How environmental randomness can reverse the trend

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–
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SIAM DS19, Snowbird

20 May 2019

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$$\begin{aligned} \frac{dx_t}{dt} &= (1-x_t)(ax_t + by_t) - \alpha x_t \\ \frac{dy_t}{dt} &= (1-y_t)(cx_t + dy_t) - \beta y_t \end{aligned}$$

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  with some initial condition $x_0 = x, y_0 = y$.
- The **Environment** corresponds to the coefficients $a, b, c, d, \alpha, \beta > 0$.

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Theorem (Lajmanovic-Yorke)

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2. $\lambda(A) > 0 \Rightarrow \exists (x^*, y^*) > 0, \forall (x_0, y_0) > 0, (x_t, y_t) \to (x^*, y^*)$, *i.e* (x^*, y^*) *is Globally Asymptotically Stabe*

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(x^*, y^*) is the *Endemic Equilibrium*

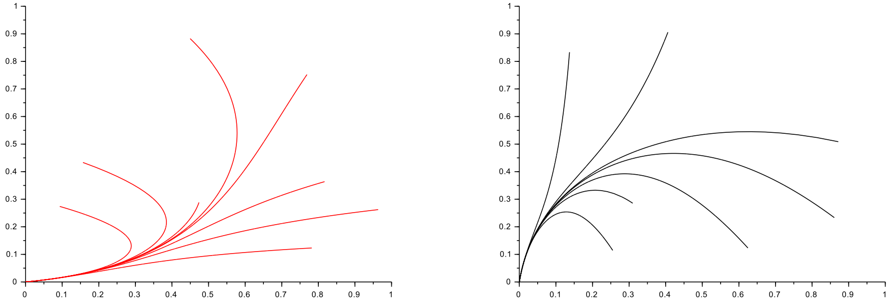
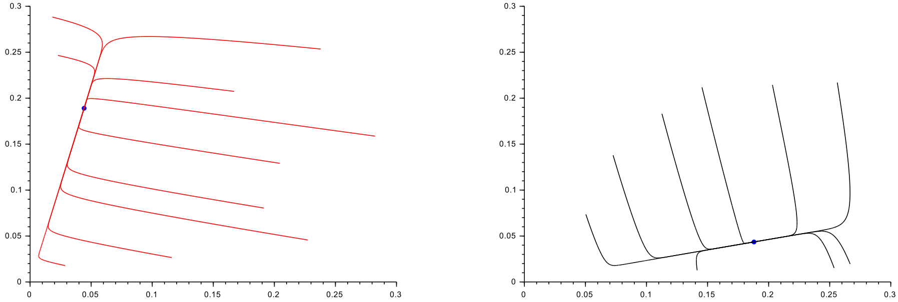


Figure – Examples of environments in which the disease disappears.



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Environment is in state 0 during a time T_0^1...
when it switches to state 1 for a time T_1^1...
then it goes back to 0 for a time T_0^2, and so on.

An example

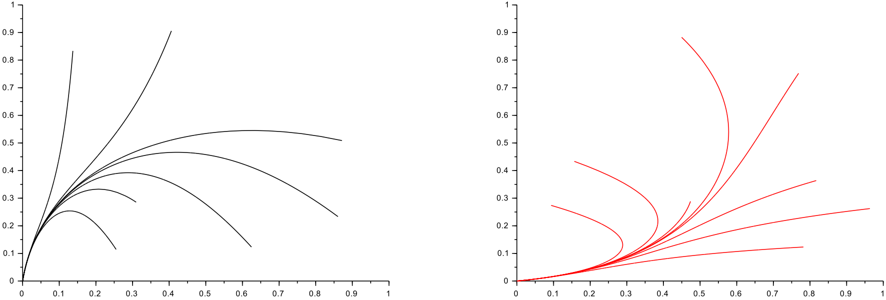
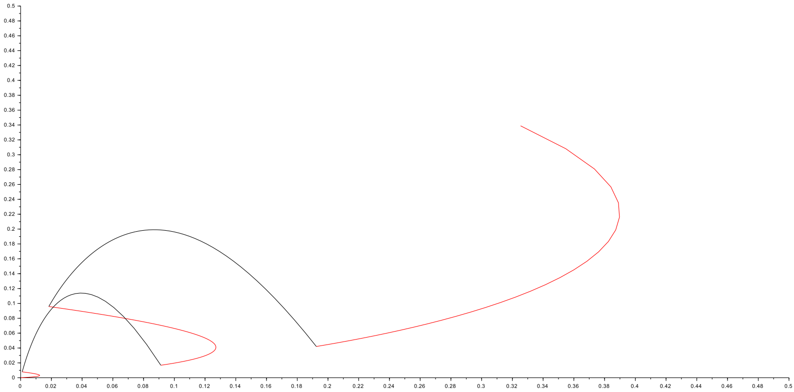


Figure – Environment $\mathcal{E}_0$ and $\mathcal{E}_1$ in which the disease eventually disappears

If T_0^n and T_1^n are big, i.e. few switches per unit of time...



