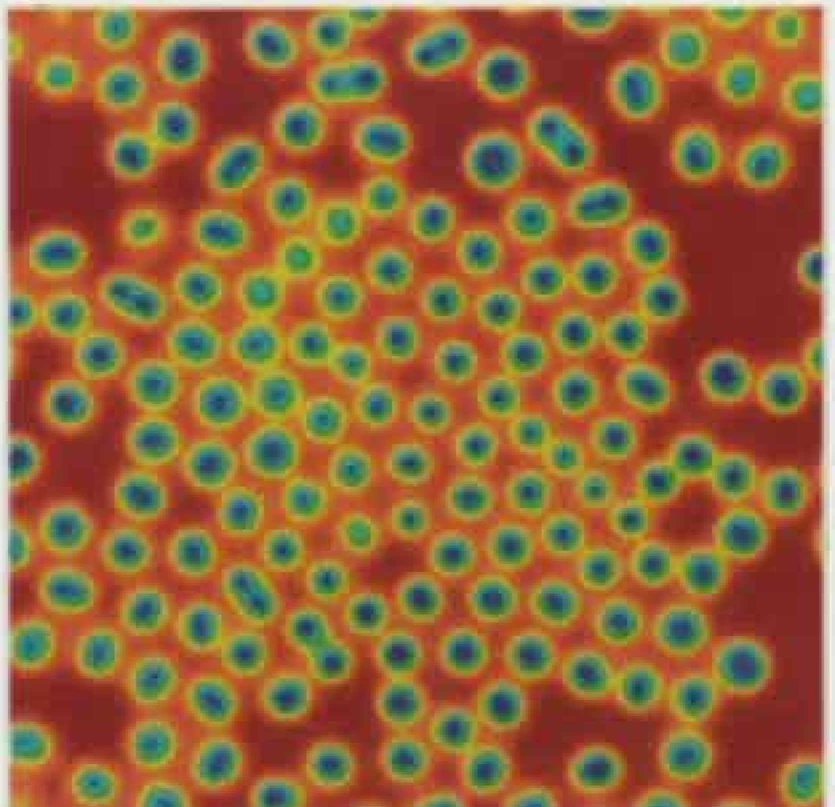
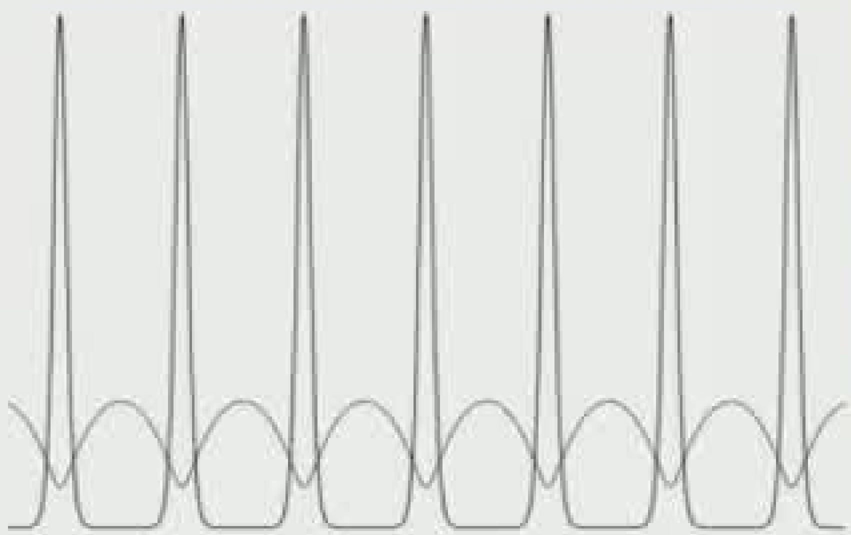
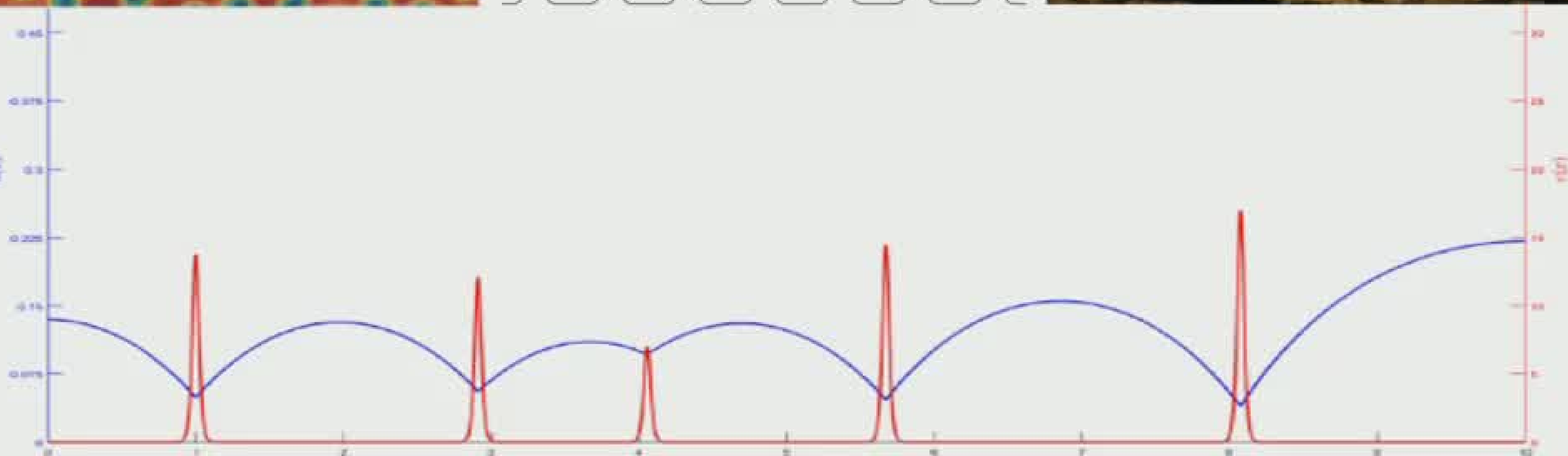
Interactions, Deformations and Bifurcations of Singular Patterns

Arjen Doelman
Leiden, the Netherlands







Thanks to friends and collaborators

Gianne Derks	Yasumasa Nishiura	Max Rietkerk
Rob Gardner	Keith Promislow	….
Tasso Kaper	Jens Rademacher	

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	Lotte Sewalt	
Peter van Heijster	Eric Siero	Tom Bellsky
Geertje Hek	Koen Siteur	Greg Hayrapetyan
David Iron	Sjors van der Stelt	Matt Holzer
Olfa Jaïbi	Nienke Valkhoff	Chris Knight

Universiteit Leiden
The Netherlands

NWO Netherlands Organisation for Scientific Research

What is a 'singular pattern'?

A singular pattern is a(n evolving) solution of a multiple scale system – typically a singularly perturbed PDE – where $0 < \varepsilon \ll 1$ 'measures' the perturbation.

Why would one study singular patterns?

- Multiple scale systems – or SP PDEs – appear naturally in 'applications'.
- While exhibiting behavior of a richness comparable to general, non-SP systems, the SP nature of these systems provides a framework – based on various $\varepsilon \to 0$ limits – by which the behavior of the patterns can be unraveled.

The theme of today:

The strong cross-fertilization between applications and the development of mathematical theory.

Reversing desertification [Ehud Meron]

Structure of the talk

- Autocatalytic reactions & Gray-Scott dynamics.

→ Semi-strong pulse interactions.

- Spatial ecology, vegetation patterns and desertification.

→ Pattern dynamics of under slowly varying circumstances.

- The Busse balloon: turbulence ↔ desertification.

→ Slowly nonlinear singularly perturbed equations.

→ A fine-structure of the boundary of the Busse balloon.

- Pattern dynamics under slowly varying conditions.

← The impact of the speed of change.

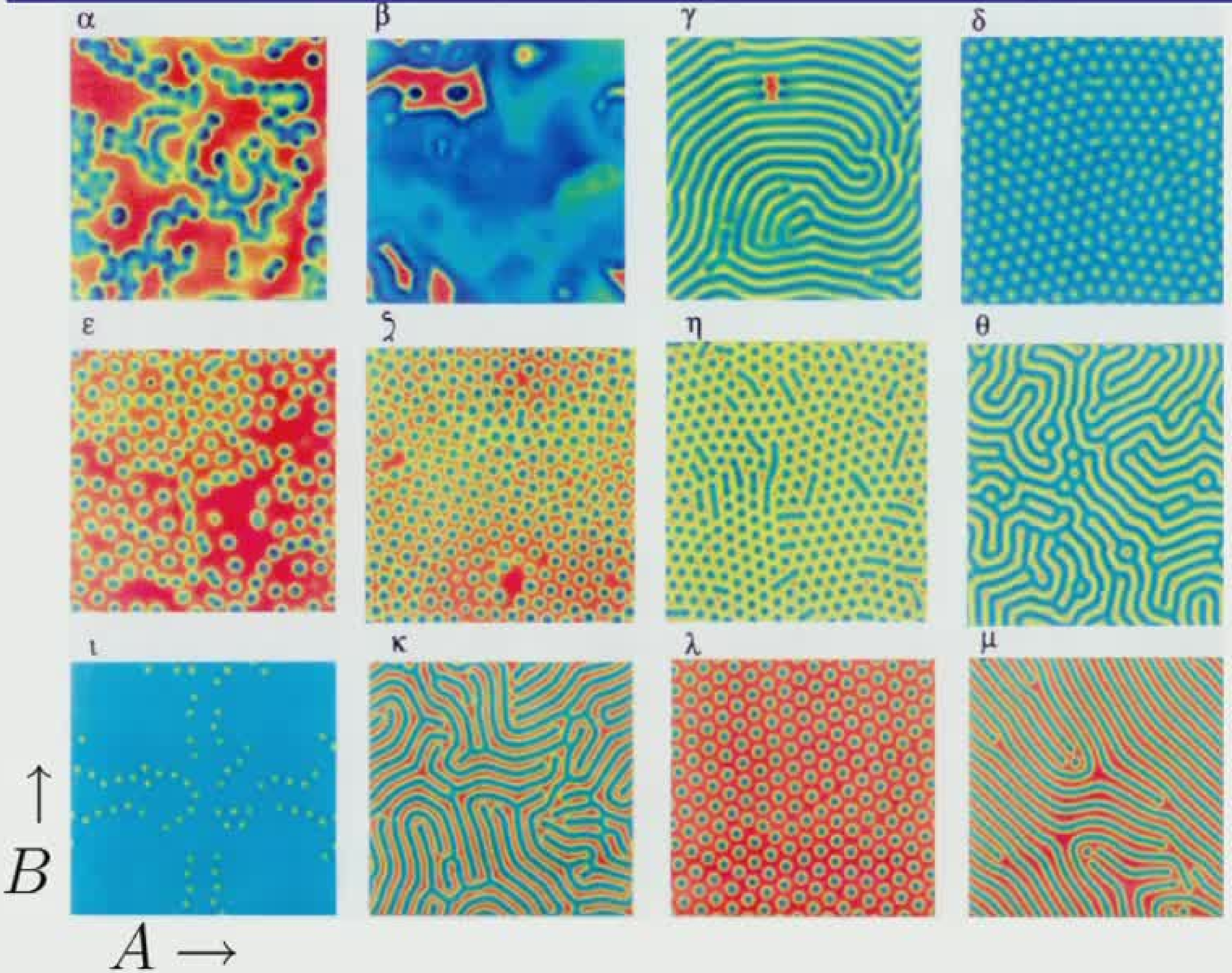
← Catastrophic ↔ gradual decline.

- Conclusions & Discussion.

+ some intermezzos

CHEMISTRY: Autocatalytic chemical reactions

(I) J.E. Pearson, Complex patterns in a simple system, *Science* (1994).



Computer simulations of the Gray-Scott RDE.

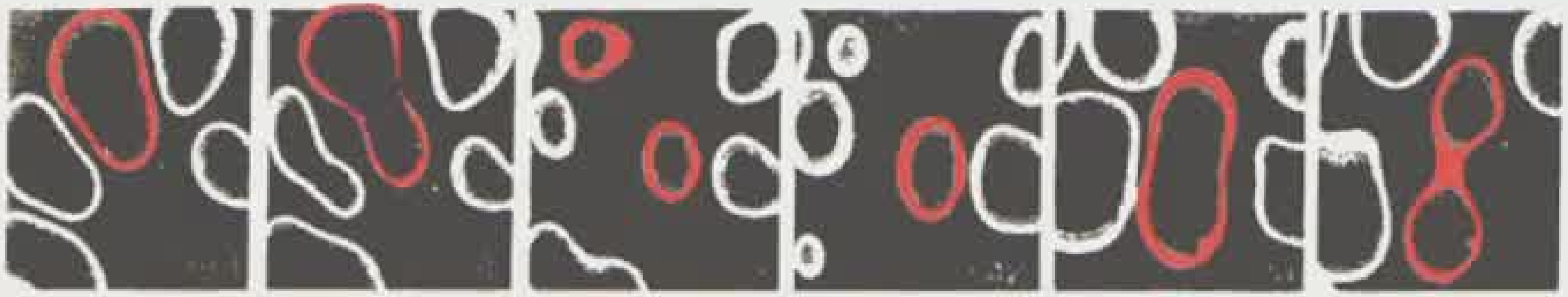
Q: What's going on?

Diffusion coefficient

$$D = 0.5$$
$$= \varepsilon^2??$$

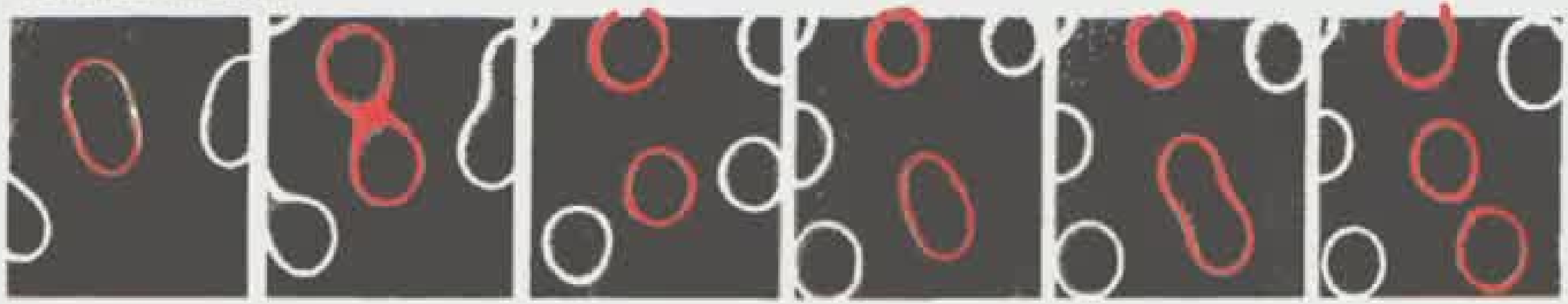
(II) K.J. Lee, W.D. McCormick, J.E. Pearson, H.L. Swinney, Experimental observation of self-replicating spots in a reaction-diffusion system, *Nature* (1994).

Laboratory experiment



Dynamics of an autocatalytic chemical reaction.

