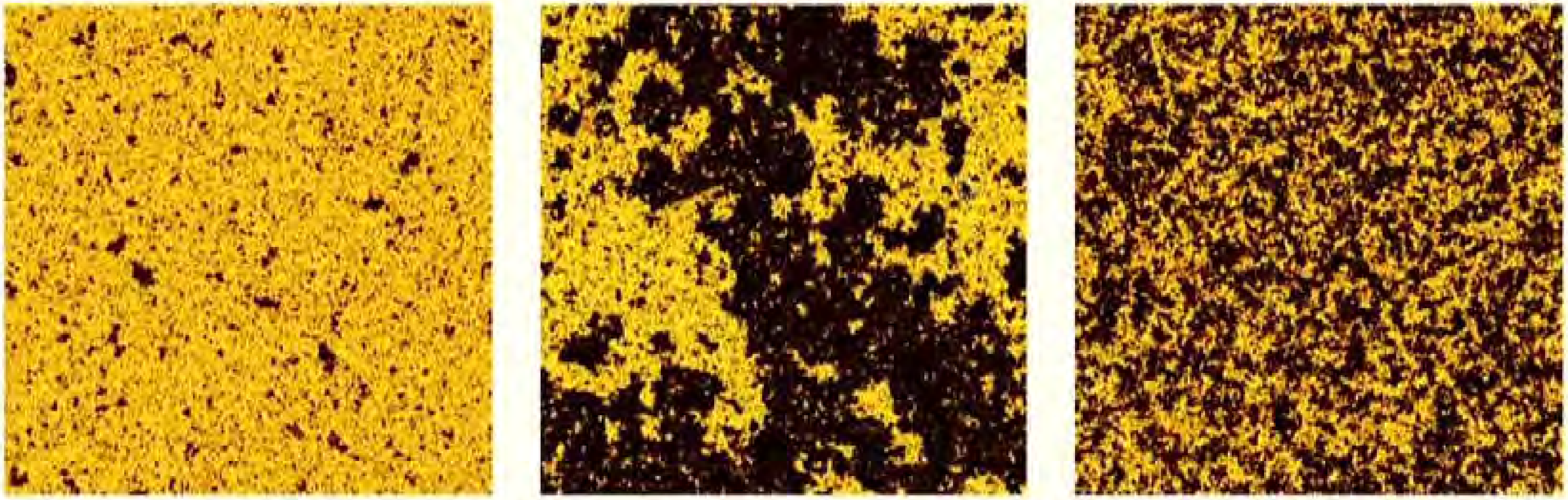




Formally, $\mathbf{P}(dh) \propto \exp(-\int |\nabla h|^2\, dx)$ "dh".

Beyond "free" systems

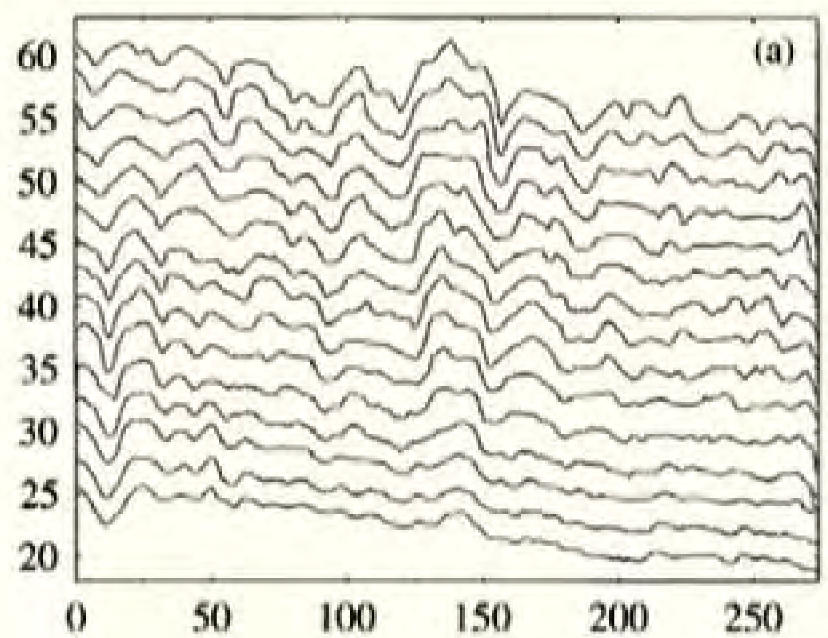
Ising model: state space $\sigma : \Lambda \to \{\pm 1\}$. Probability to see σ proportional to $\exp(\beta \sum_{x \sim y} \sigma_x \sigma_y)$.



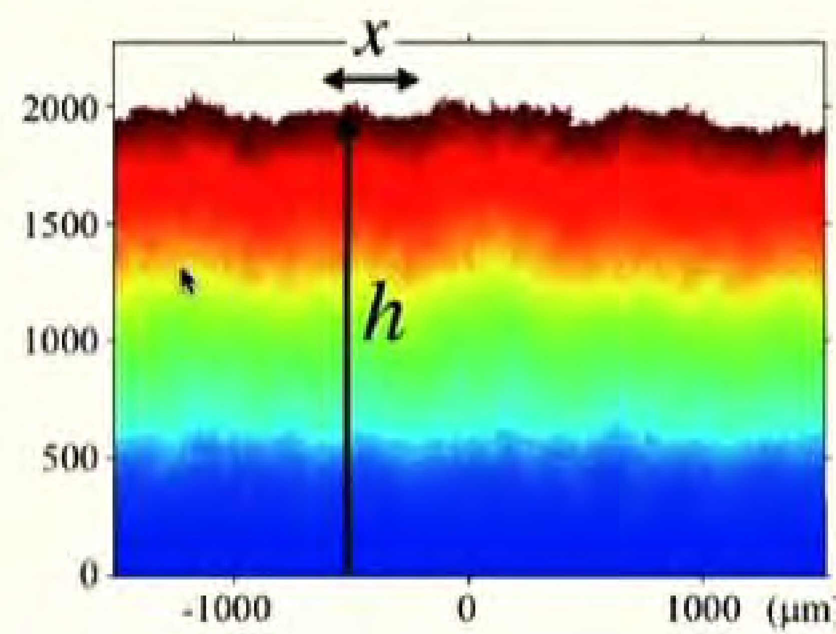
At critical temperature, one has a non-Gaussian scaling limit (rigorous proofs only over last few years), conjectured to be universal for many phase transition models.

Interface growth models

Context: Two "phases" of differing stability. Stable phase invades unstable phase.



Maunuksela & Al, PRL



Takeuchi & Al, Sci. Rep.

Many simple mathematical models exhibit similar features, appear to have same scaling limit. (See Borodin, Corwin, Ferrari, Johansson, Quastel, Sasamoto, Spohn, ...)

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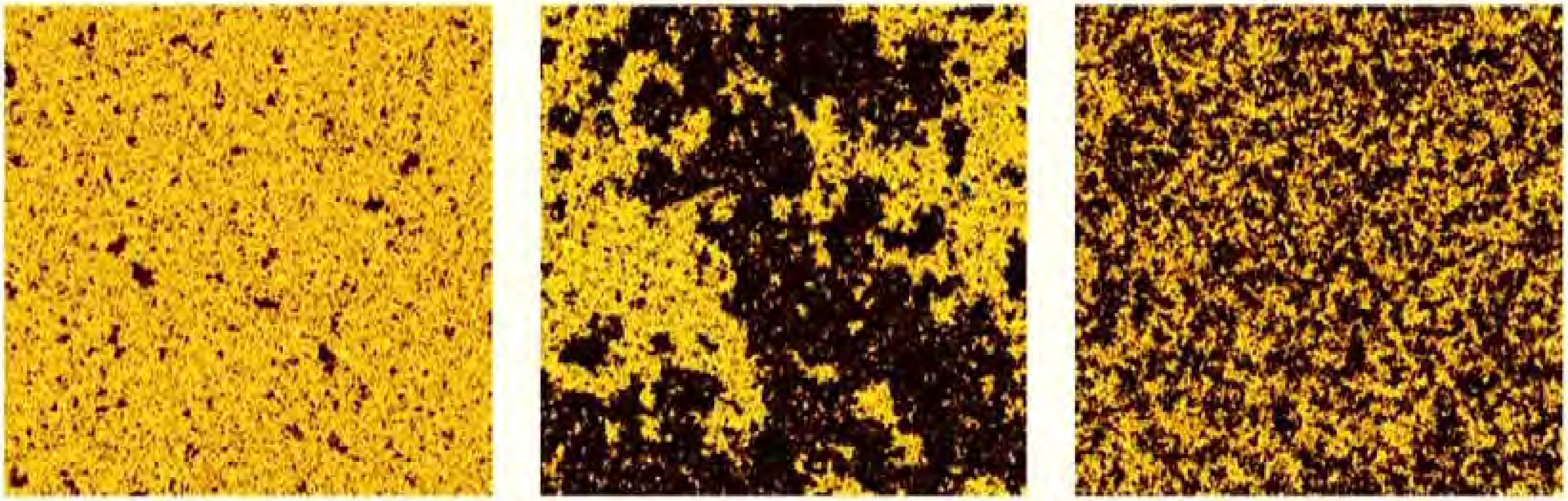

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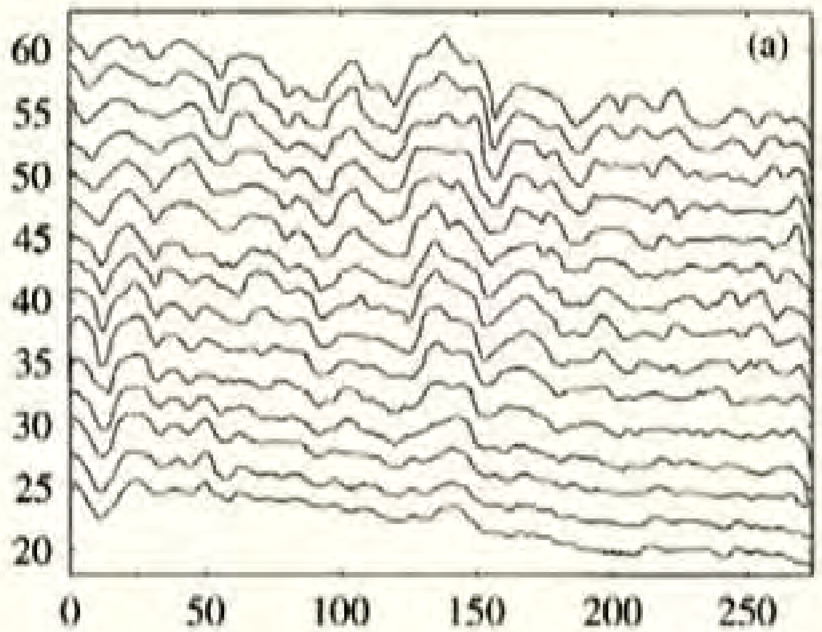