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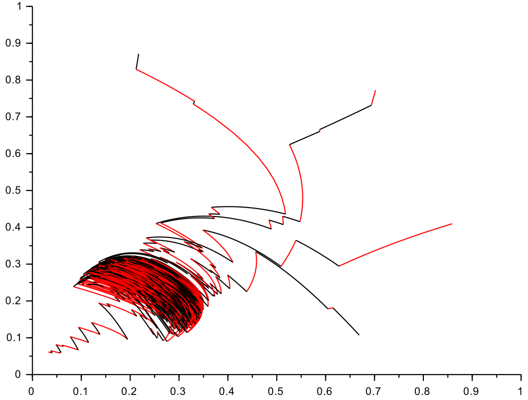

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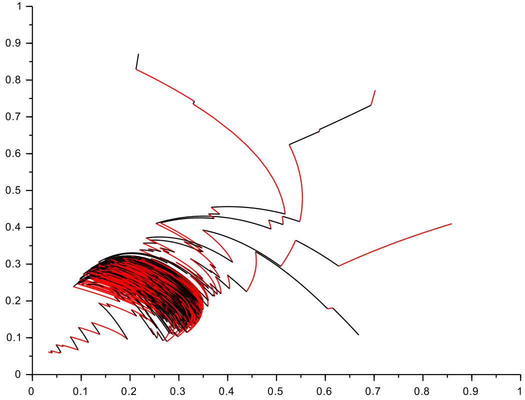

Environmental randomness can reverse the trend : here, it promotes the persistence of the disease in the population.

Converse example

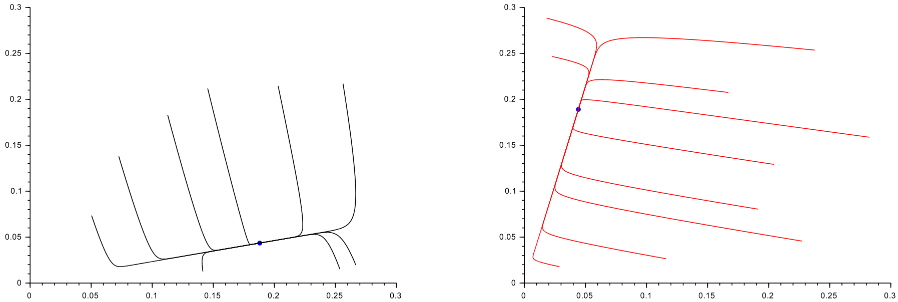

Figure – Environments $\mathcal{E}_0$ and $\mathcal{E}_1$ in which the disease persists in the population.

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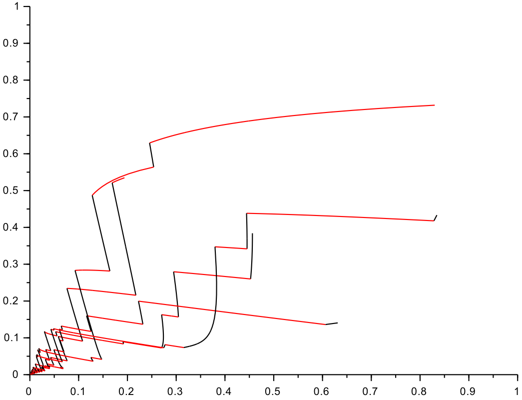

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- Is there a way to predict the random behaviour ?

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- $\forall i, j$, $Q(i, j) \in [0, 1]$: *giving the probability to switch from environment* i *to environment* j.

Construction

Flow

$$\begin{cases} \frac{d}{dt}x_t = F^i(x_t) \\ x_0 = x \end{cases}$$

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The first jump time

If $I_0 = i$, $\mathbb{P}(T_1 > t) = \exp(-\alpha_i t)$: $T_1 \sim \mathcal{E}(\alpha_i)$ and $\mathbb{E}(T_1) = \alpha_i^{-1}$.

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In short, $\dot{X}_t = F^{I_t}(X_t)$, with I_t a Markov Chain on E.

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Remark : the Markov property comes from the definition of T_1 :

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Question : What happens if $X_0 \neq 0$?

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Note : if $\lambda \in Sp(A)$, then $\exists y_0$ such that

$$\lim \frac{1}{t} \log \|y_t\| = Re(\lambda).$$

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$$\text{Look at } \lim \frac{1}{t} \log \|Y_t\| \ldots$$

Theorem

Under general conditions, $\exists \lambda \in \mathbb{R} : \forall (y_0, i)$,

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Main Results : the local behaviour of X near 0 is given by the sign λ.