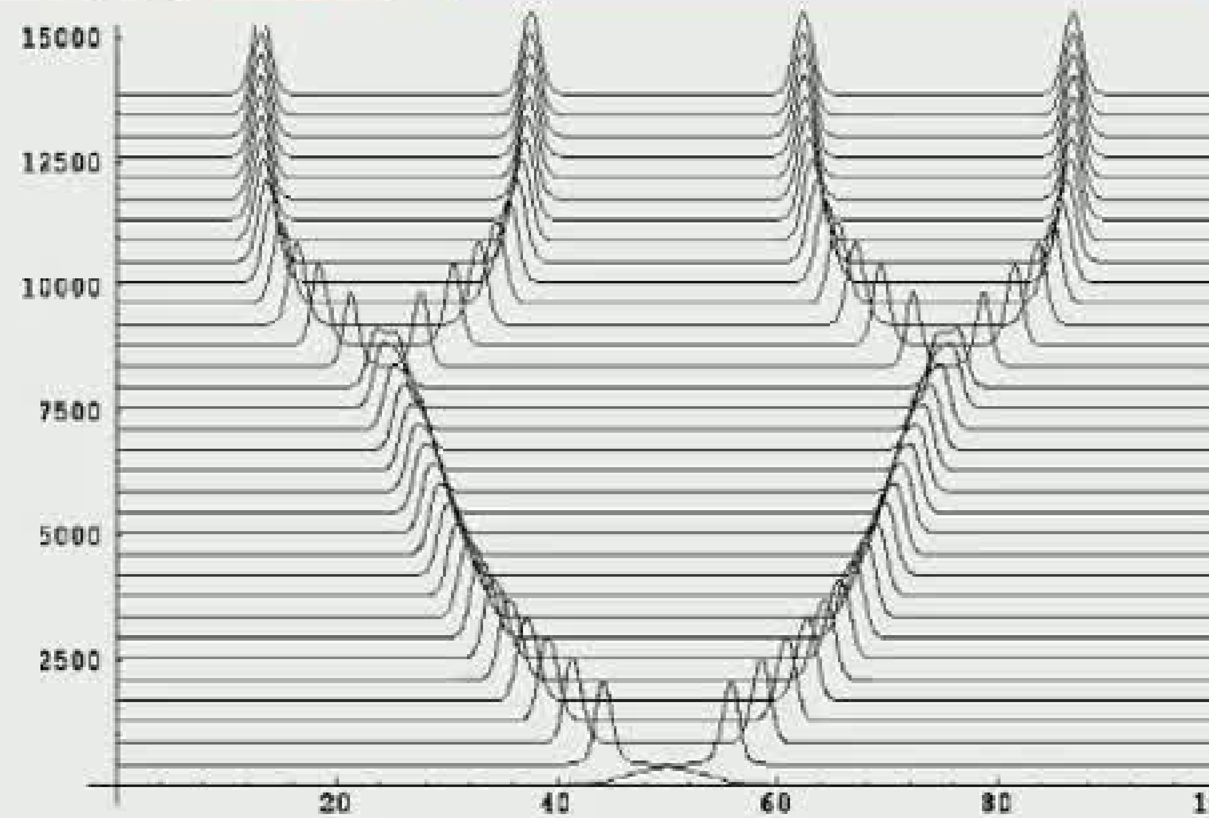
Numerical simulation



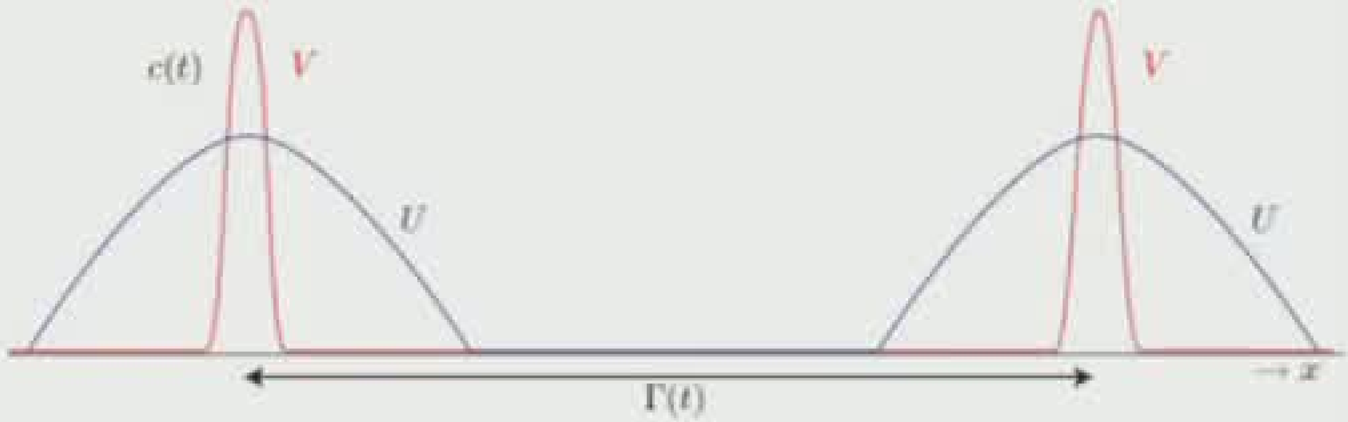
Numerical simulations of the Gray-Scott model.

The dynamics of ‘self-replicating’ spots and pulses.

→ The interaction of localized structures beyond the weak interaction limit.



Weak $\leftrightarrow$ Strong interactions



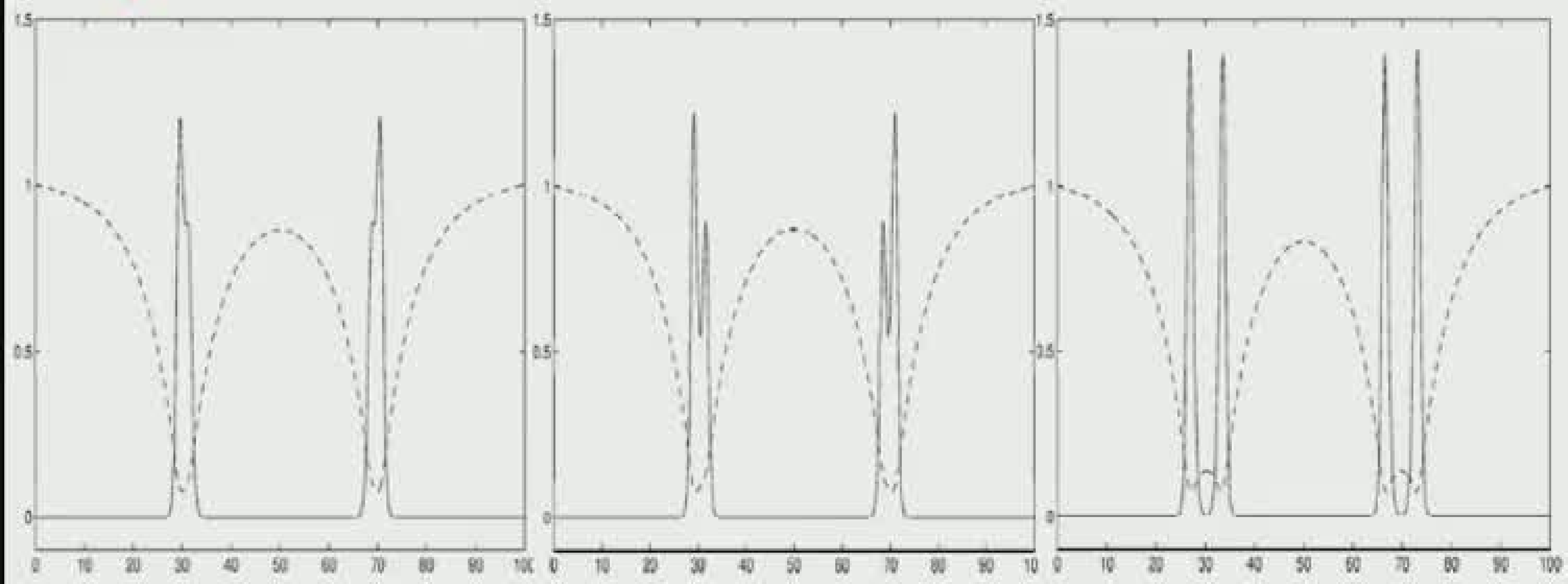
Assumption: $|\Gamma(t)| > C \gg 1$

WEAK interactions: pulses are so far apart that they only 'communicate' through exponentially small 'tail-tail' interactions $\longrightarrow$ pulses behave as 'particles'.

[Ei, Mimura, Promislow, Sandstede, Zelik, …]

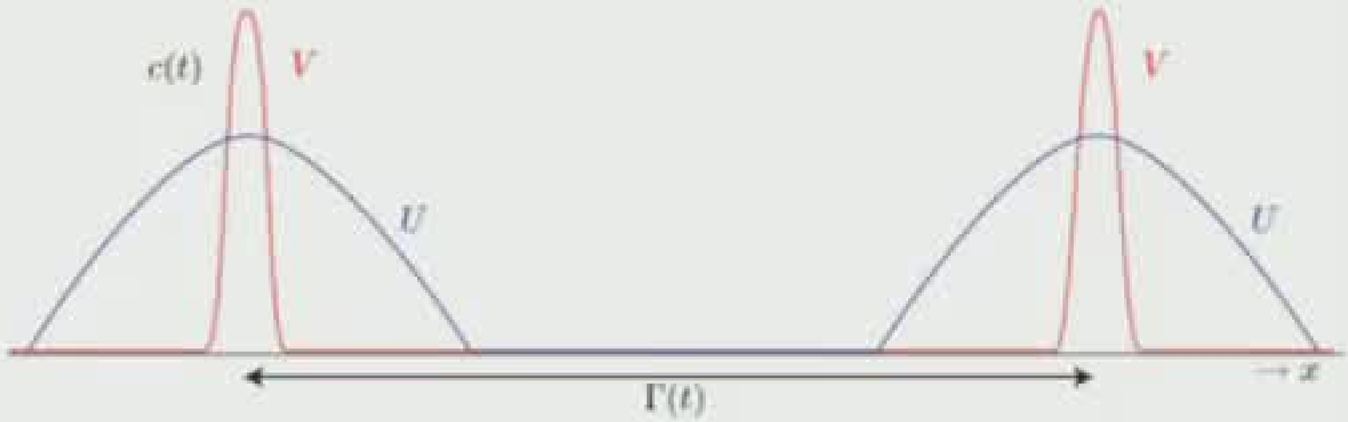
$\Updownarrow$

'Exploit' the singular structure: **SEMI-STRONG interactions**.



STRONG interactions: the pulse splitting process $\longrightarrow$ all components change.

Weak ↔ Strong interactions



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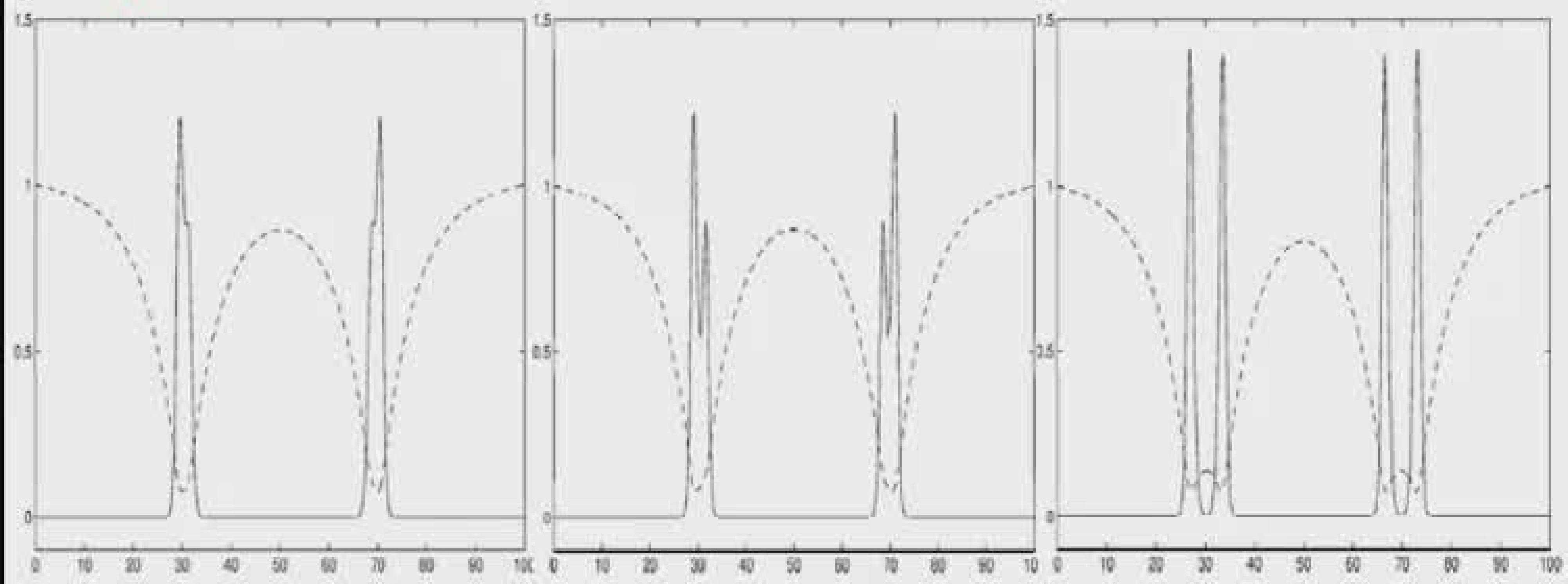
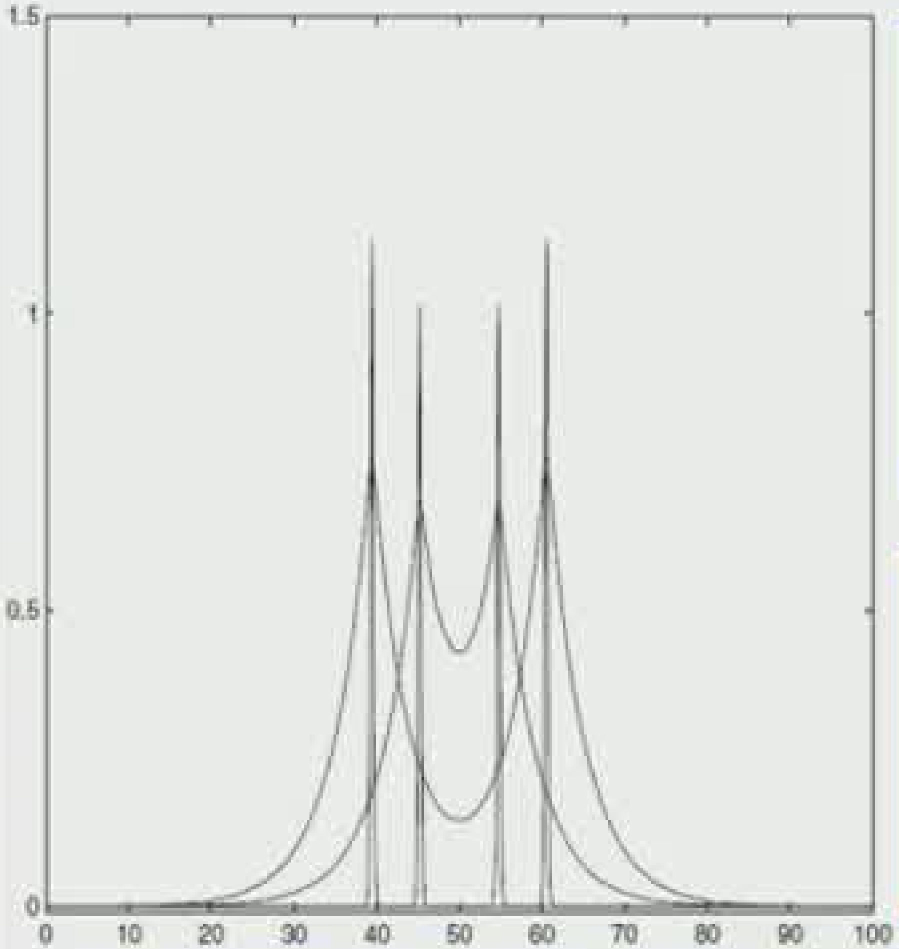


STRONG interactions: the pulse splitting process → all components change.