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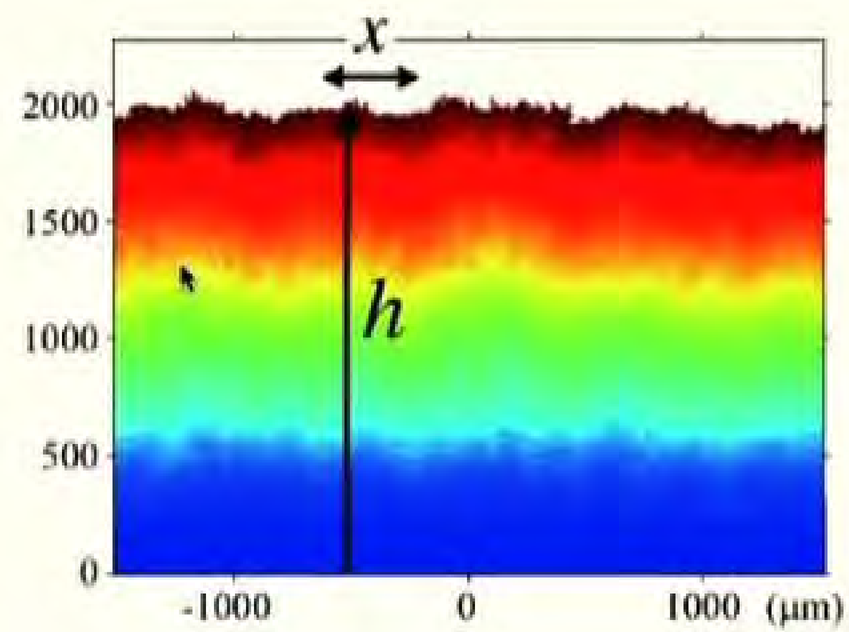

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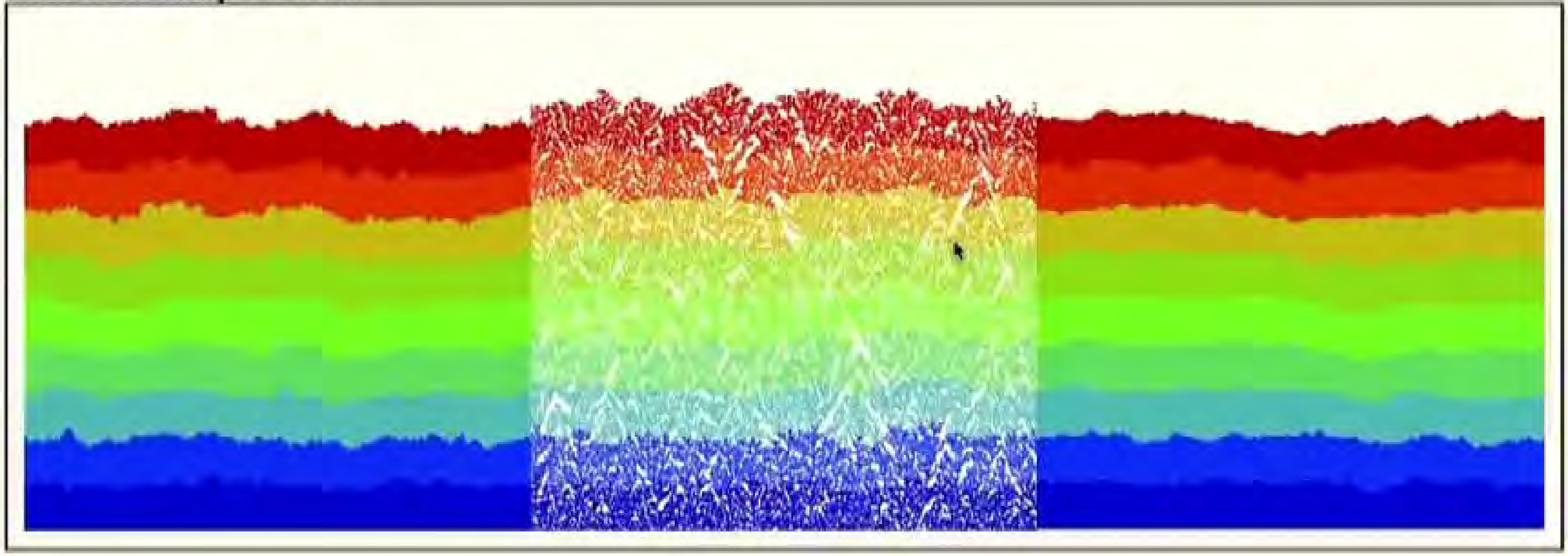

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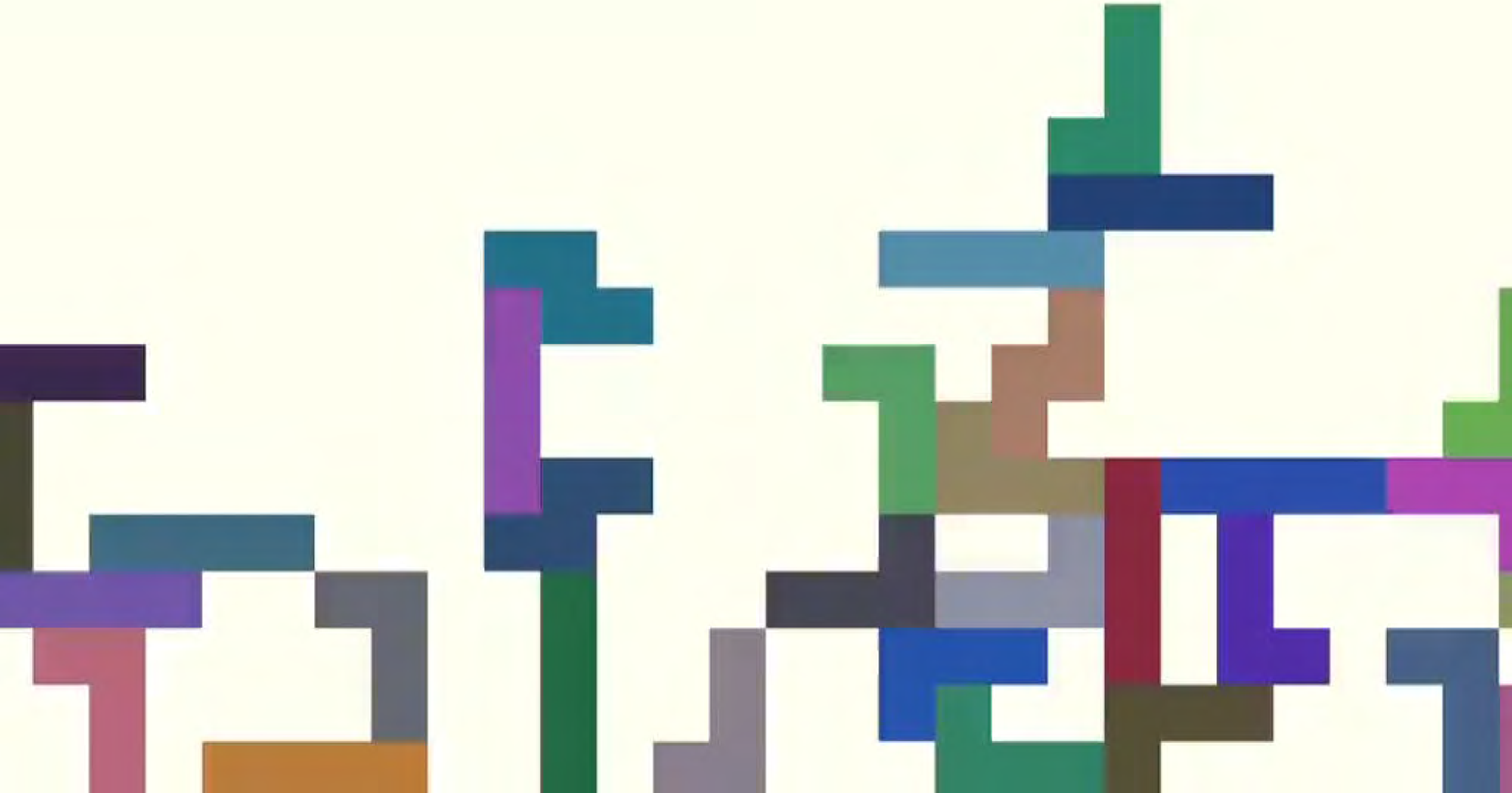