Theorem (with M. Benaïm)

1. Assume $\lambda < 0$. Then $\forall \alpha \in (\Lambda^+, 0)$, there exists $\eta > 0$ and a neighbourhood $\mathcal{U}$ of 0 such that

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3. Let $\tau^\varepsilon = \inf\{t \geq 0 : \ \|X_t\| \geq \varepsilon\}$. Then there exist $\varepsilon > 0$, and $a > 0$ such that for all $X_0 \neq 0$,

$$\mathbb{E}(\mathrm{e}^{a\tau^\varepsilon}) < \infty.$$

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- Irreducibility : disease can spread from group k to group l
- Some monotonicity and sublinearity assumption.

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2. If $\lambda > 0$, Z admits a unique invariant probability μ such that $\mu(\{0\} \times E) = 0$. Moreover,the law of Z_t converges to μ

Example in dimension 3

F^0 and F^1 two SIS vector fields in dimension 3 such that, for all $s \in [0, 1]$, 0 is globally asymptotically stable for $F_s = sF^1 + (1 - s)F^0$. Jump rate : $\alpha_1 = \alpha_2 = \beta > 0$.

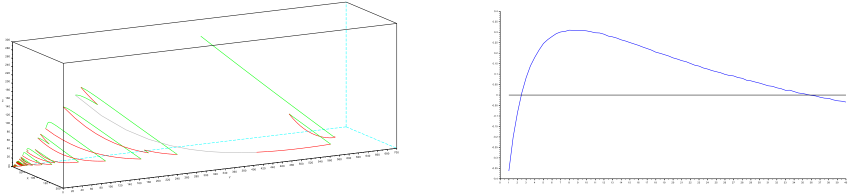


Figure – Simulation of Y_t for $\beta = 10$ and simulation of $\lambda(\beta)$

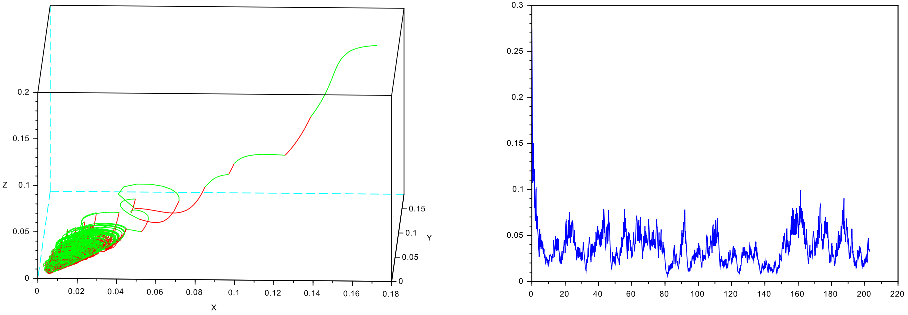


Figure – Simulation of X_t for $\beta = 10$ and simulation of $\|X_t\|$

Another example : Lotka - Volterra prey-predator

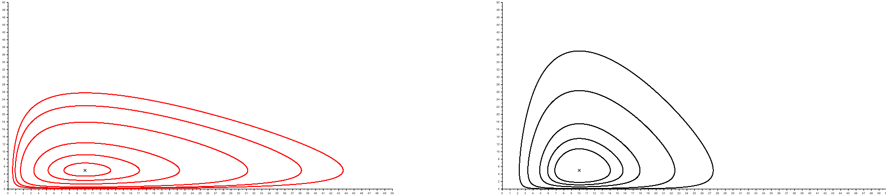


Figure – Periodic orbits around the same point x

$$F^i(x,y) = \begin{pmatrix} x(a_i - b_i y) \\ y(-c_i + d_i x) \end{pmatrix}, i = 0,1 \quad p = (\frac{c_i}{d_i}, \frac{a_i}{b_i}).$$

Theorem (with Alex Hening)

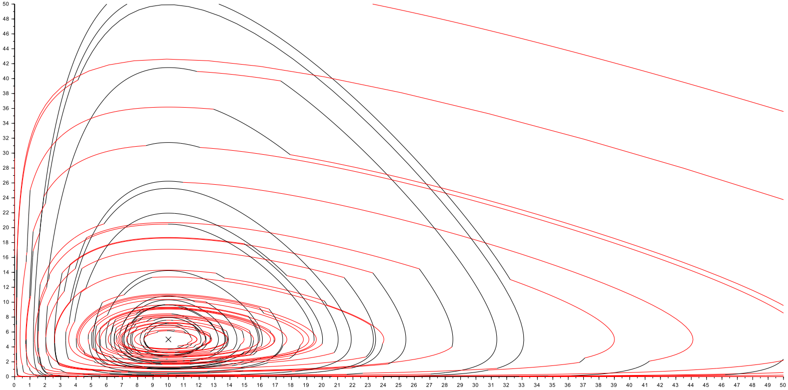
If F^0 and F^1 are not proportional ;

$$\lambda > 0,$$

and

$$\limsup x_t = \limsup y_t = +\infty, \quad \liminf x_t = \liminf y_t = 0 \quad p.s.$$

Environmental Randomness



Thank you !