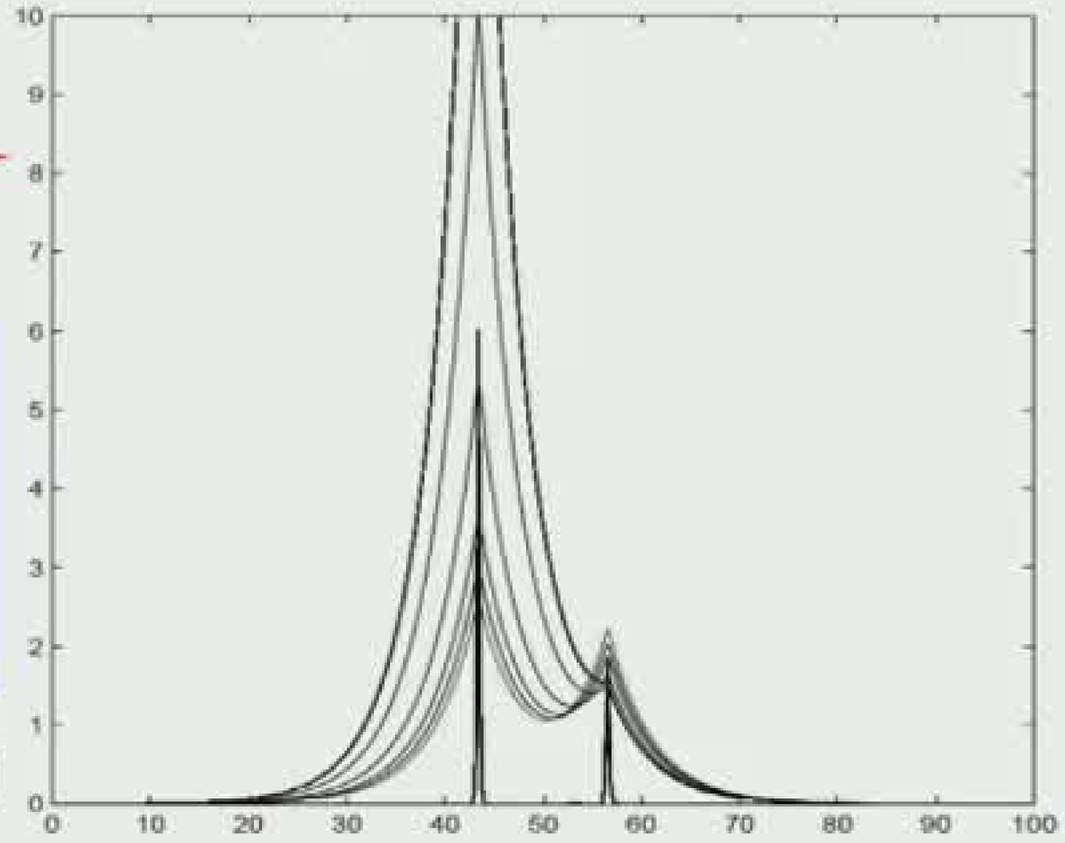
Semi-strong interactions (of singular patterns!)



A pair of pulses in semi-strong interaction: the 'fast' component is exponentially small between pulses, the 'slow' component varies 'at leading order'.

→ Pulses change in amplitude & shape during the evolution/interaction, they may even 'push' each other through a 'dynamic bifurcation'.

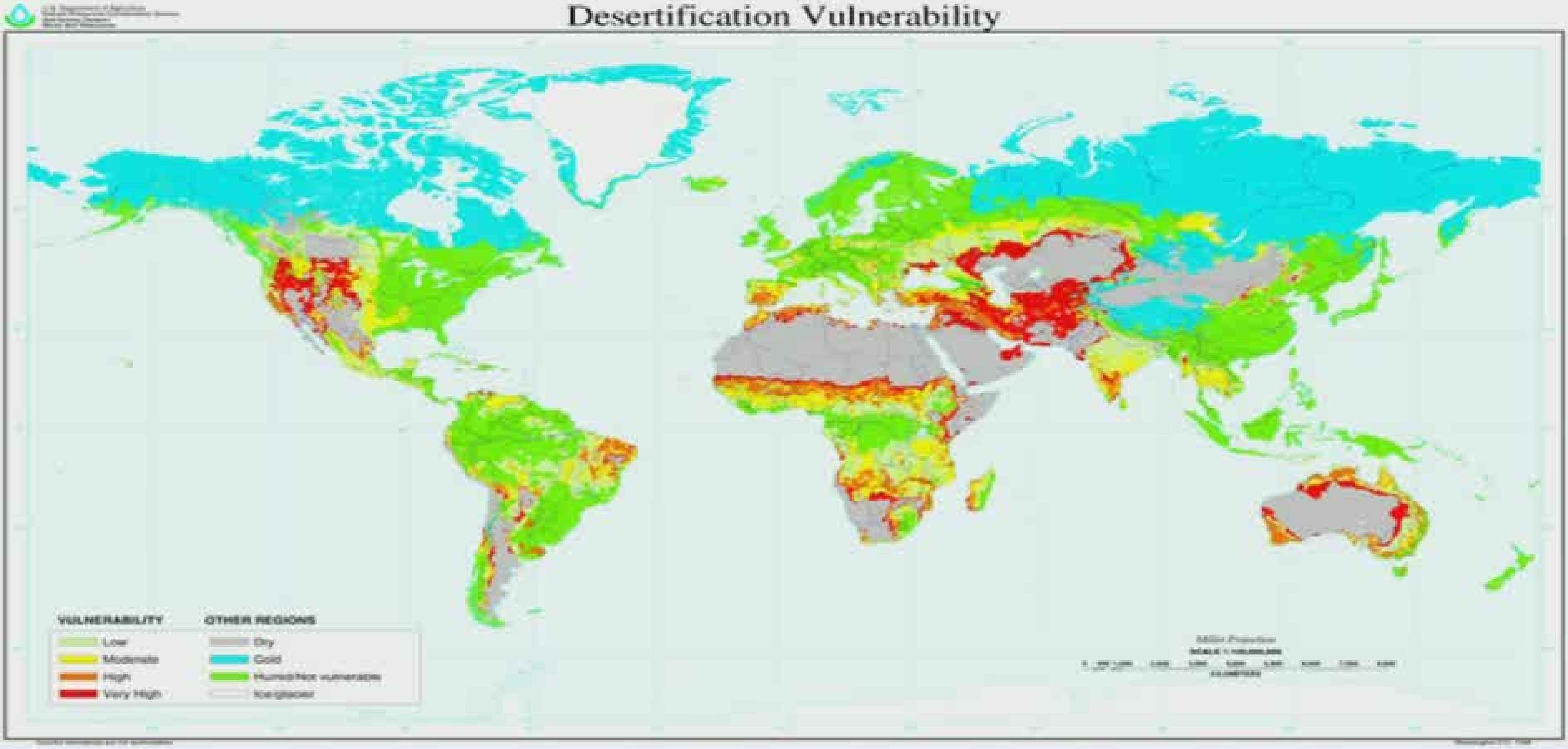
Finite-time blow-up induced by semi-strong pulse interactions.



Biology! Methods developed in context of (the GS and) the Gierer-Meinhardt (GM) model for morphogenesis.

[Ward, Kolokolnikov, Nishiura, …, D., Kaper & students]

SPATIAL ECOLOGY: vegetation patterns & desertification



"Drylands occupy approximately 40–41% of Earth's land area and are home to more than 2 billion people. It has been estimated that 10–20% of drylands are already degraded, and that a billion people are under threat from further desertification."

↔ Mary Silber *Pattern Formation in the Drylands* on Monday

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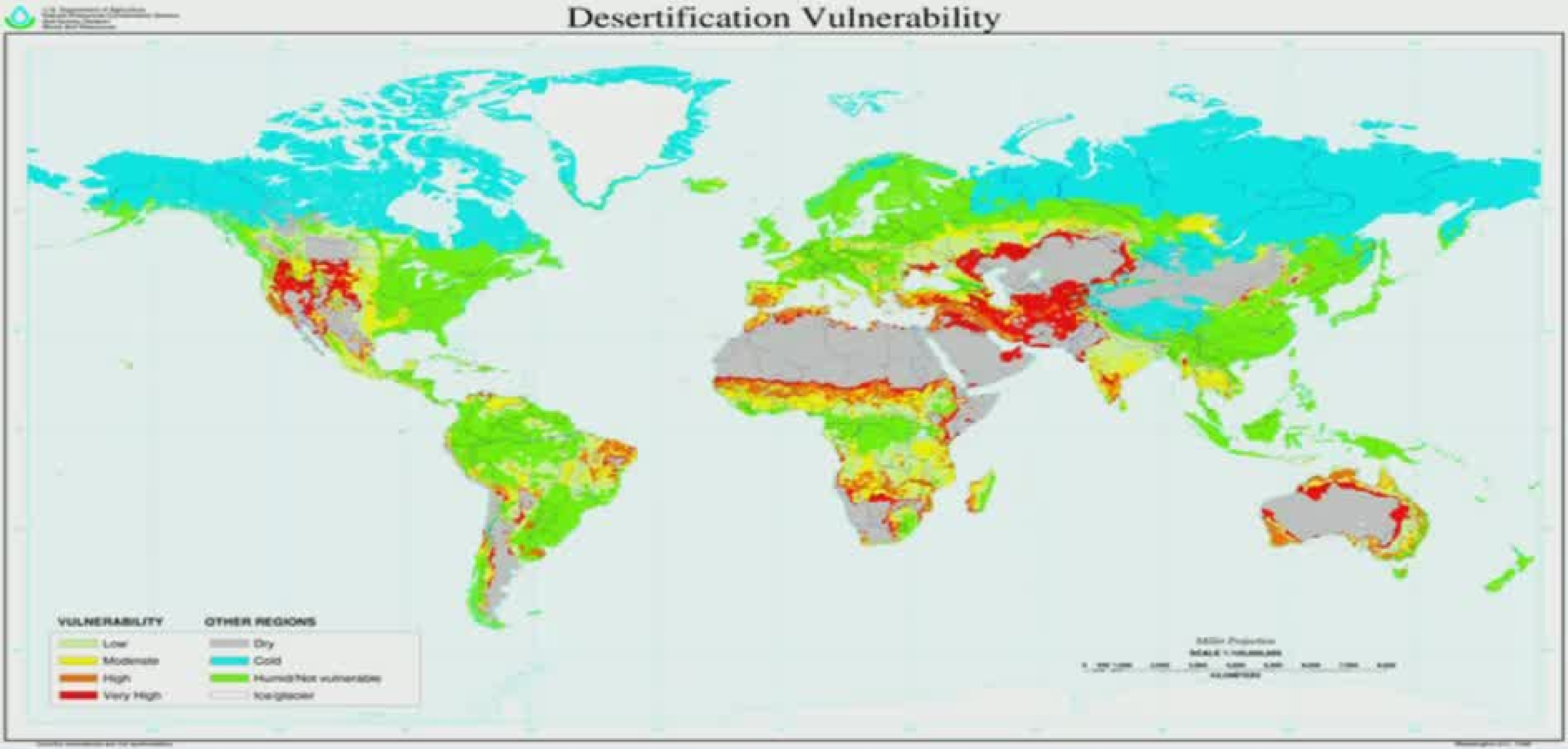

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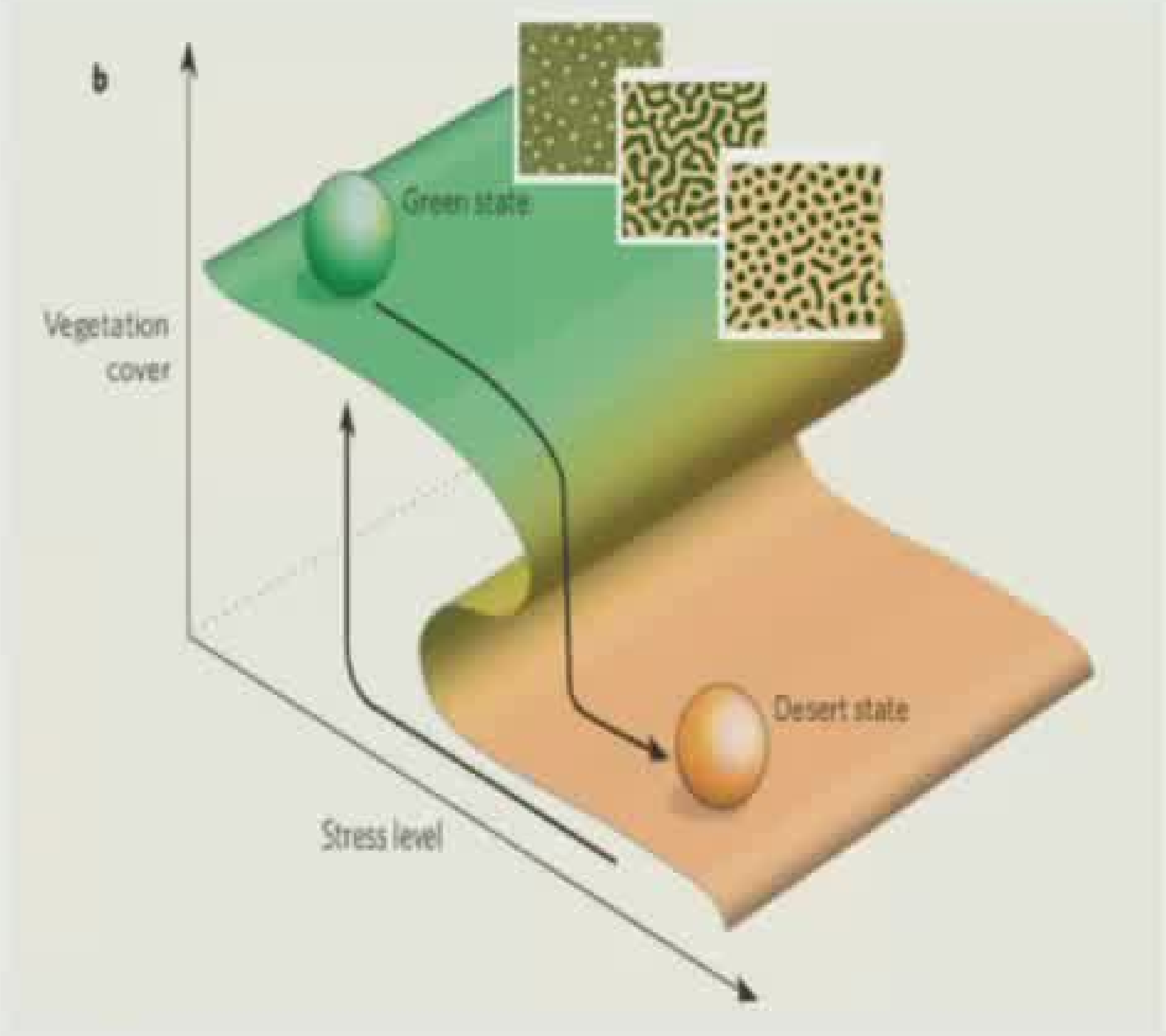
Desertification = The process by which a vegetated state transforms – or collapses – into a bare soil state.