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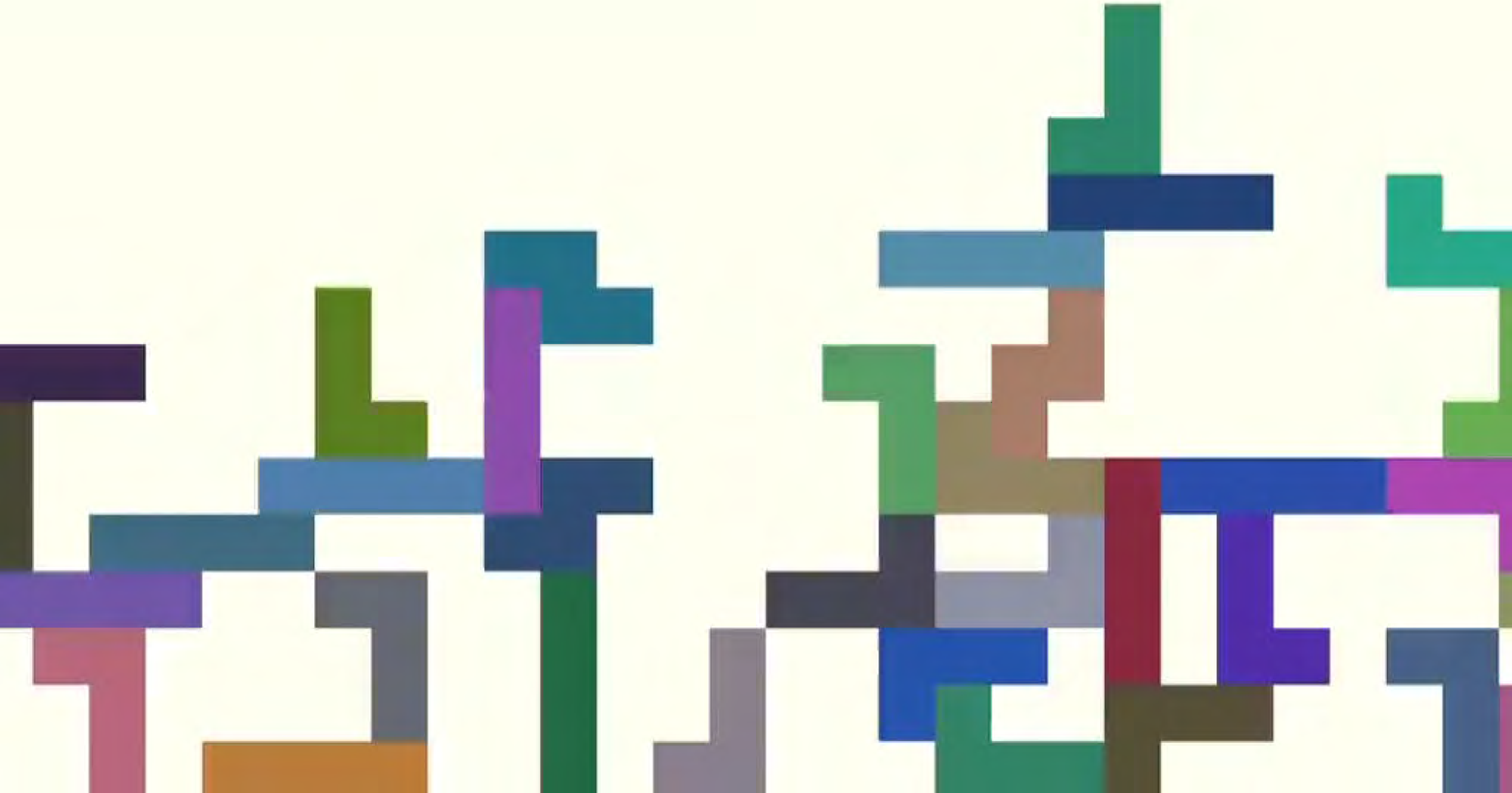