- Caused by a slow change in the 'environment'.
- Patterns appear as 'early' warning signal.
- Partly discontinuous/fast – catastrophic – partly gradual.
- Irriversible – Hysteresis

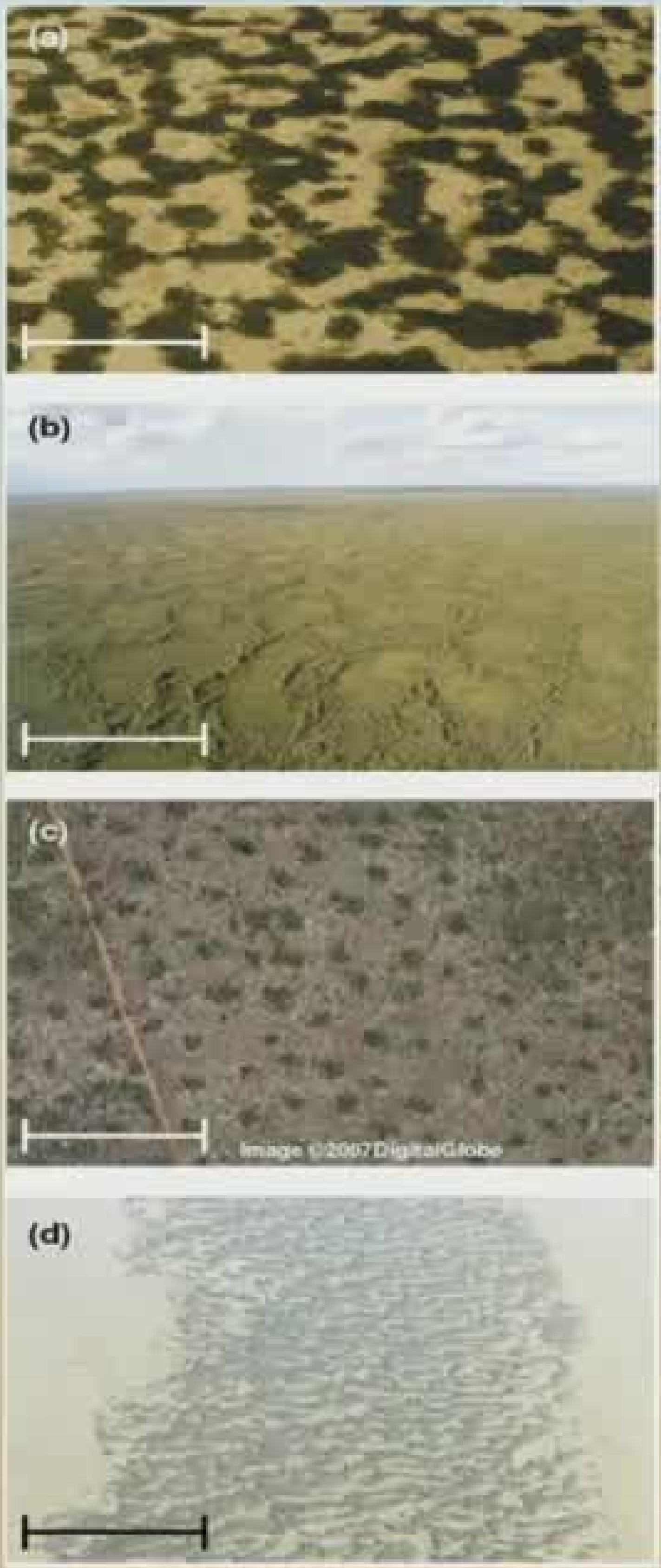


Questions from ecology:

- When is the process catastrophic and when is it gradual?
- Can we predict the 'collapse'?
- Can we measure 'how far' the system is from the desert state?

The mathematical perspective:
The dynamics of singular patterns under slowly varying circumstances (→ parameters).

Intermezzo: spatial ecology & singular patterns



[Rietkerk & van de Koppel, '08]: Pattern formation in ecological systems is driven by counteracting feedback mechanisms on widely different spatial scales.

Mathematics: The dynamics of singular patterns.

The generalized Klausmeier-Gray-Scott model

$$\begin{cases} U_t = D_u \Delta U + C U_x - UV^2 + A(1-U) \\ V_t = D_v \Delta V + UV^2 - BV \end{cases}$$

with

$C = 0$: Gray–Scott, $U, V \sim$ concentrations
$D_u = 0$: Klausmeier, $U, V \sim$ water and biomass

Both models are highly simplified/conceptual:

- (K) $A \sim$ rainfall, constant in time and space??
- (K) $UV^2 \sim f(U,V) \times V$ uptake of water.
- (GS) $UV^2 \sim$ 'reduction' of several reaction steps.

Singularly perturbed?!

- (GS) $0 < D_v = \varepsilon^2 \ll 1$??
- (K) $D_u \neq 0 \sim$ spread of water on flat terrains.
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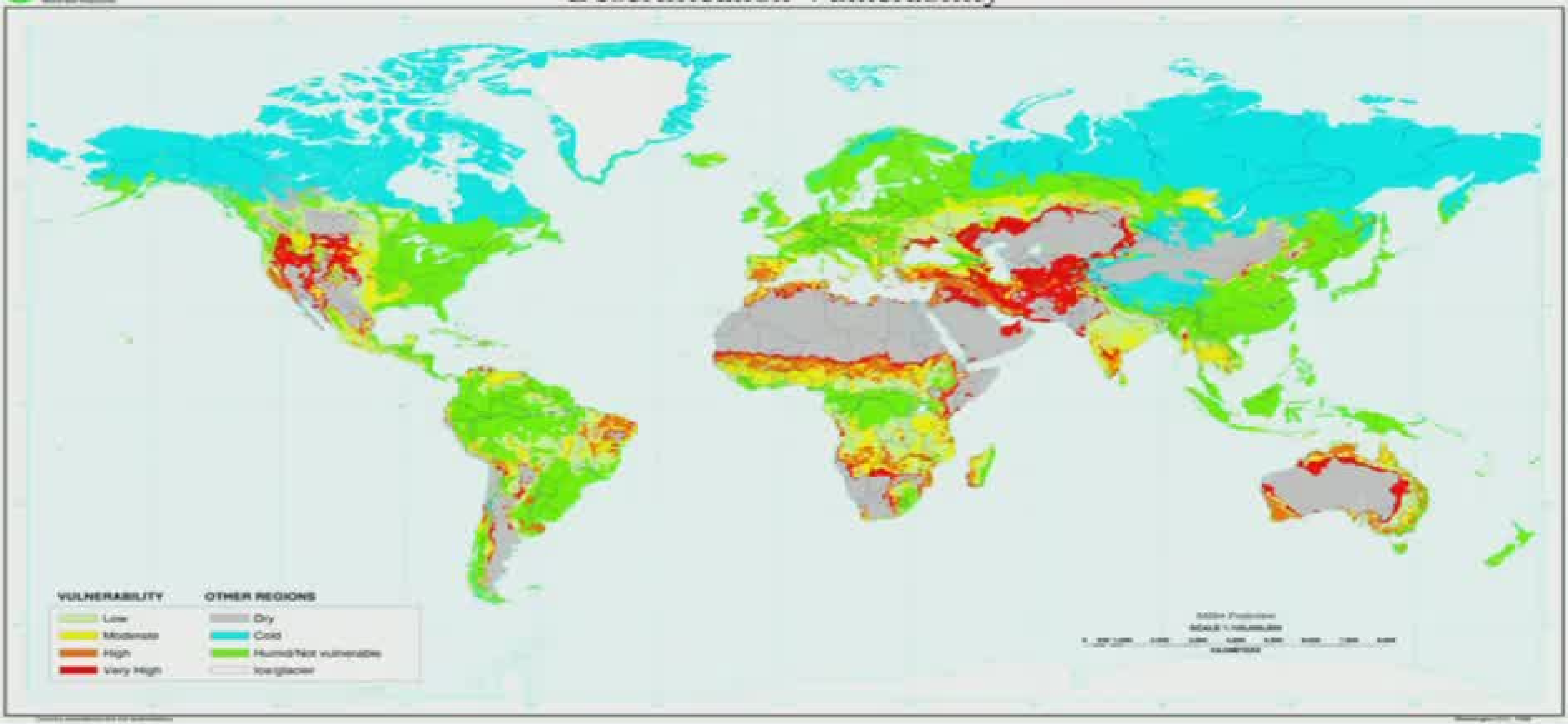
SPATIAL ECOLOGY: vegetation patterns & desertification

Desertification Vulnerability



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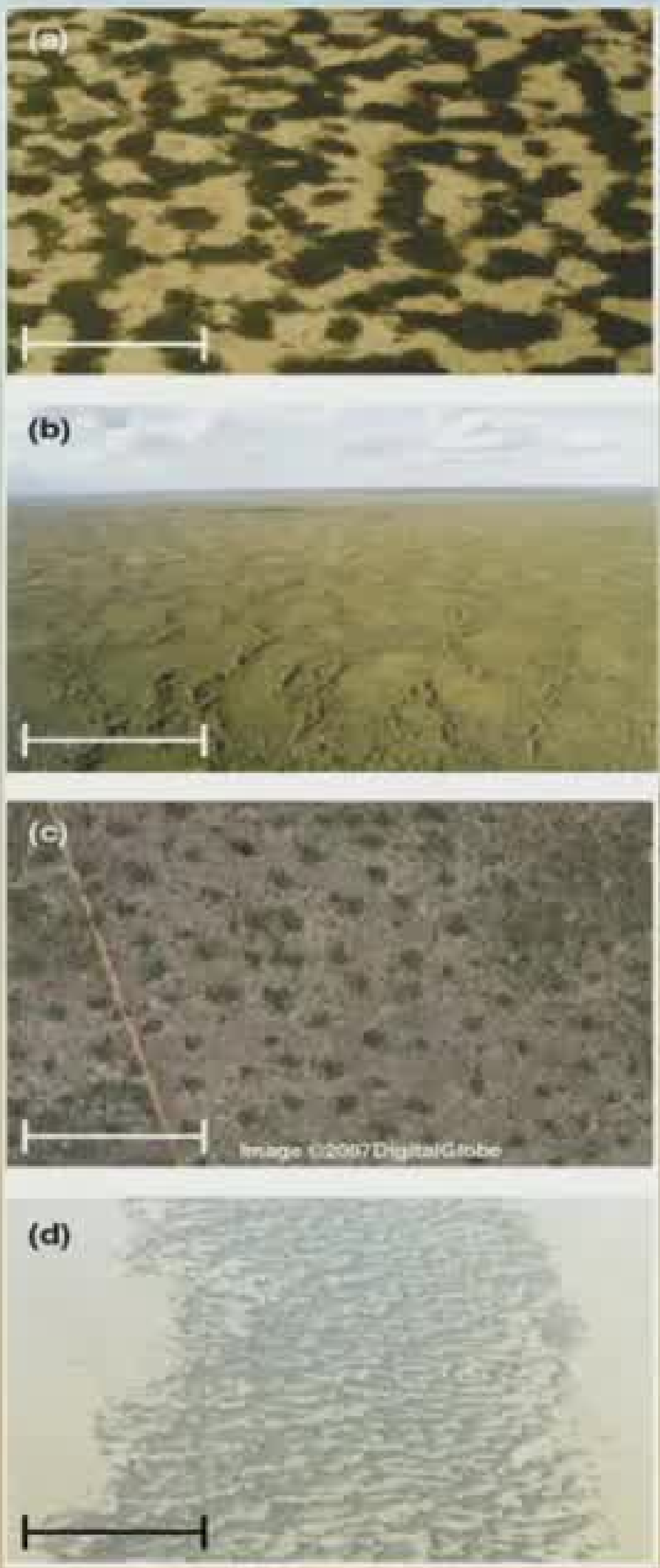

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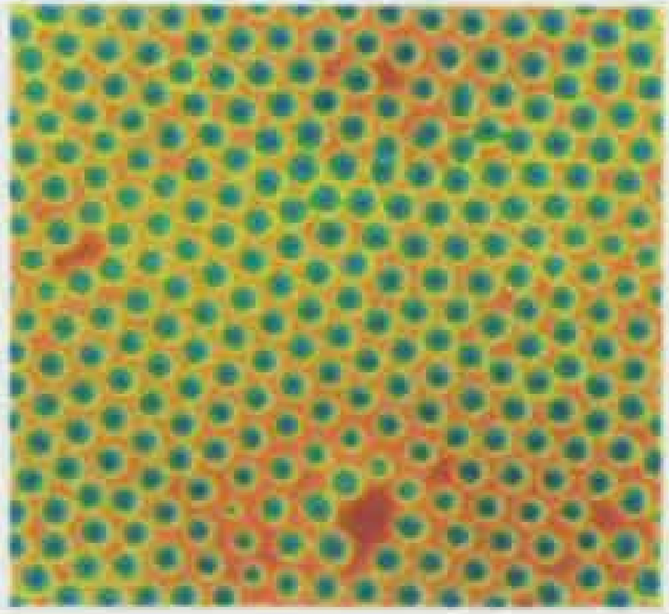
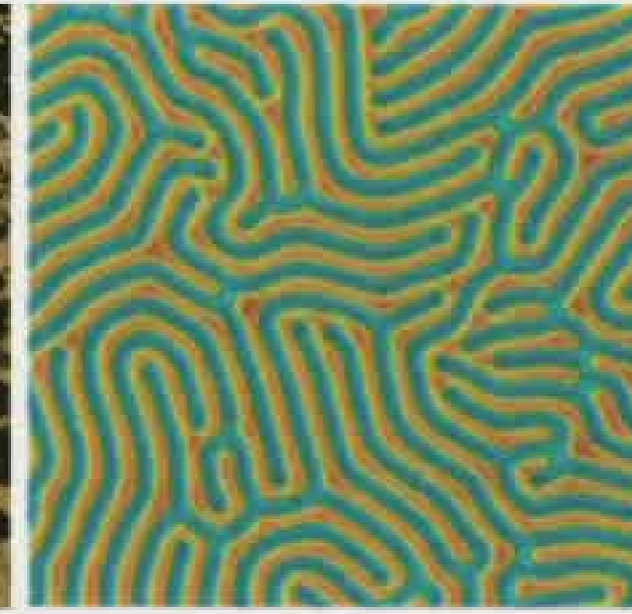
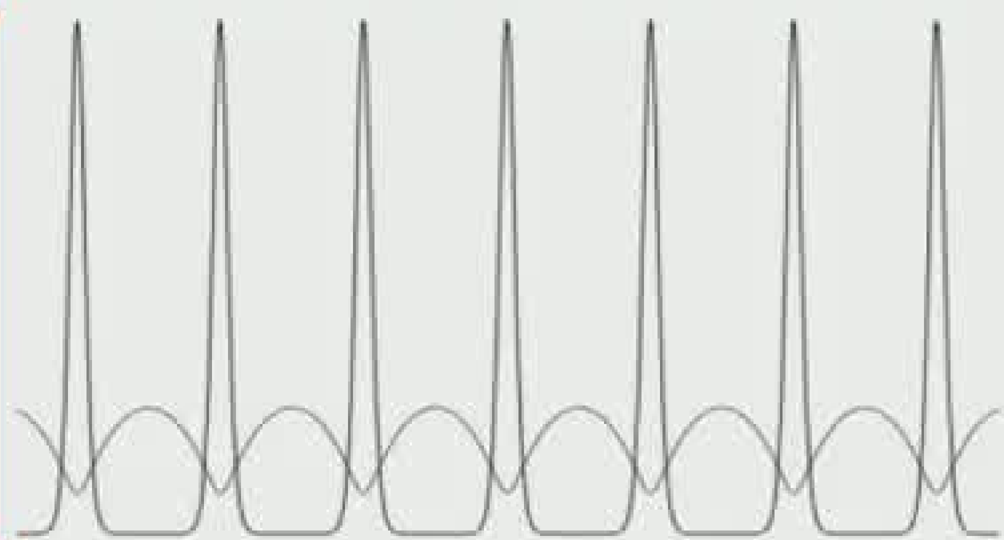
There is a remarkable – expected!? – similarity between observed vegetation patterns and GS simulations.