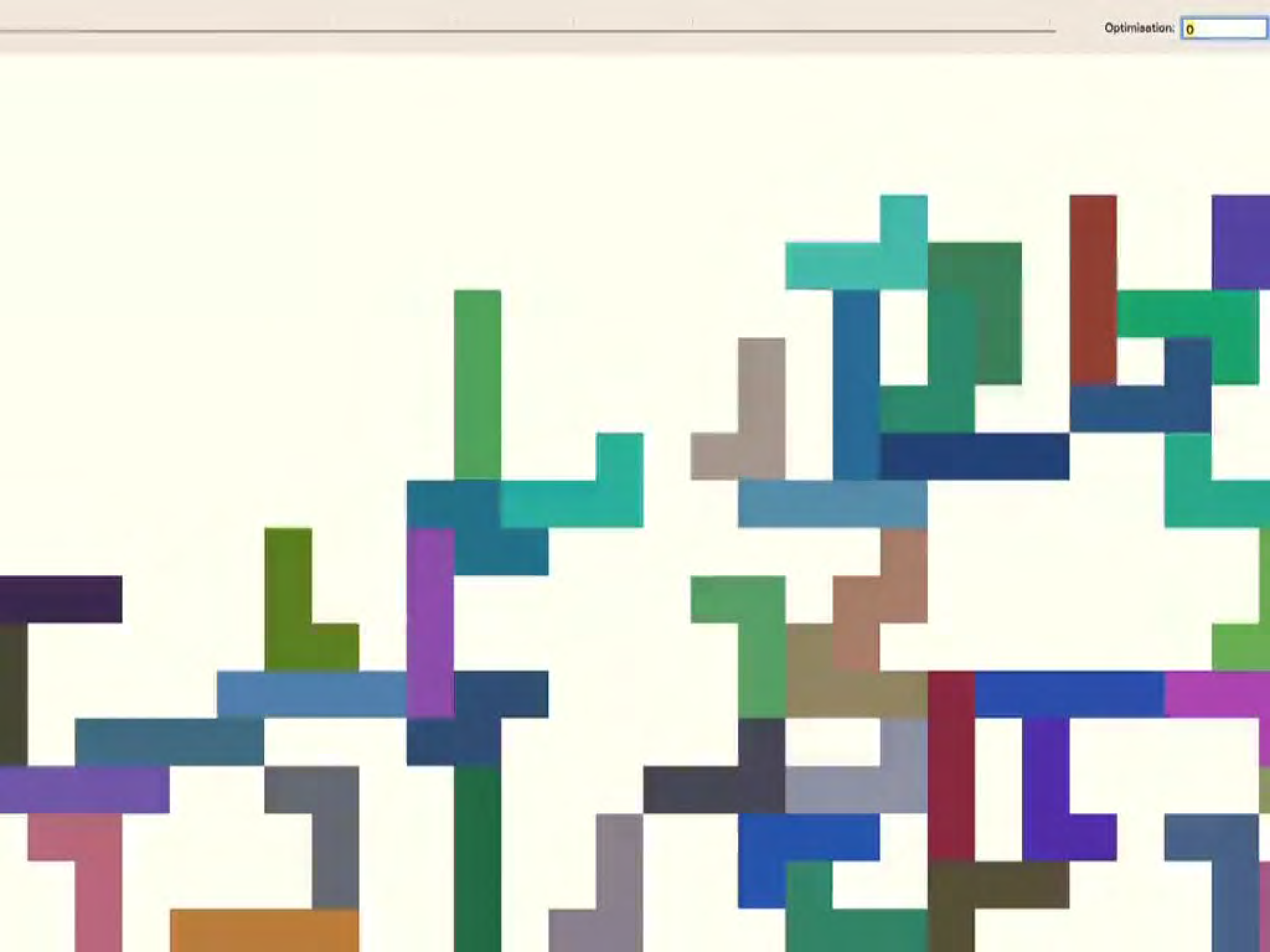
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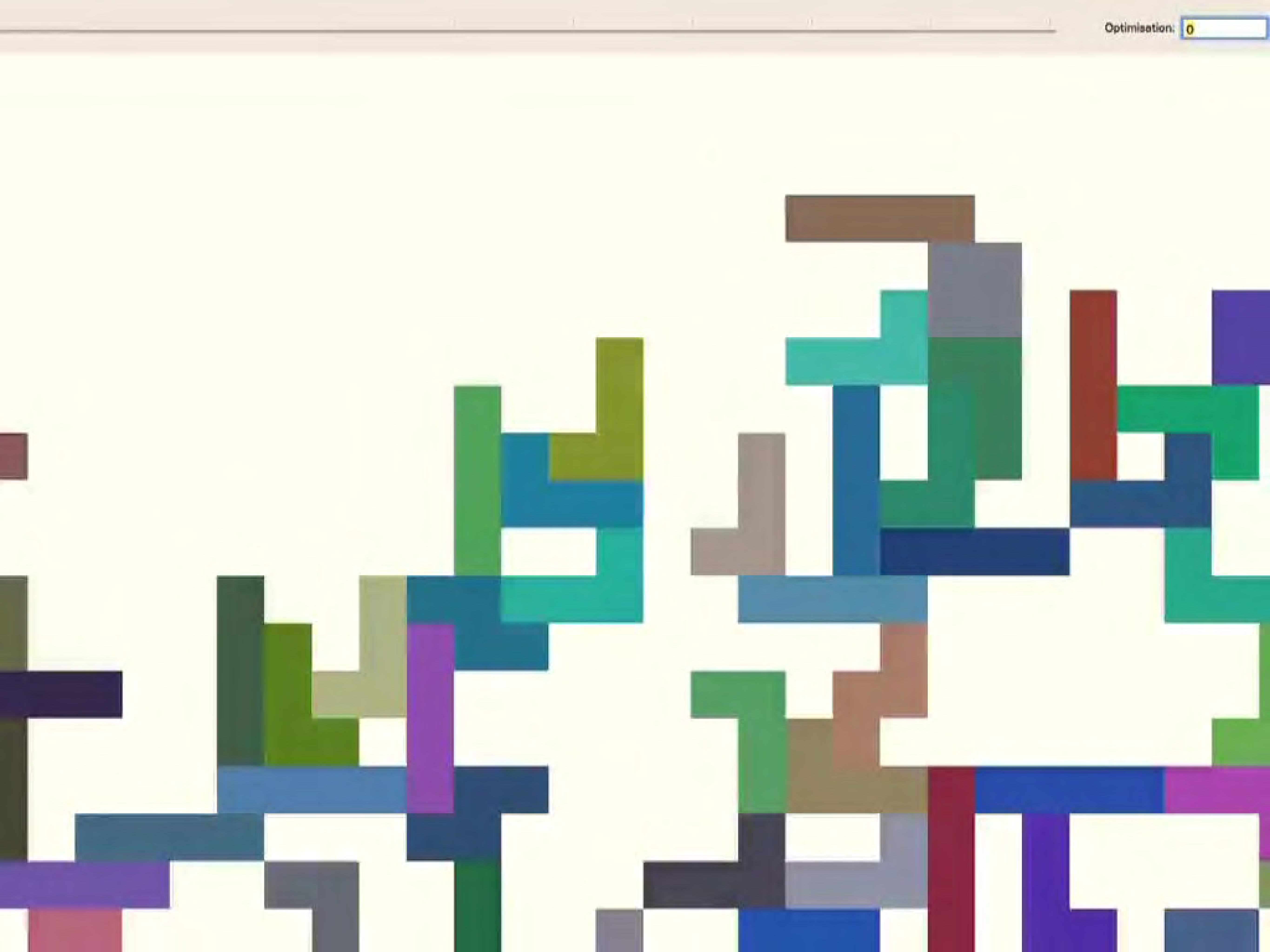
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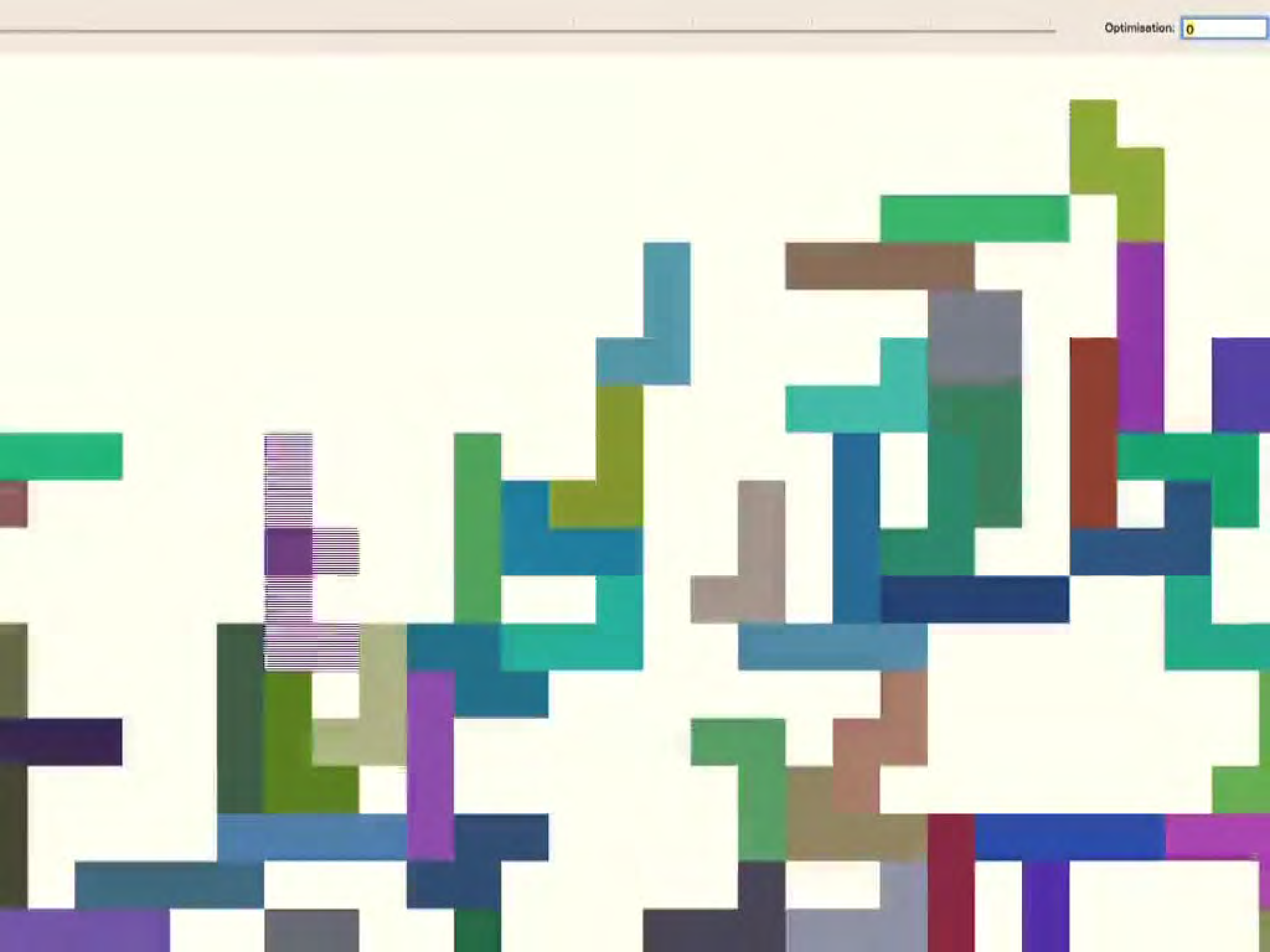
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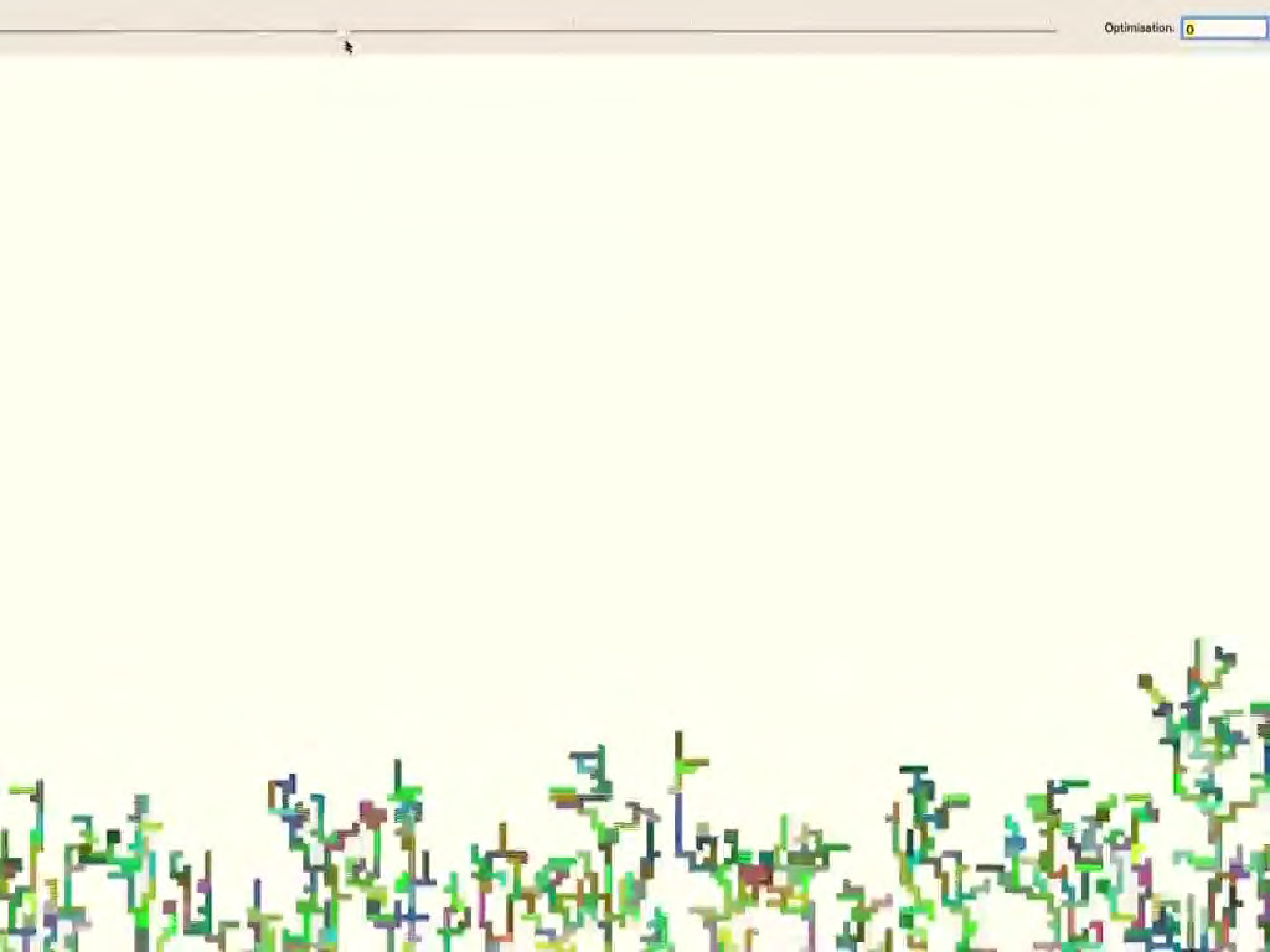
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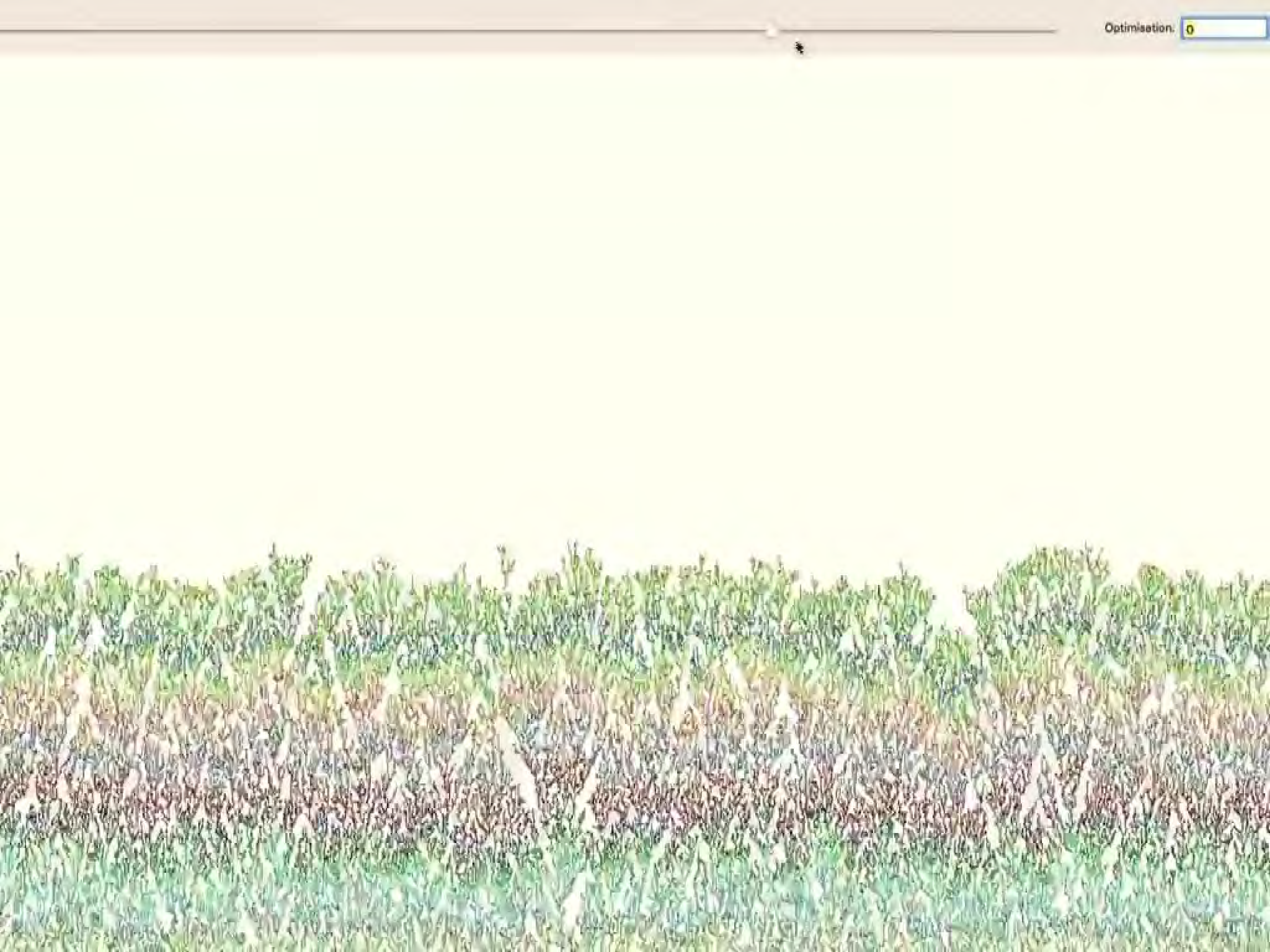
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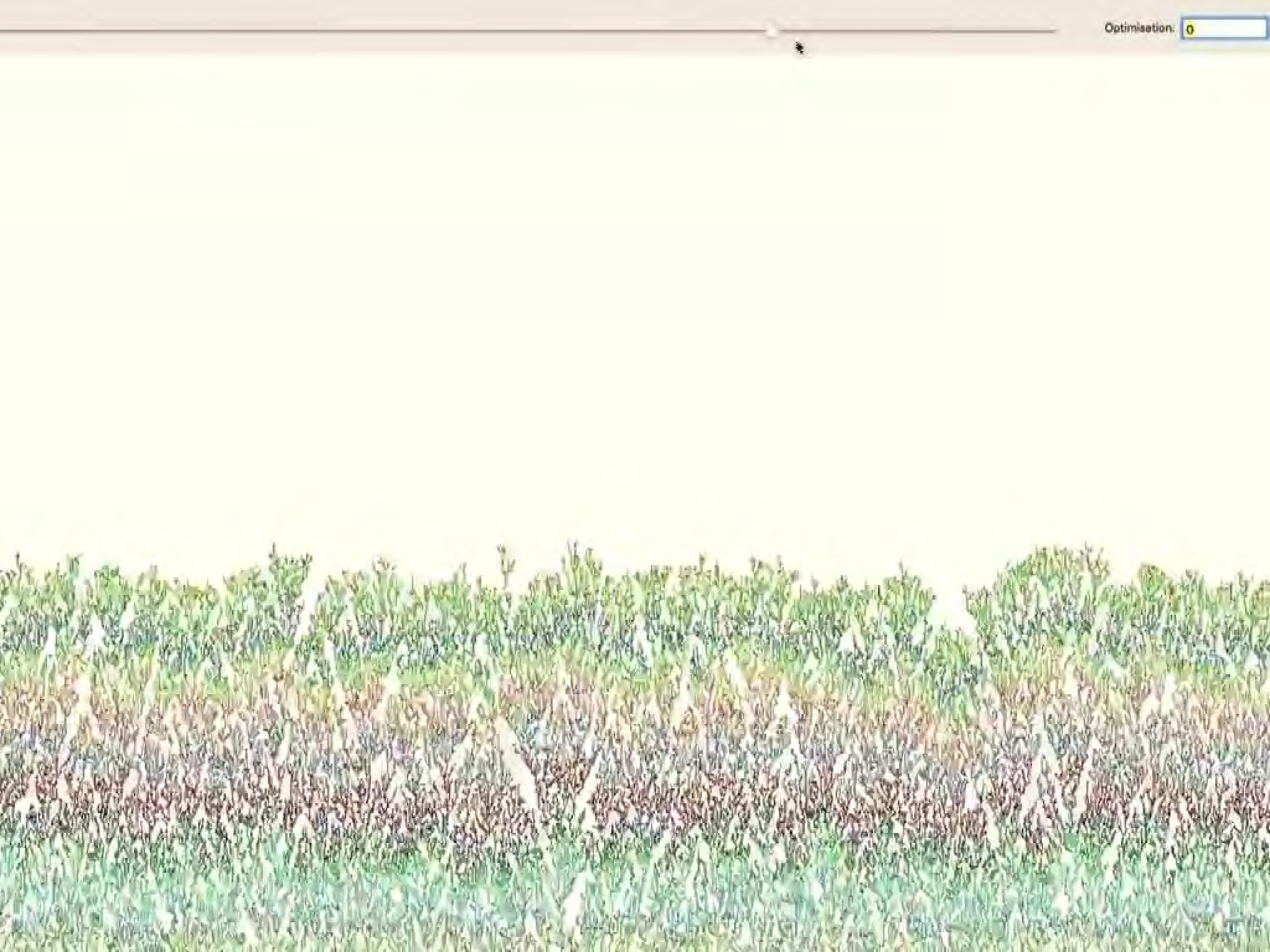
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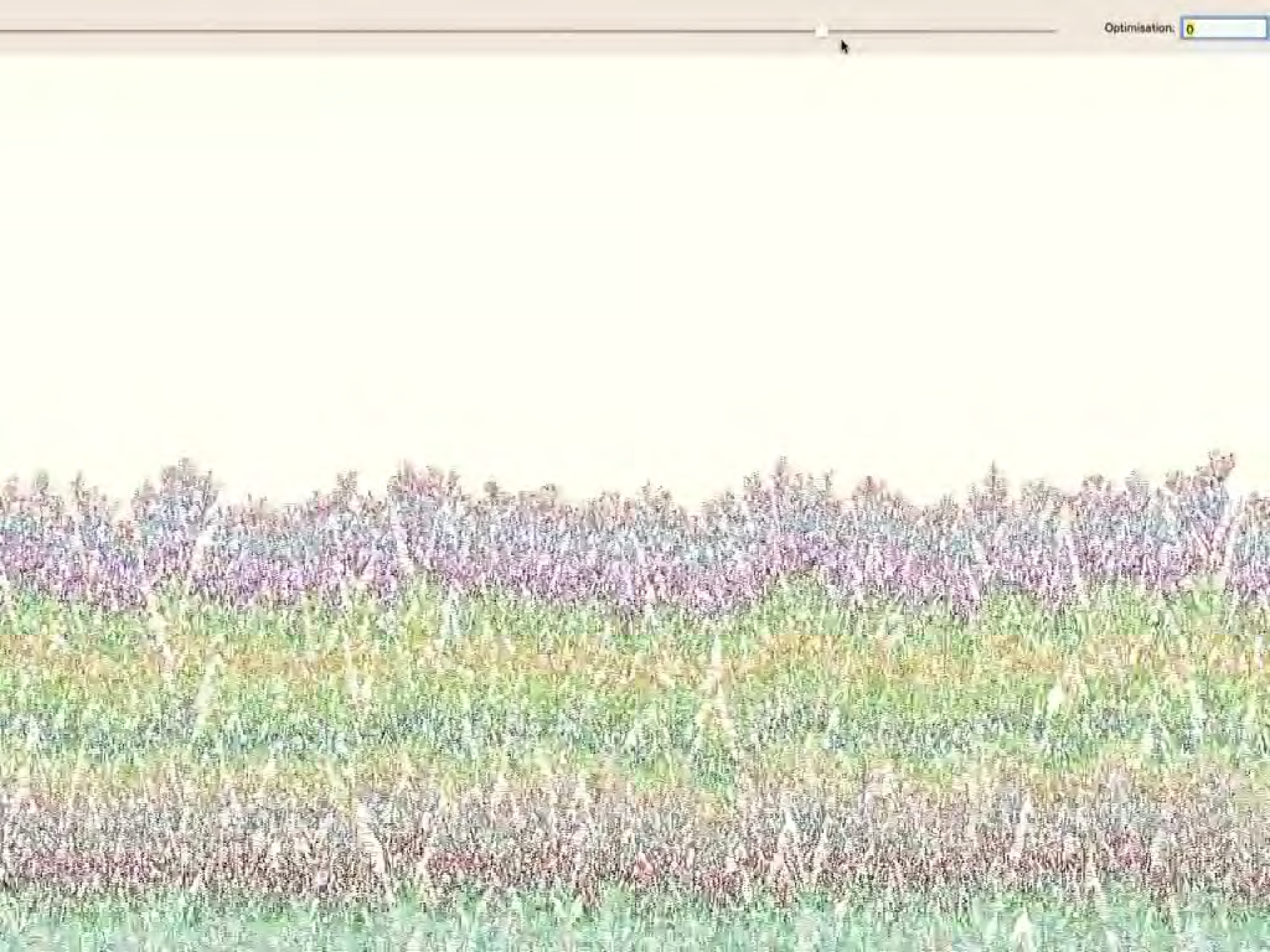
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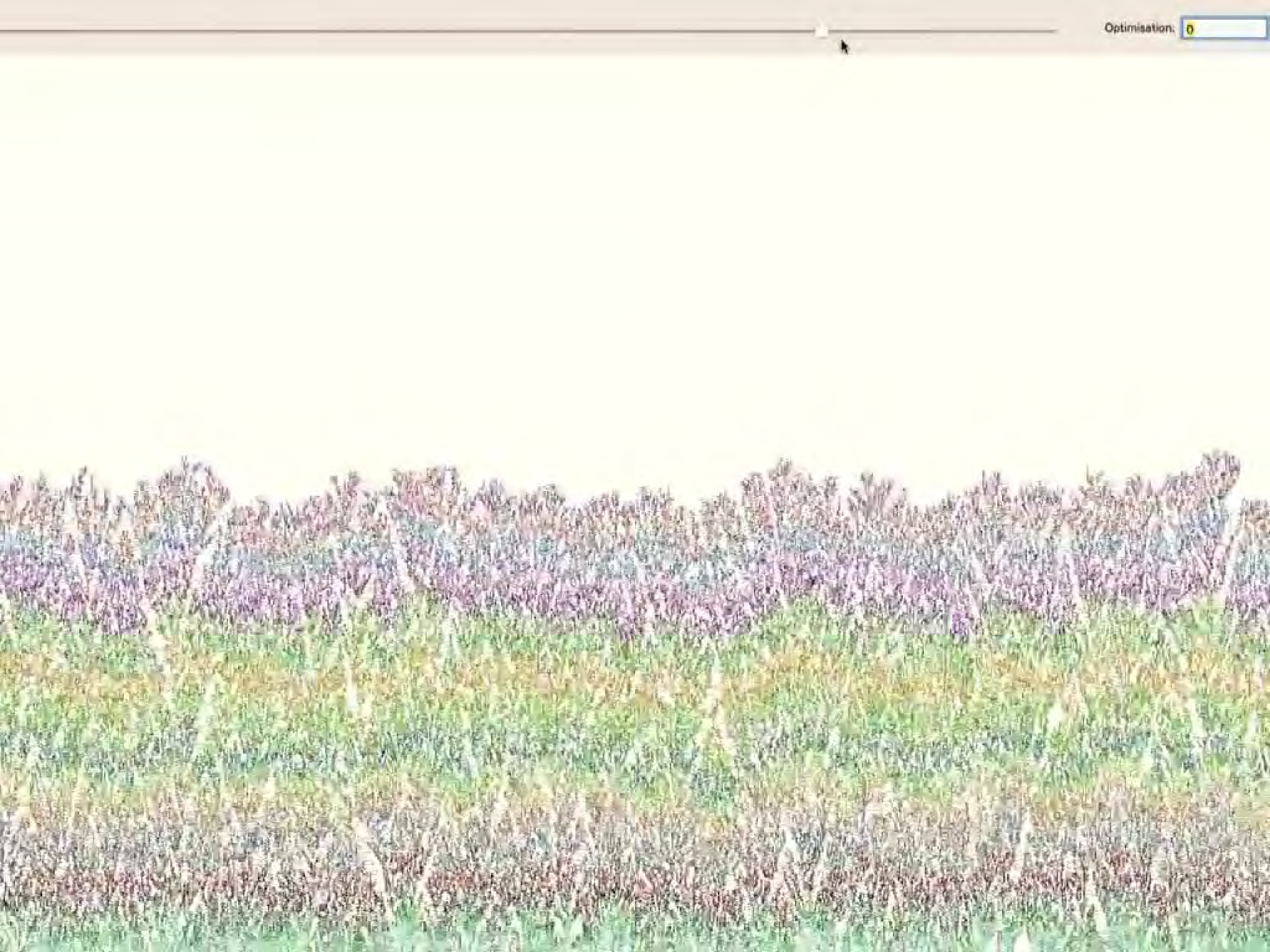