- Can we use our mathematical insights in systems of SP RDEs to obtain insight in the process of desertification?
- **Cross-fertilization**: novel mathematical theory motivated by questions from and observations in ecology.

- Consider – for simplicity – patterns in 1 space dimension.
- Fundamental question: What kind of patterns can be exhibited by the model?



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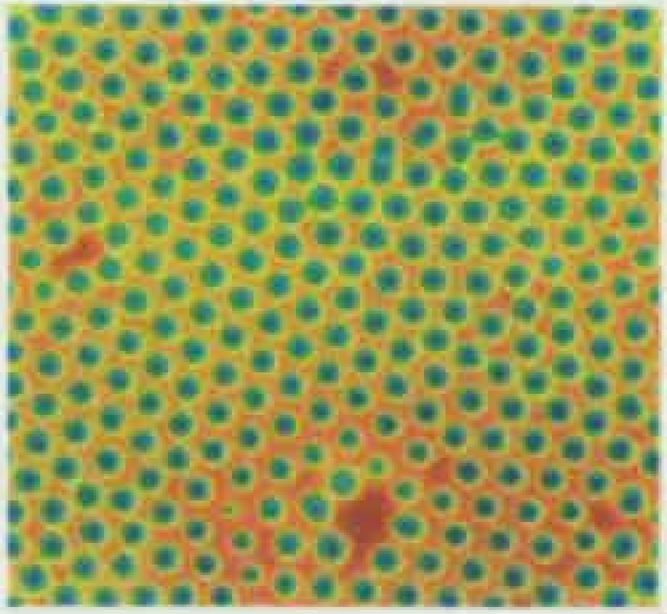

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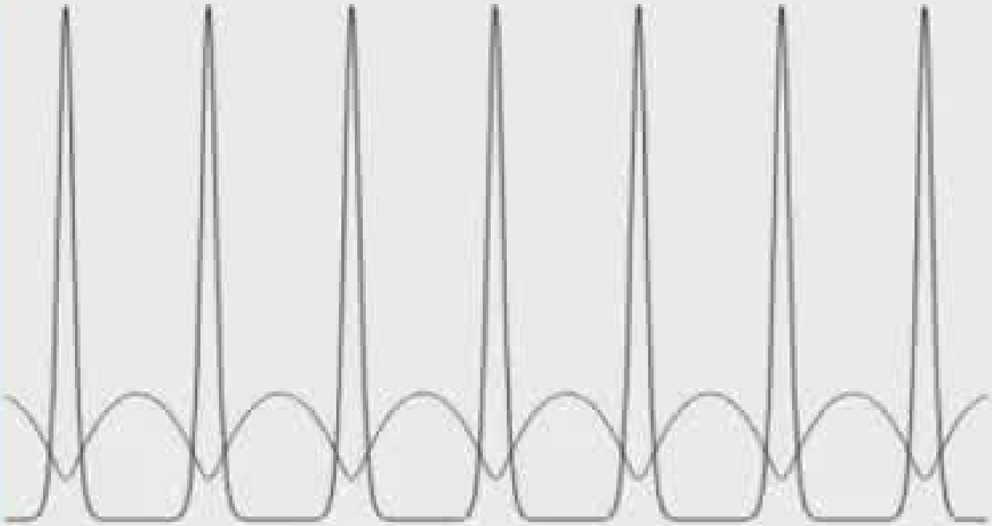
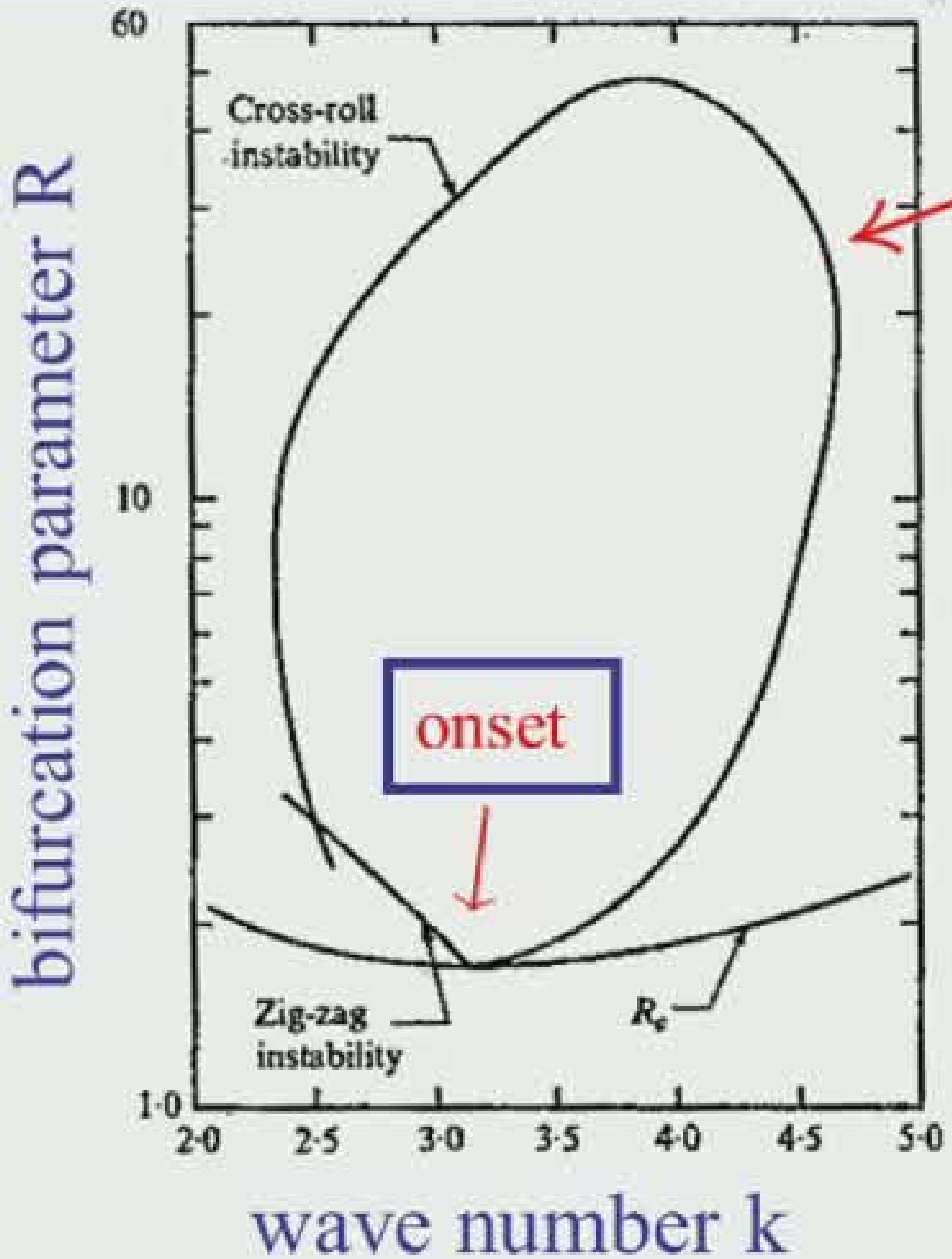


First sub-question: For which parameter values can spatially periodic patterns be observed?

This question was first answered (numerically) by Fritz Busse in 1978 for roll patterns in convection.



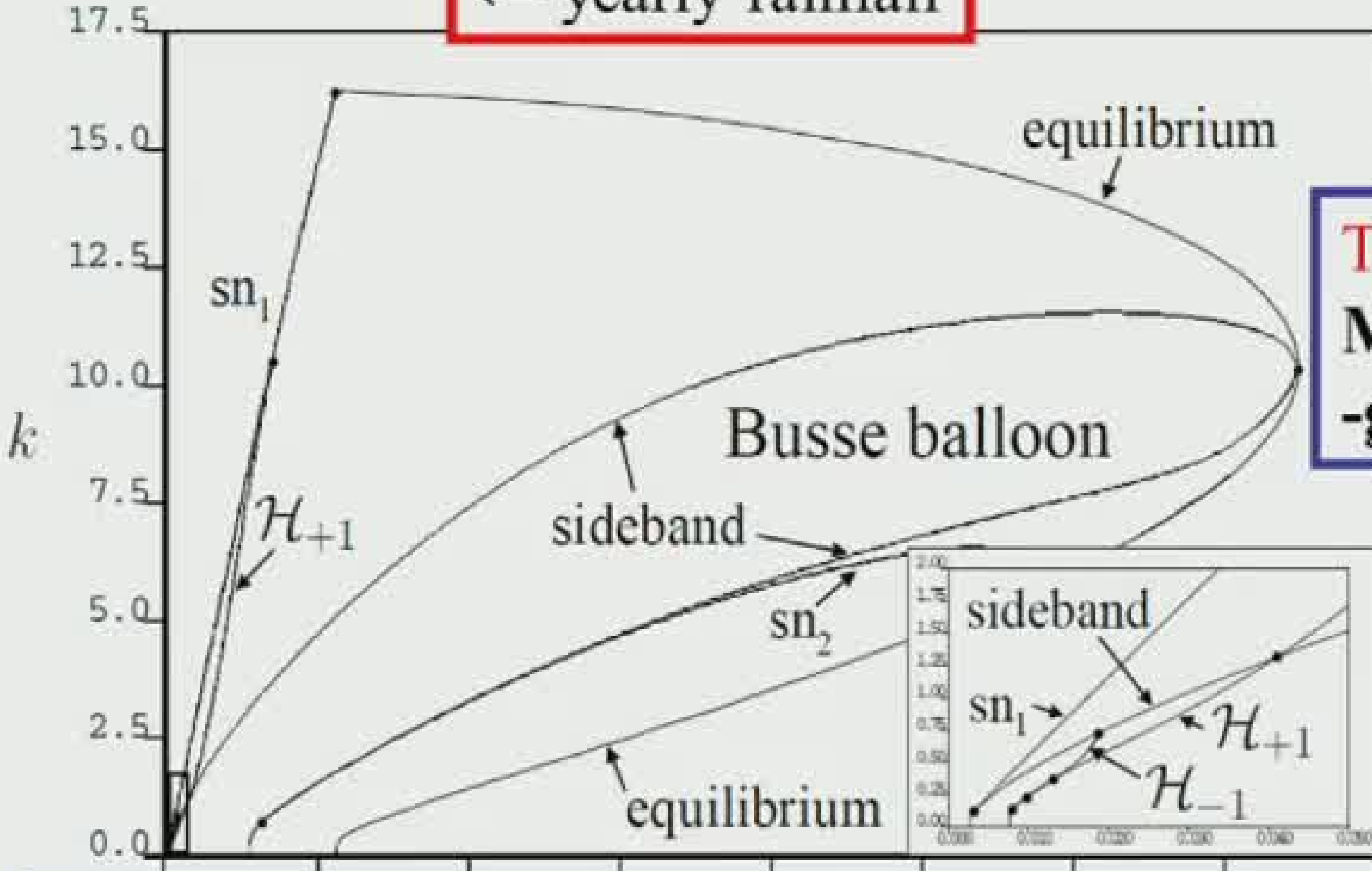
The Busse balloon

Definition: a Busse balloon = a region in (wavenumber, parameter)-space in which stable spatially periodic patterns exist

In convection, the Busse balloon is the first step from fluid-at-rest to (eventually) turbulence.

A Busse balloon for the Gray-Scott/gKGS model

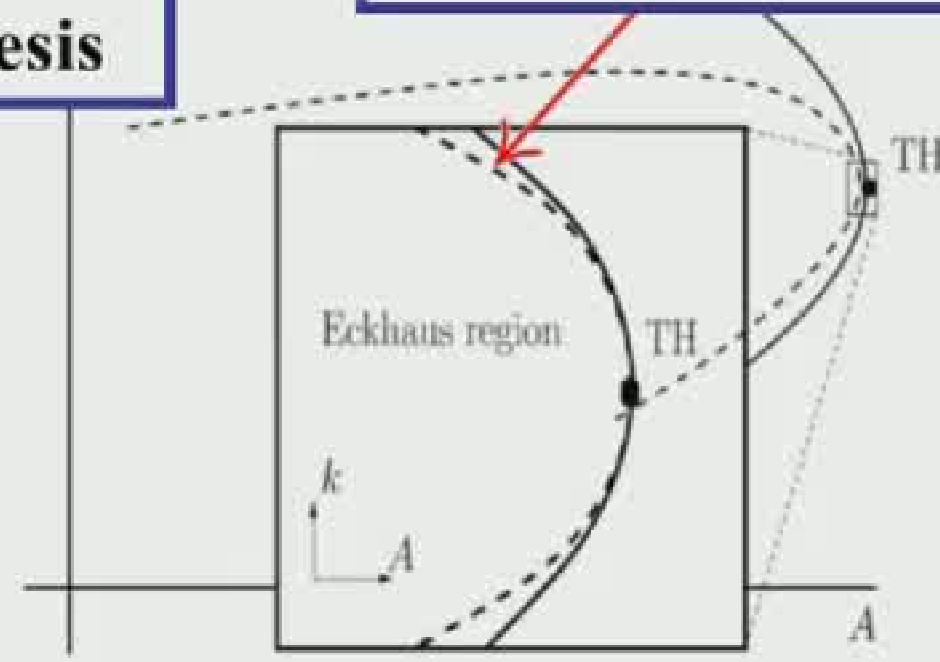
← yearly rainfall



MATH at the 'nose':

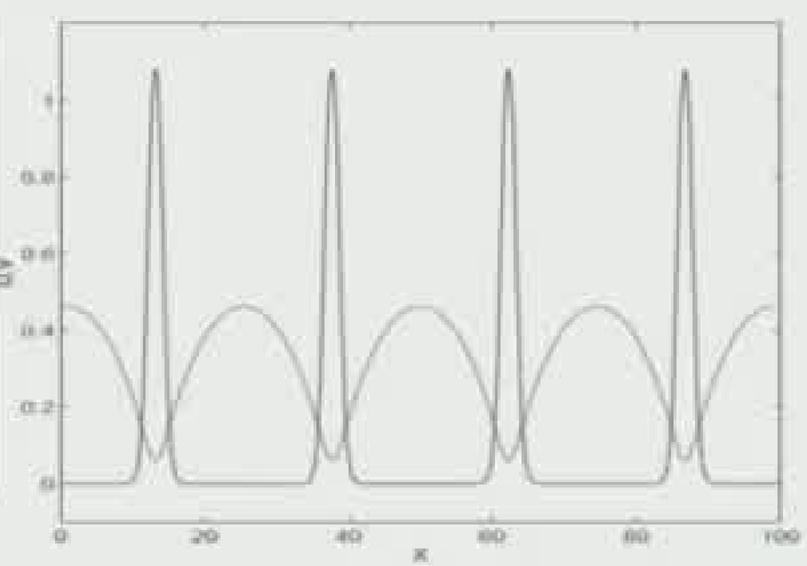
TURING Morpho -genesis

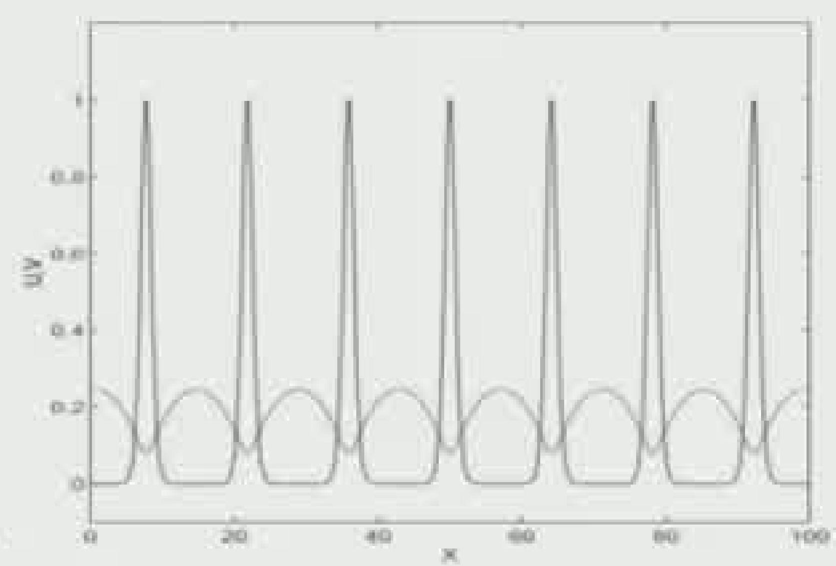
Eckhaus parabola



Fall of patterns at k=0/Morphothanatos

MATH at the 'homoclinic tip': SP localized pulses in semi-strong interaction.





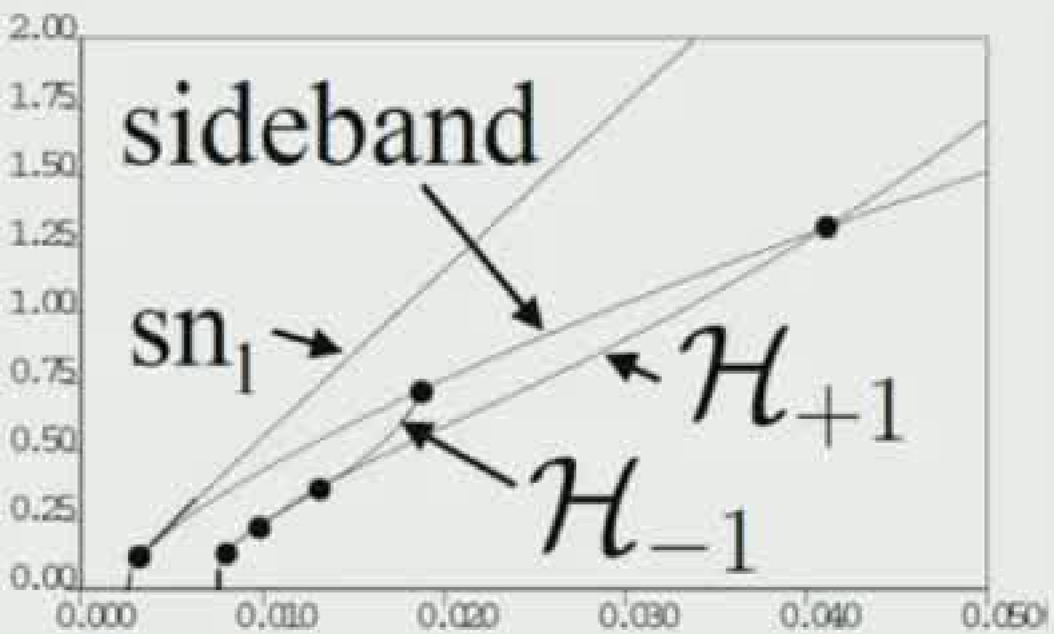
(Non-ecological) Observations/Questions

- Unlike in fluid dynamics, there are no stable patterns of increasing complexity bifurcating from (the boundary of) the Busse balloon?

  [From morphogenesis to morphothanatos & nothing else??]

- There is a fine-structure of ‘intertwining’ boundary curves near the homoclinic tip of the Busse balloon?

[The same behavior is exhibited by the Gierer-Meinhardt model!]



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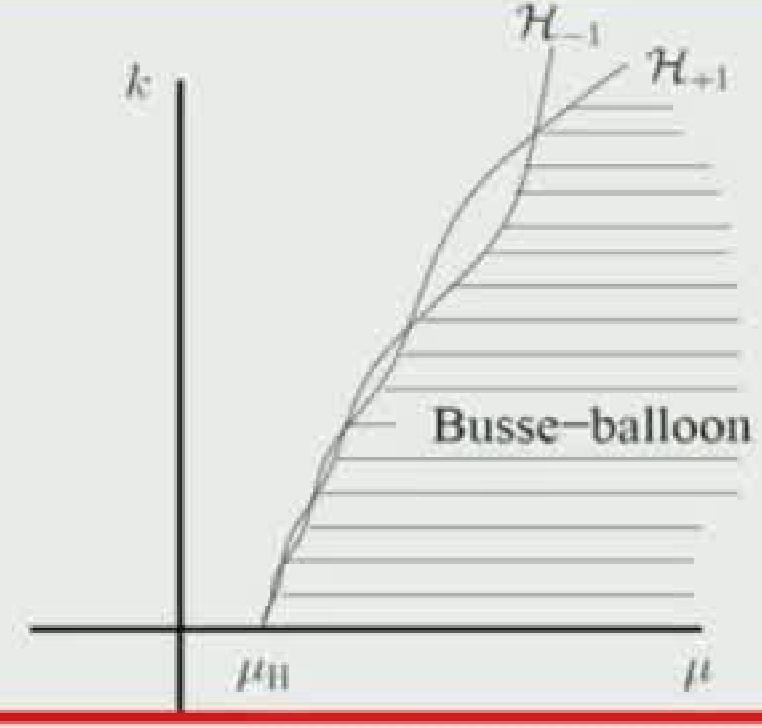

Is this perhaps due to the special nature of these models??

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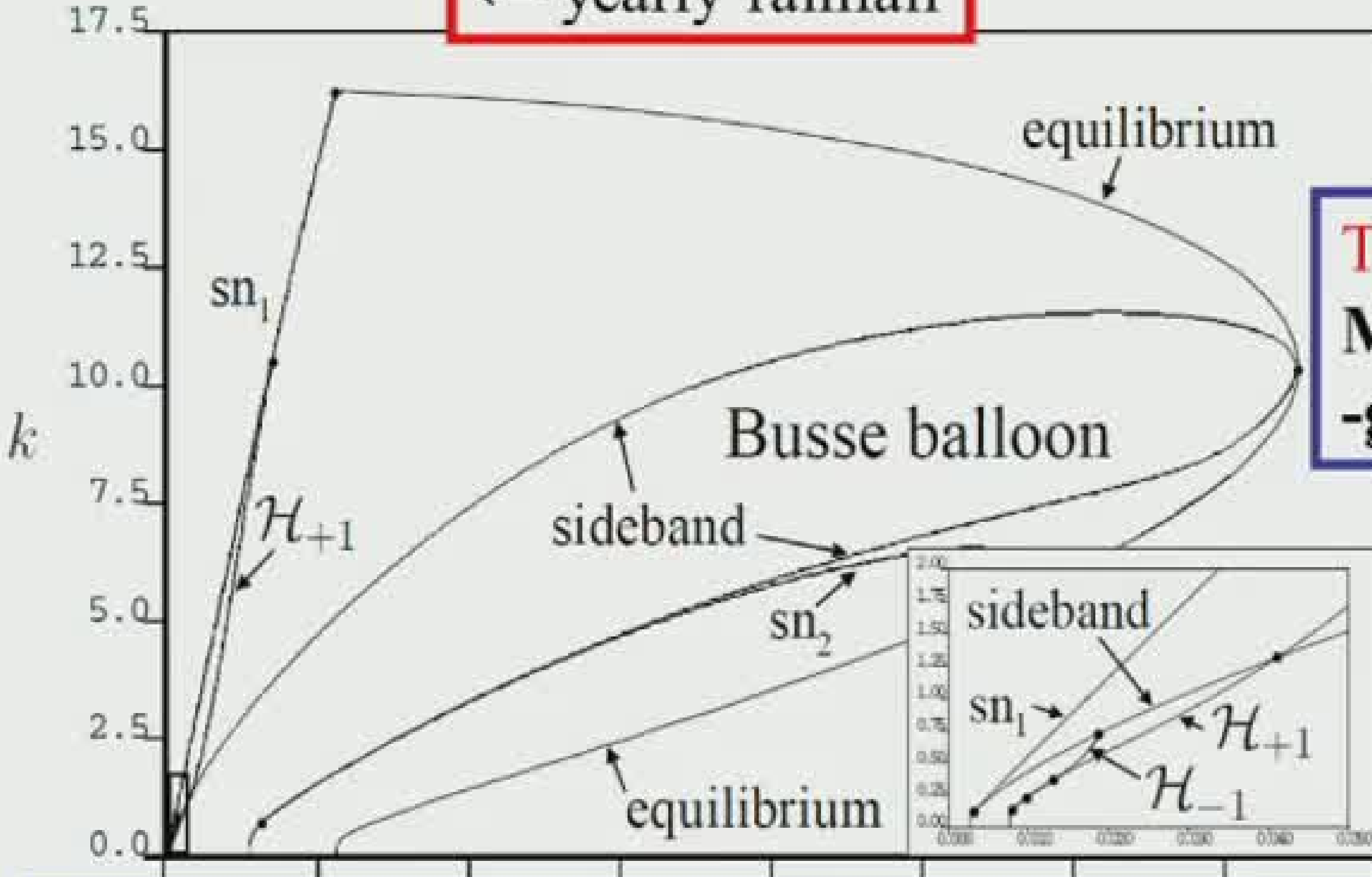

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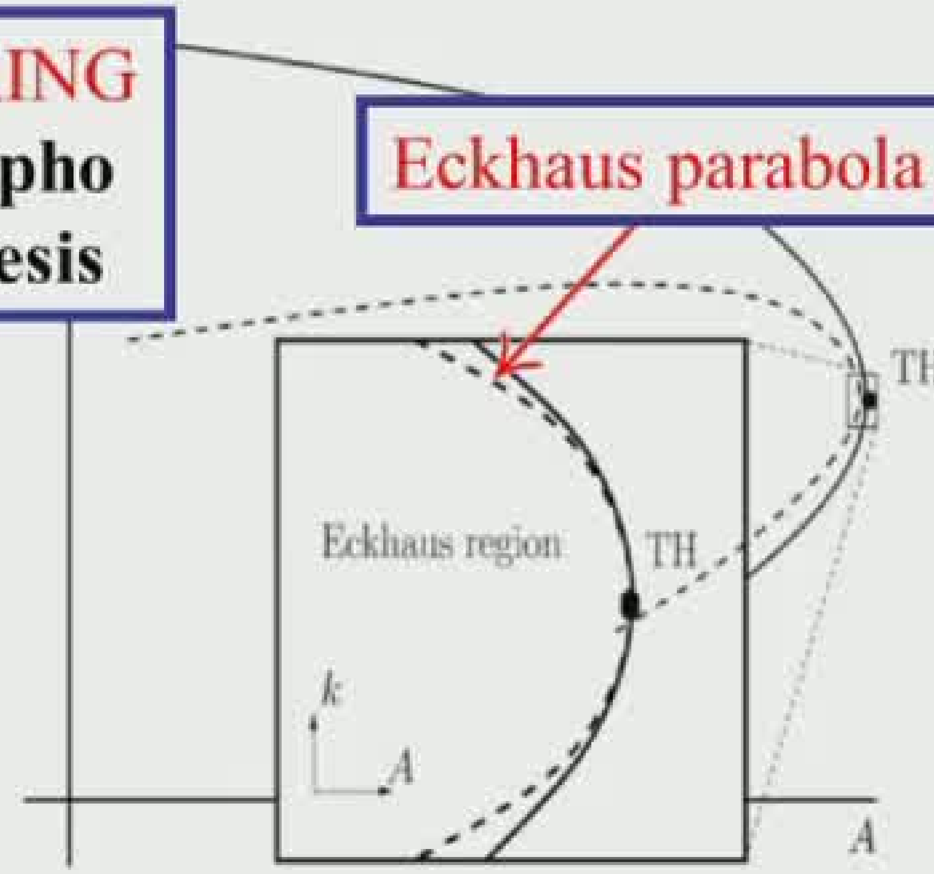

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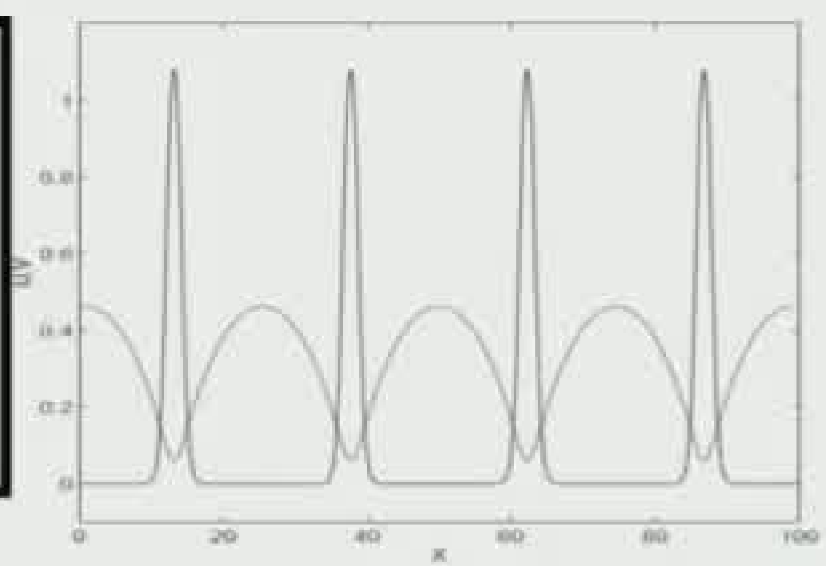