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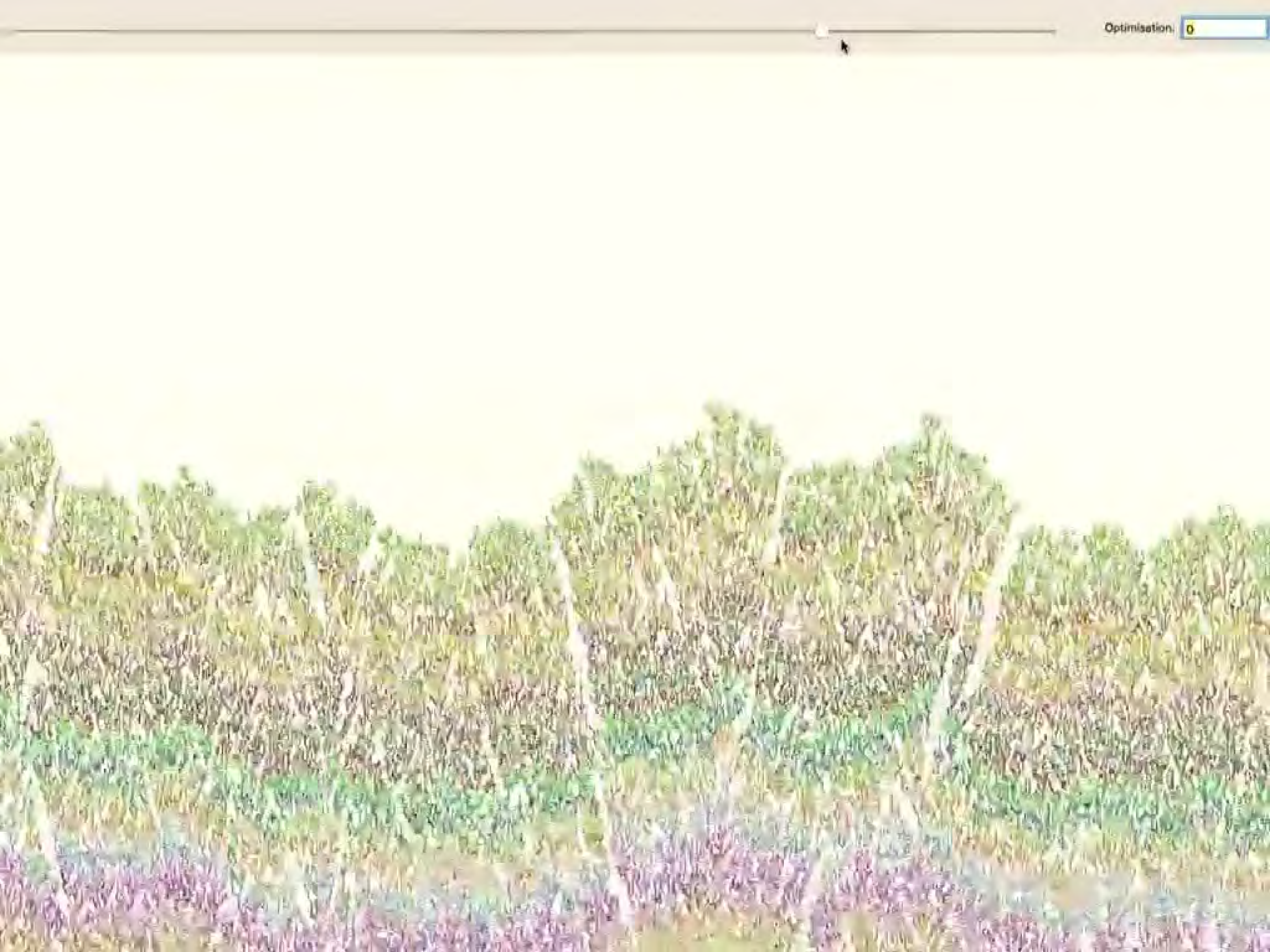
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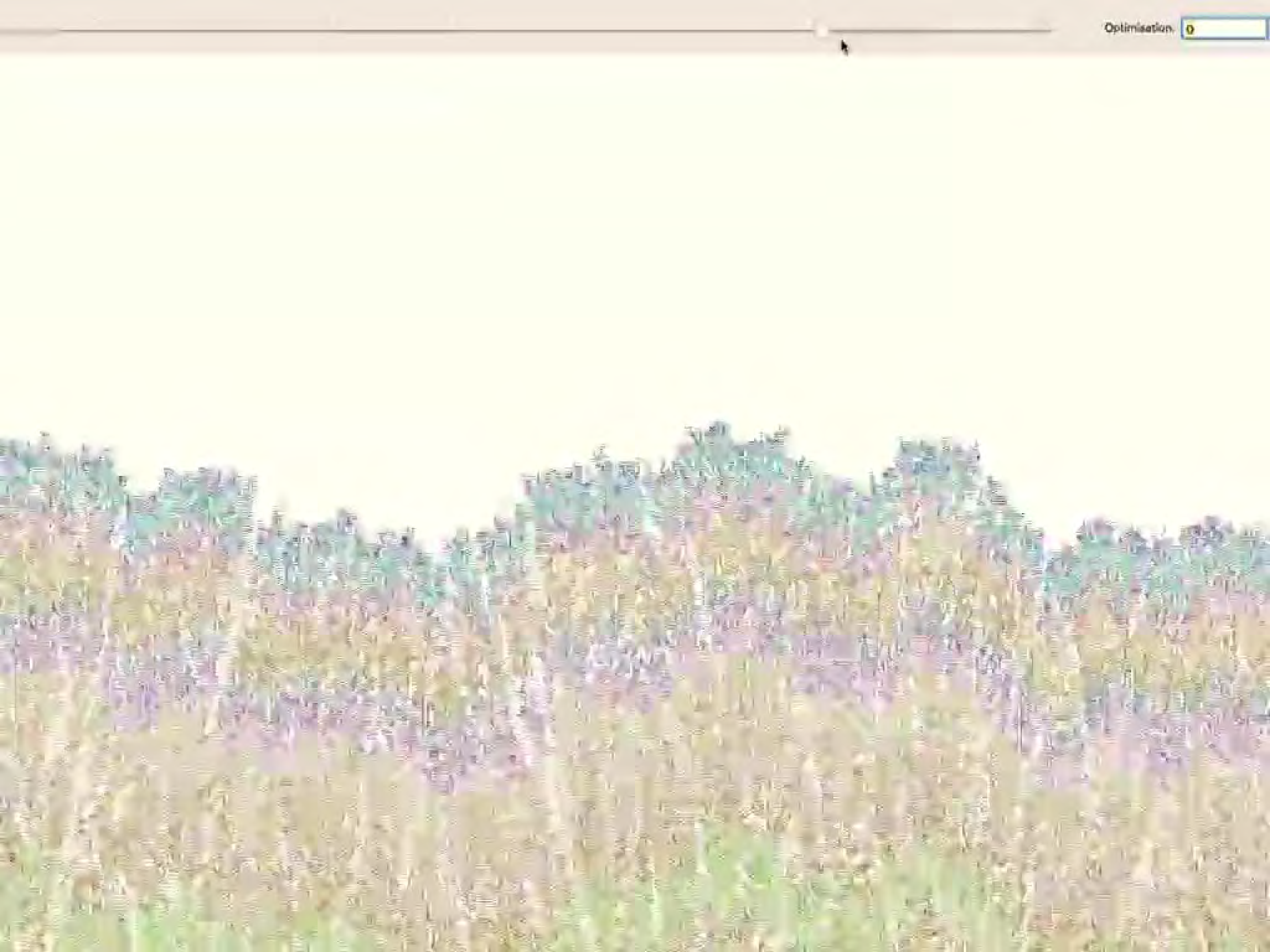
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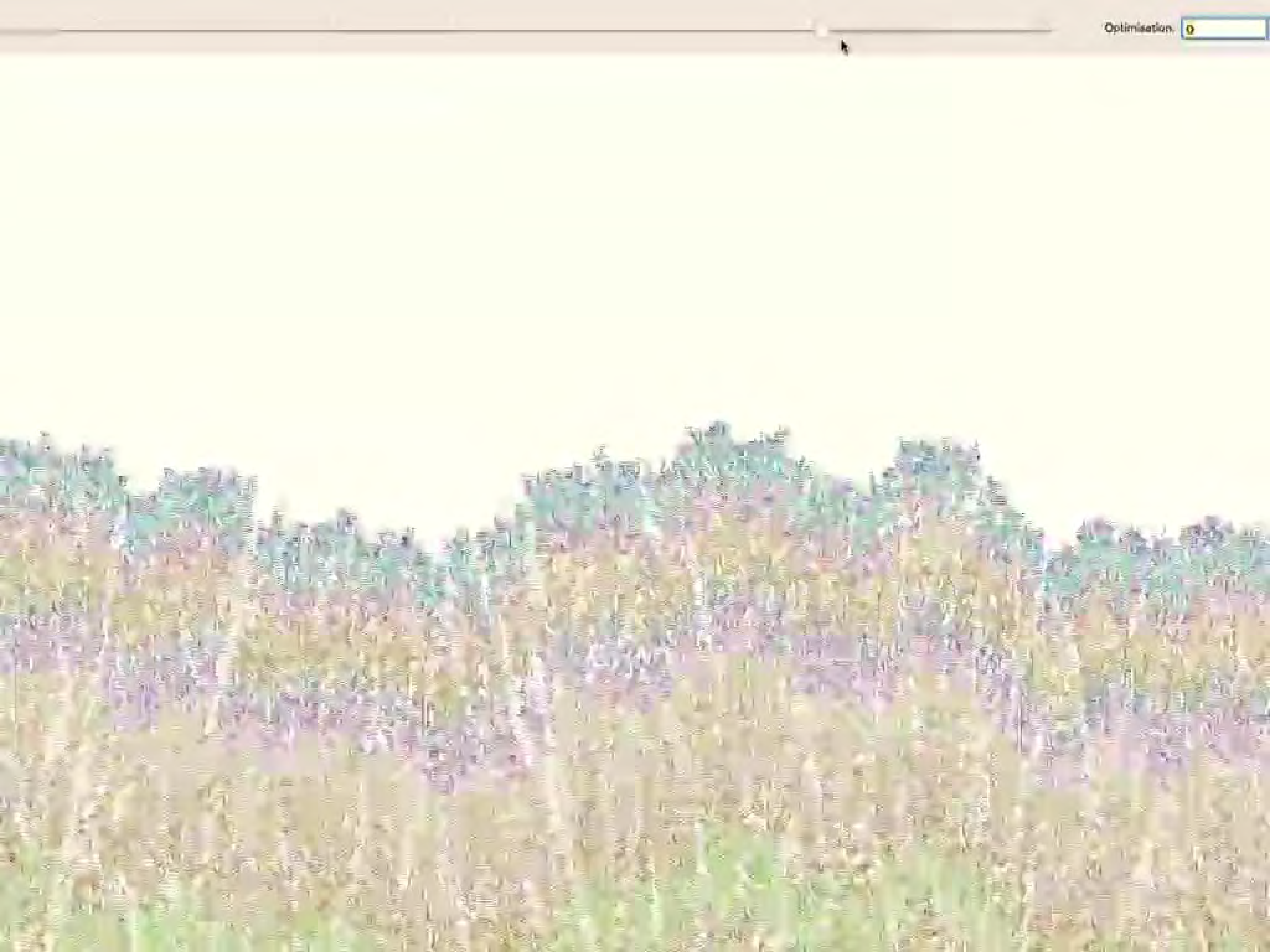
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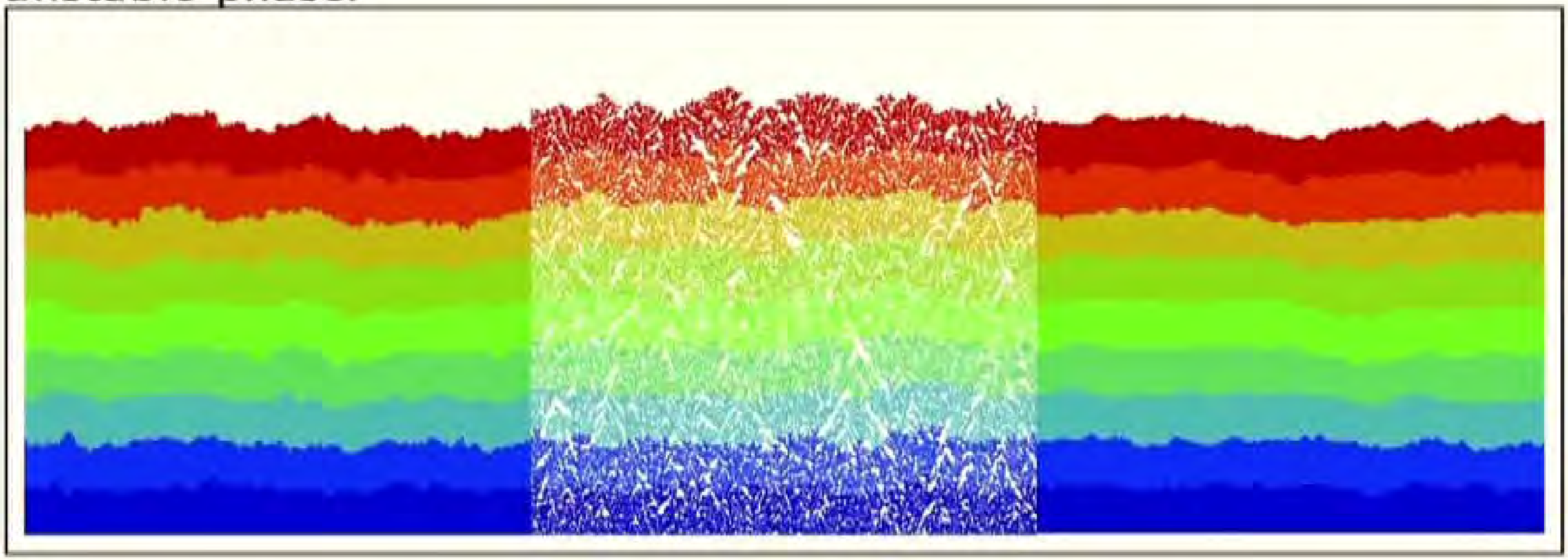