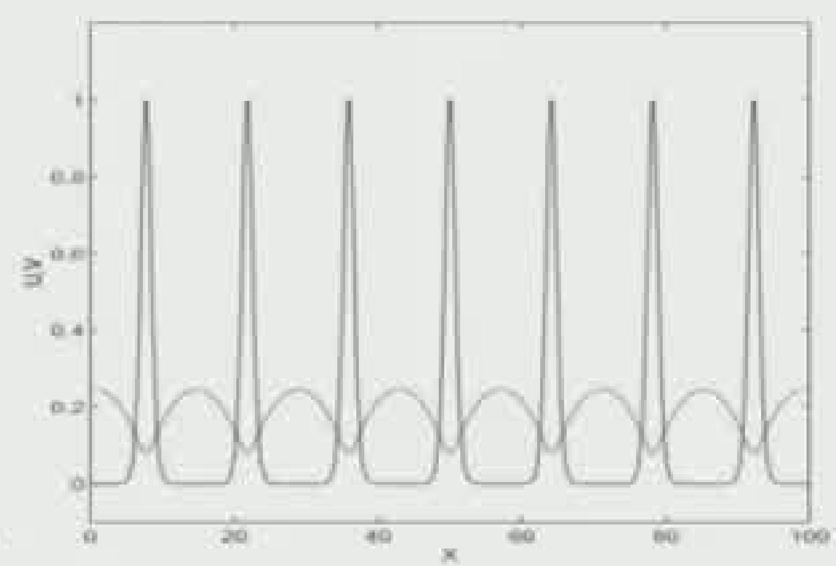

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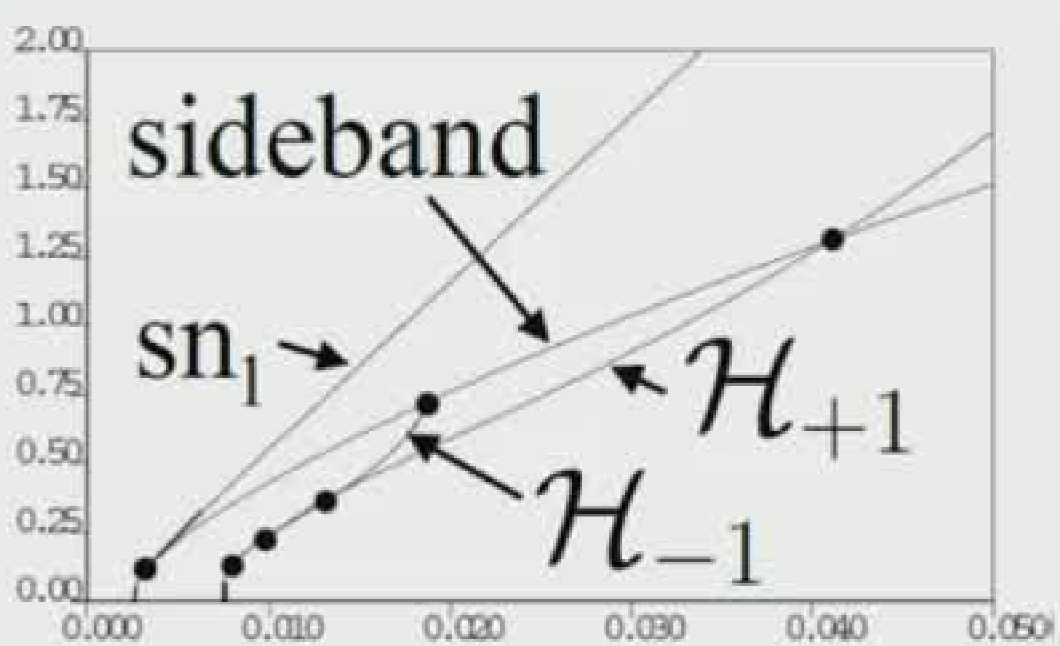

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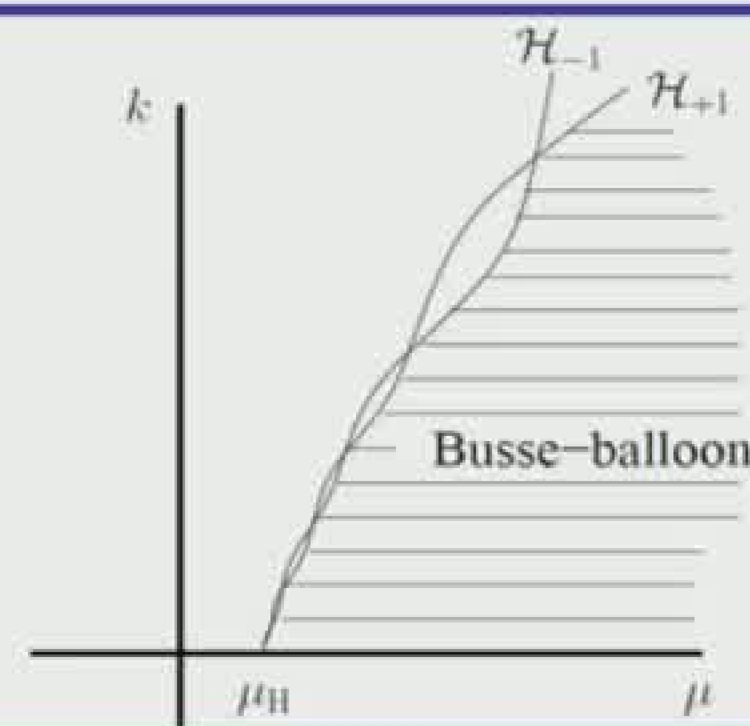

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Pulse dynamics in slowly nonlinear SP RDEs

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- [Veerman, '15] A general method to construct the center manifold reduction associated to the destabilization of a homoclinic pulse in a slowly nonlinear SP 2-component RDE: the Hopf bifurcation may be supercritical. (The Hopf bifurcation is subcritical in the GM RDE.)
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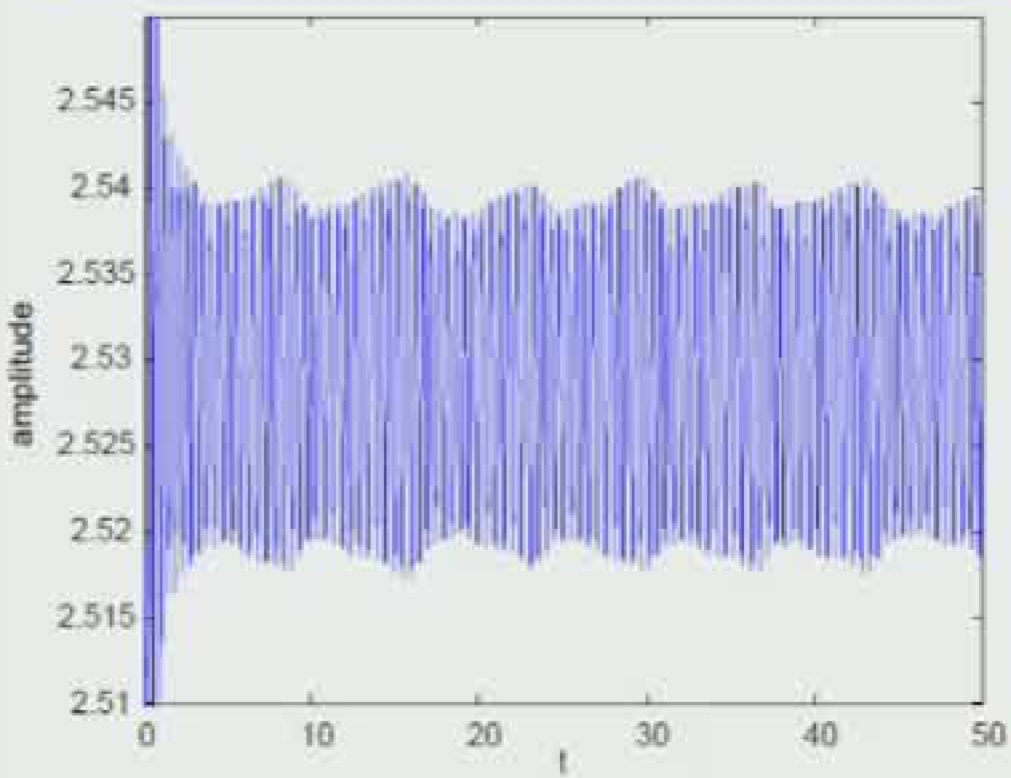
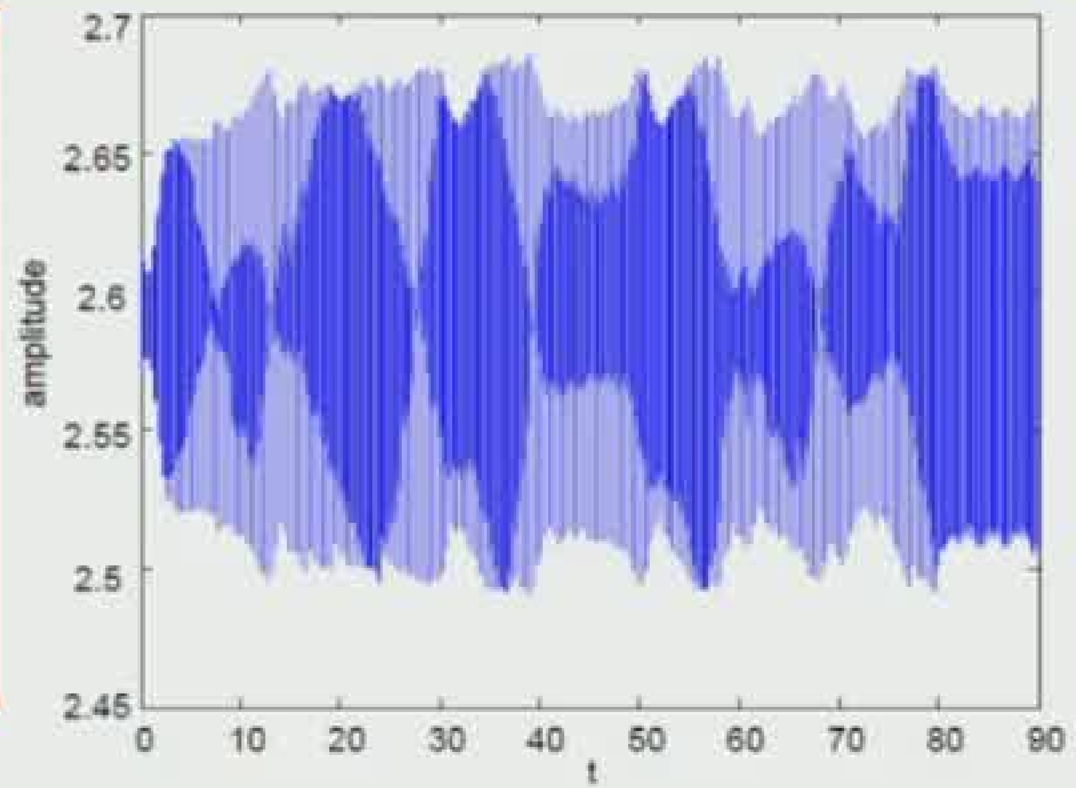


The dynamics of the tip of a solitary, standing, homoclinic pulse



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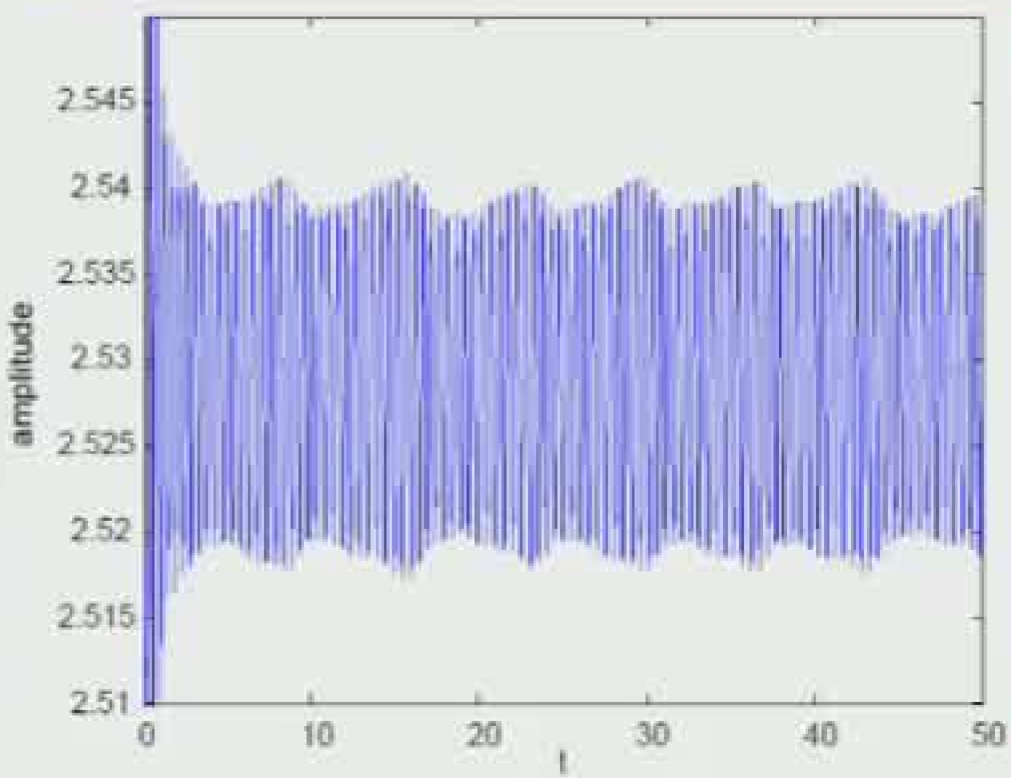