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Crossover regimes

Consider models that converge to a Gaussian fixed point when "zooming in" and a non-Gaussian FP when "zooming out".

Described by simple (looking) "normal form" equations:

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi - C\,, \qquad (\text{KPZ: } d = 1)$$

$$\partial_t \Phi = -\Delta(\Delta\Phi + C\Phi - \Phi^3) + \nabla\xi\,. \quad (\Phi^4\text{: } d = 2, 3)$$

Here ξ is space-time white noise (think of independent random variables at every space-time point).

KPZ: universal model for weakly asymmetric interface growth.

Φ^4: universal model for phase coexistence near mean-field.

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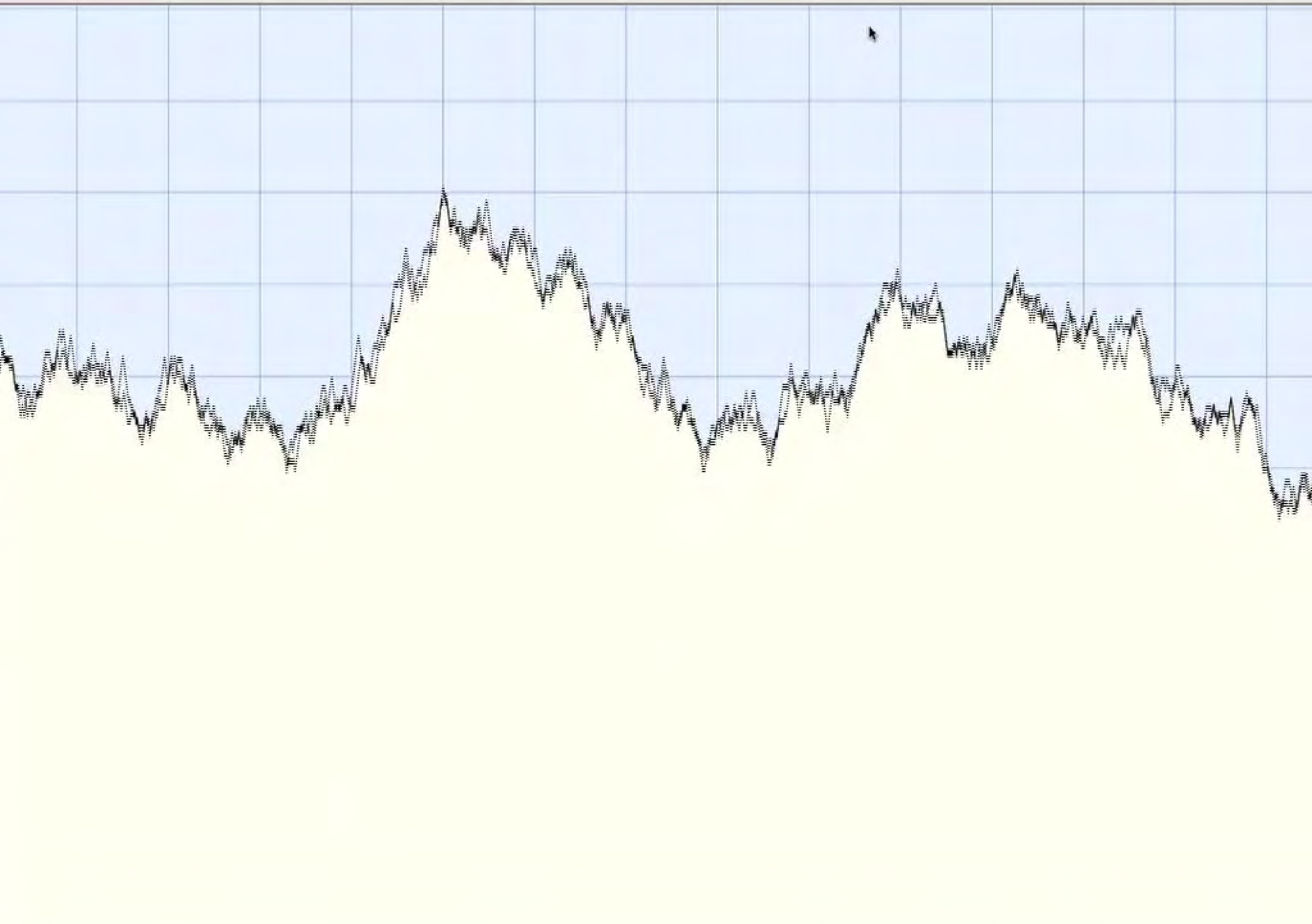

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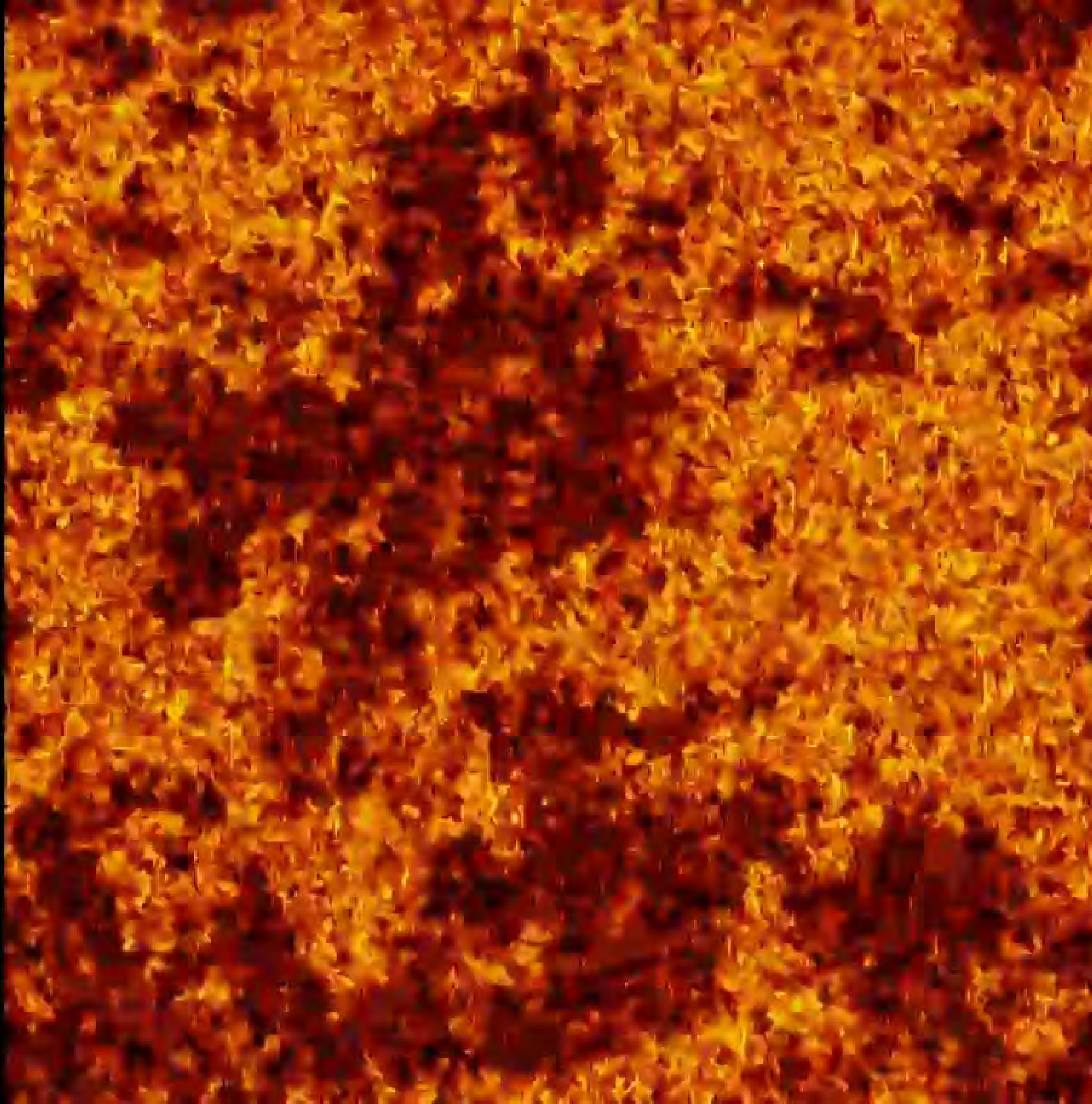

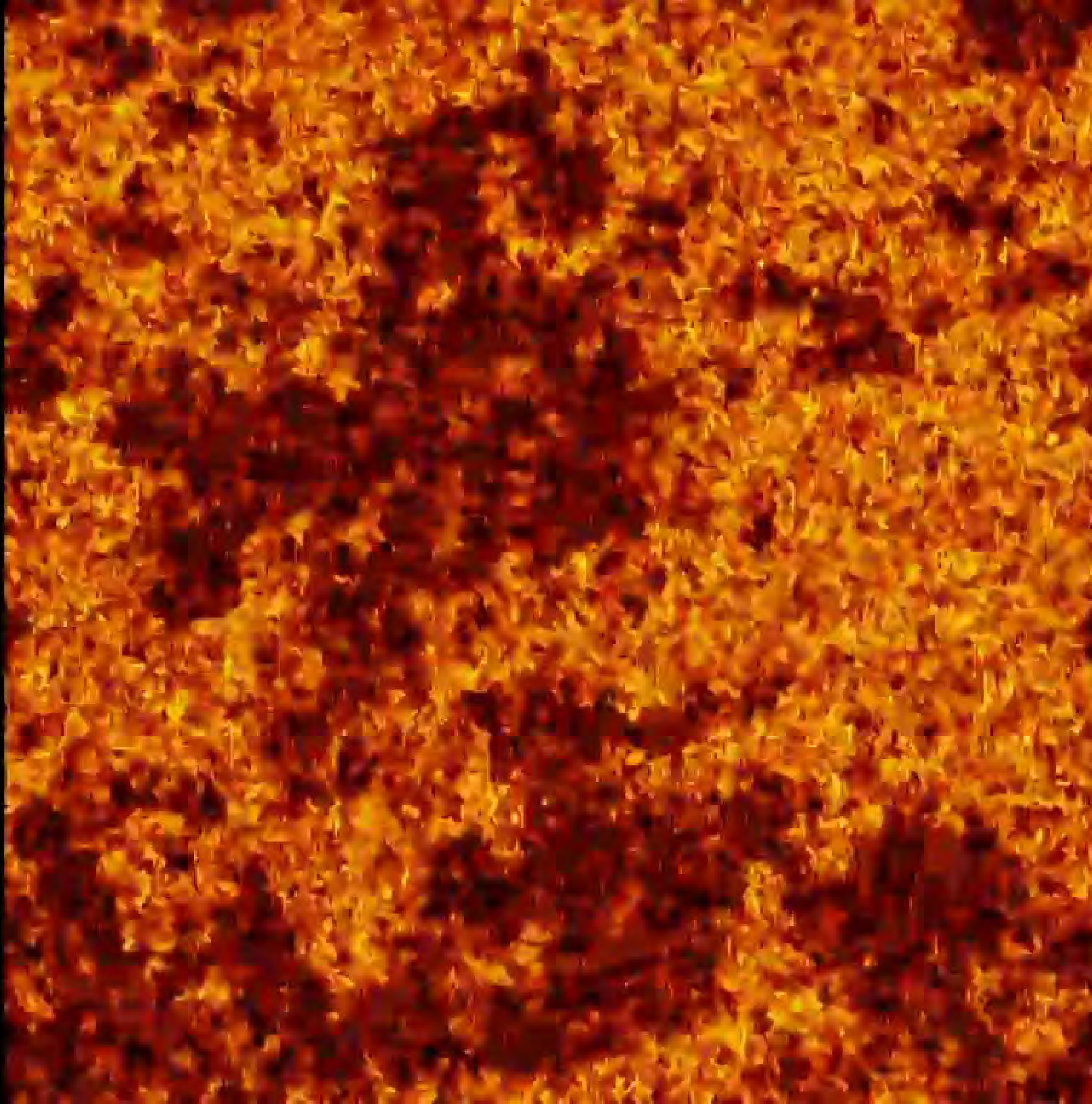

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A general theorem

Joint with Y. Bruned, A. Chandra, I. Chevyrev, L. Zambotti.

Consider a system of semilinear stochastic PDEs of the form

$$\partial_t u_i = \mathcal{L}_i u_i + G_i(u, \nabla u, \ldots) + F_{ij}(u)\xi_j \;, \qquad (\star)$$

with elliptic $\mathcal{L}_i$ and stationary random (generalised) fields ξ_j that are scale invariant with exponents for which $(\star)$ is subcritical.

Then, there exists a canonical family $\Phi_g \colon (u_0, \xi) \mapsto u$ of "solutions" parametrised by $g \in \mathfrak{R}$, a finite-dimensional nilpotent Lie group built from the data $(\star)$. Furthermore, the maps Φ_g are continuous in both of their arguments.

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Some clarifications

Canonicity

Family $\{\Phi_g : g \in \mathfrak{R}\}$ is canonical, but parametrisation only canonical modulo shifts: action of $\mathfrak{R}$ on (F, G) such that

$$\Phi_{g\tilde{g}}^{(F,G)} = \Phi_g^{\tilde{g}(F,G)} .$$

For smooth ξ, one has a classical solution map $\Phi^{(F,G)}$ and $\Phi_g^{(F,G)} = \Phi^{(g \circ \hat{g}(\xi))(F,G)}$.

Continuity

Measure $\mathcal{S}$ of "size" of noise. Take ξ_n with $\sup_n \mathcal{S}(\xi_n) < \infty$ and $\xi_n \to \xi$ weakly in probability. Then $\Phi_g(\cdot, \xi_n) \to \Phi_g(\cdot, \xi)$ in some $\mathcal{C}^\alpha$, locally uniformly in time and initial condition, in probability. However, $\xi \mapsto \hat{g}(\xi)$ not continuous, not even defined!