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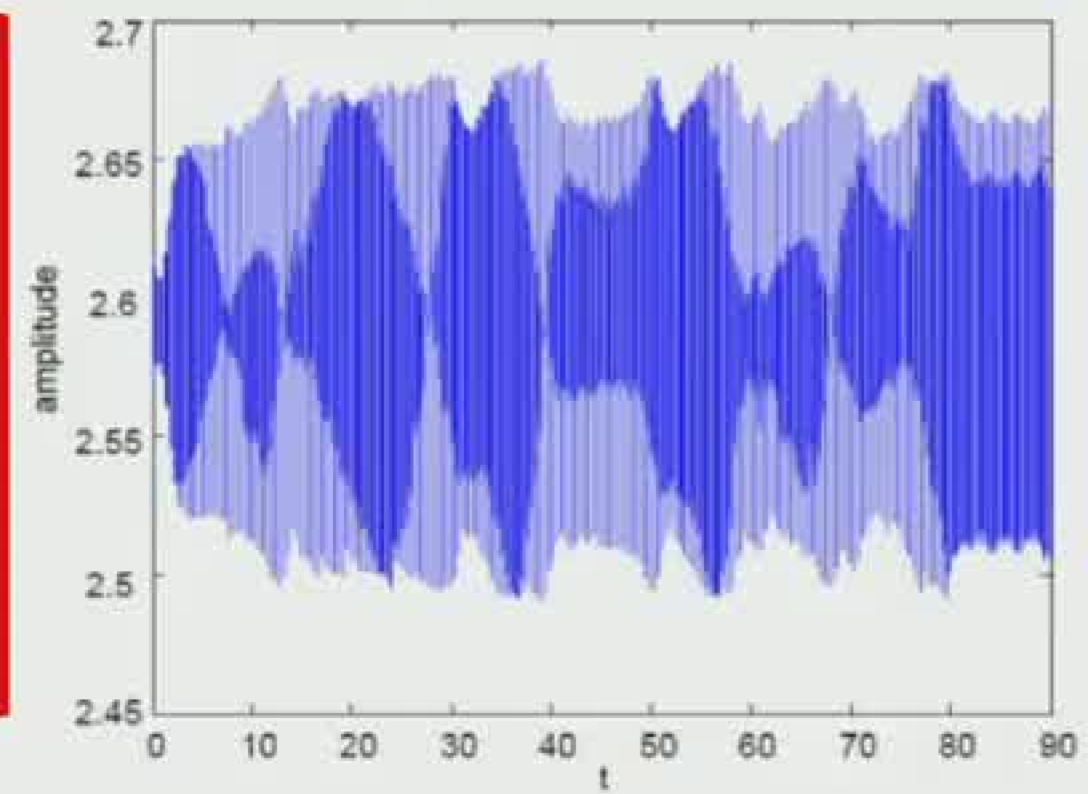
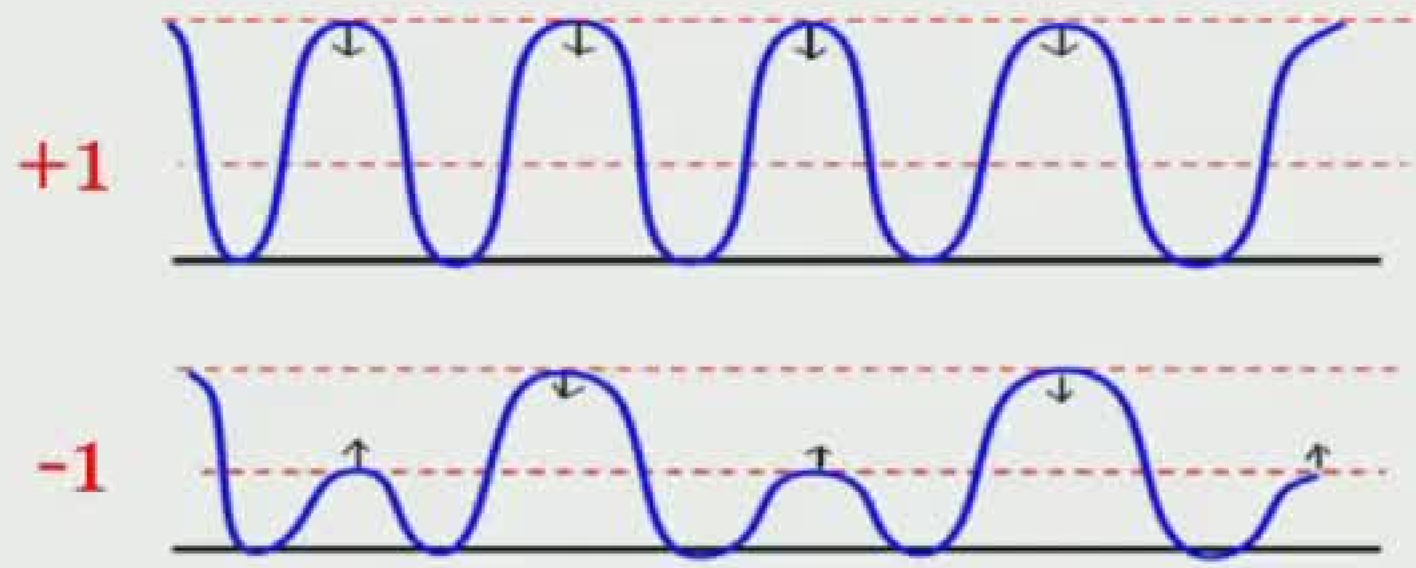
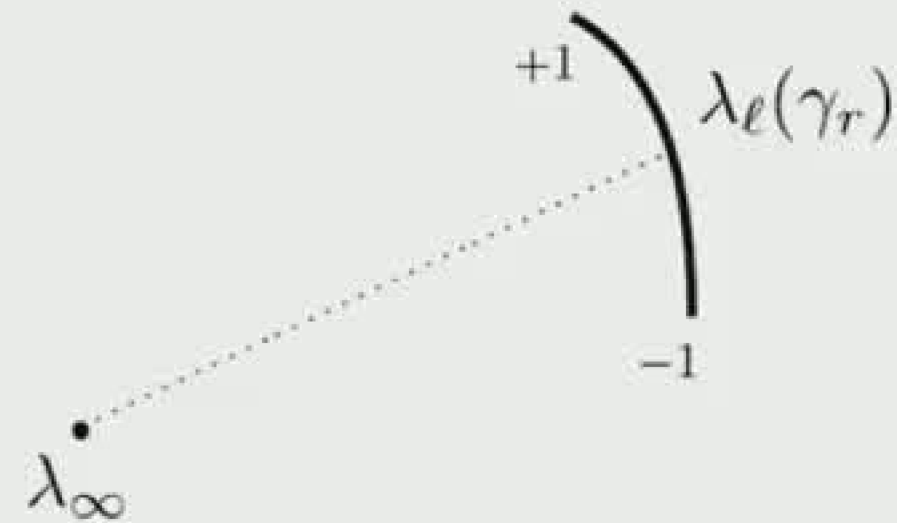


The spectrum associated to spatially periodic patterns

- The spectrum $\sigma(\Psi_\ell)$ associated to a spatially periodic pattern $\Psi_\ell(x)$ – with wavelength ℓ – consists of (up to countably many) 'closed loop images' $\lambda_\ell(\gamma)$ of S^1.
- The loops $\lambda_\ell(\gamma)$ degenerate to 'curved intervals' if $\lambda_\ell(\gamma)$ is stationary and (reversibly) symmetric, with $\gamma = \pm 1$ as endpoints.





- If $\Psi_\ell(x)$ approaches a homoclinic limit $\Psi_\infty(x)$ as $\ell \to \infty$, then $\sigma(\Psi_\ell)$ merges with $\sigma(\Psi_\infty)$, the spectrum associated to the 'localized structure' $\Psi_\infty(x)$.

[Gardner, Sandstede & Scheel]

Pulse dynamics in slowly nonlinear SP RDEs

- [Veerman & D, '13, '15] Existence & stability of homoclinic pulses.
- [Veerman, '15] A general method to construct the center manifold reduction associated to the destabilization of a homoclinic pulse in a slowly nonlinear SP 2-component RDE: the Hopf bifurcation may be supercritical. (The Hopf bifurcation is subcritical in the GM RDE.)
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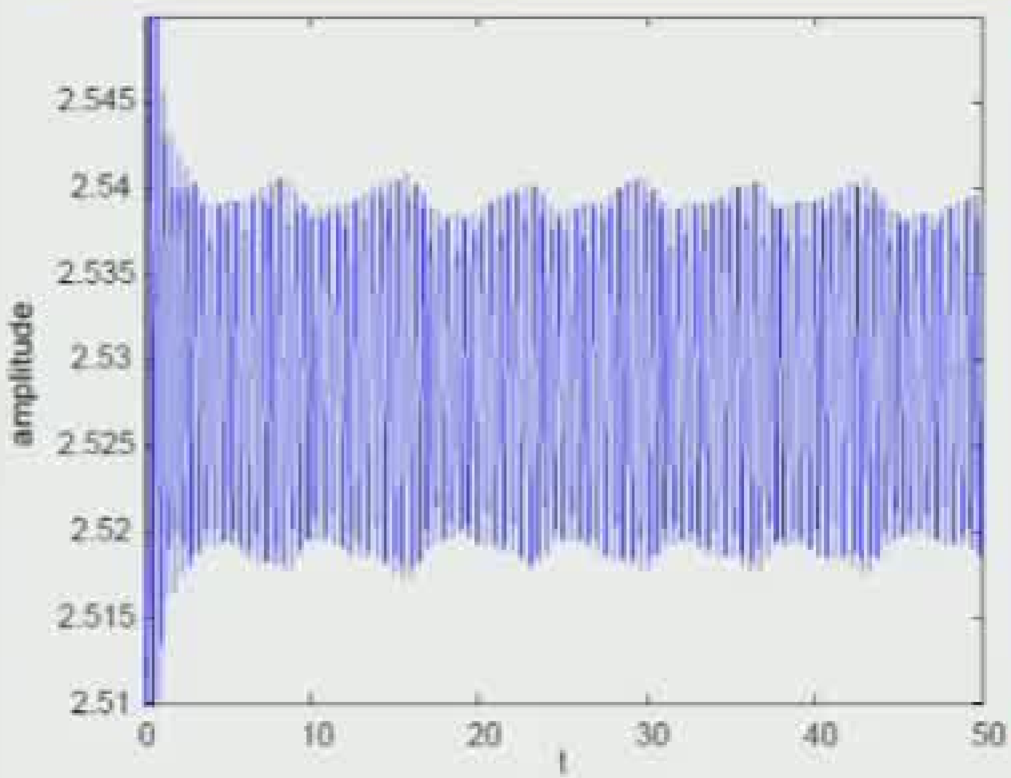
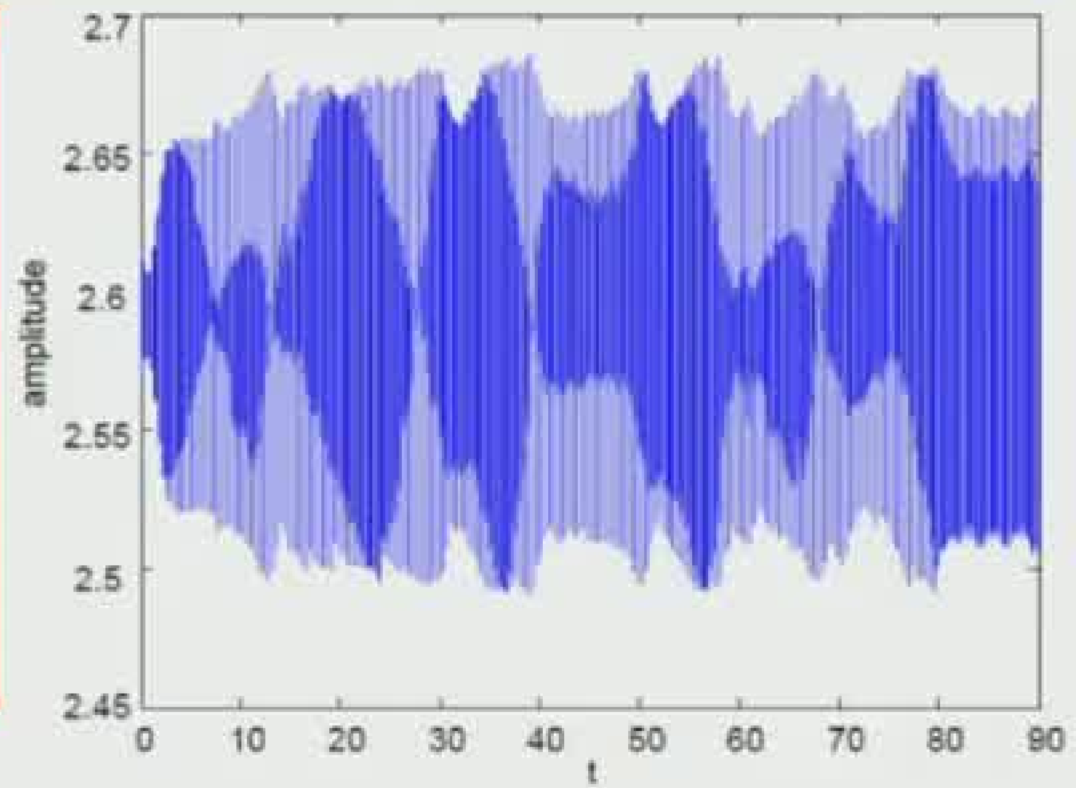


The dynamics of the tip of a solitary, standing, homoclinic pulse



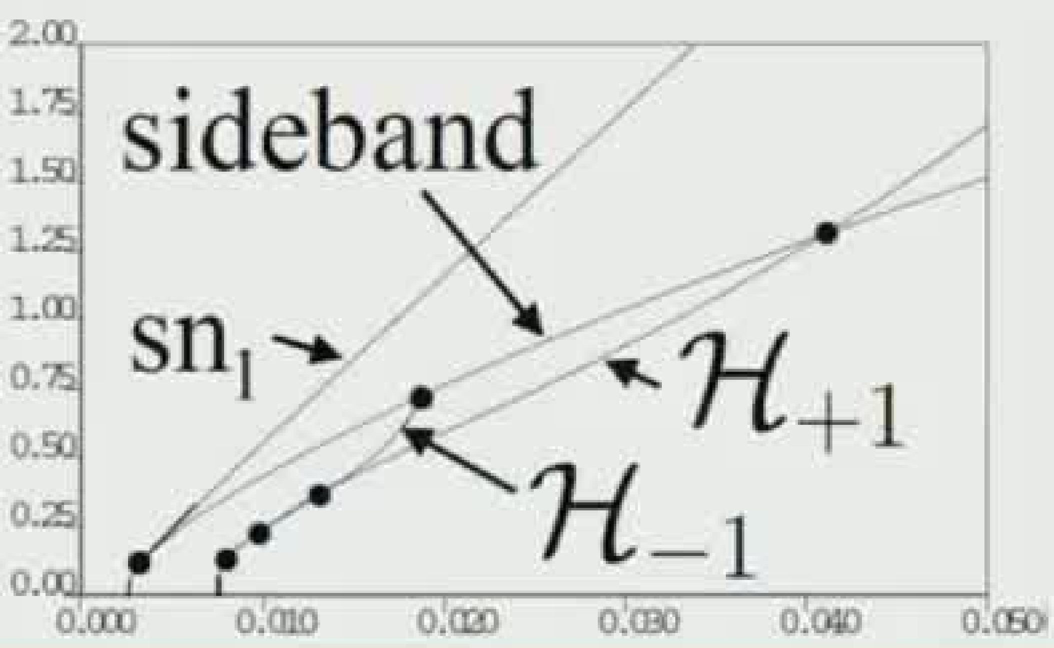
(Non-ecological) Observations/Questions

- Unlike in fluid dynamics, there are no stable patterns of increasing complexity bifurcating from (the boundary of) the Busse balloon?

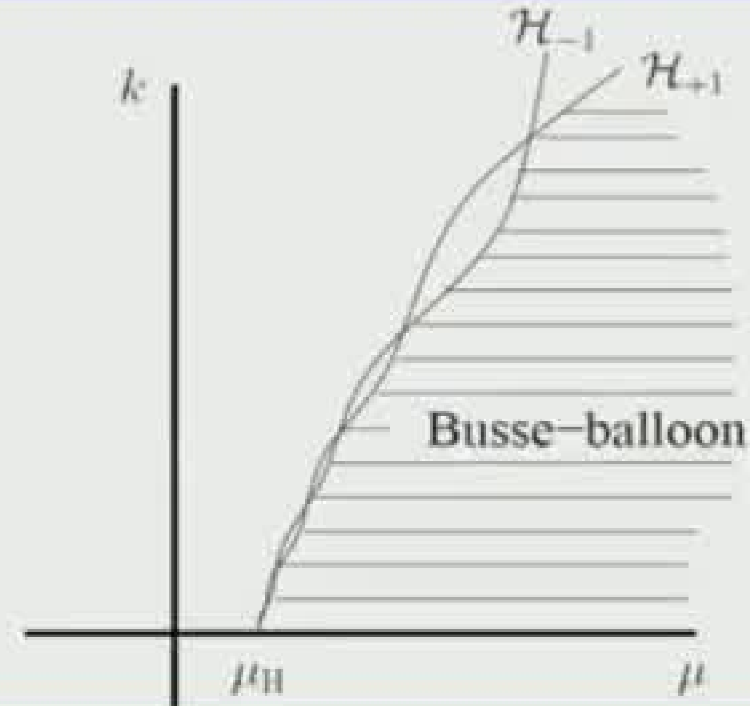
  [From morphogenesis to morphothanatos & nothing else??]

- There is a fine-structure of 'intertwining' boundary curves near the homoclinic tip of the Busse balloon?

[The same behavior is exhibited by the Gierer-Meinhardt model!]



↔



Is this perhaps due to the special nature of these models??

Slowly linear & slowly nonlinear SP RDEs

General SP RDE (for simplicity in 2 components),

$$\begin{cases} U_t &= U_{xx} + F(U,V;\varepsilon) \\ V_t &= \varepsilon^2 V_{xx} + G(U,V;\varepsilon) \end{cases}$$

Outside the 'fast' V-pulses (or fronts) – where, by translation, $V = 0$ (+ exp. small) – the dynamics are governed by,

$$U_t = U_{xx} + F(U,0;\varepsilon)$$

$F(U,0;\varepsilon)$ is linear in U for all (??) explicit models in the literature (GS, GM, gKGS, FH-N, Schnakenberg).

The 'prototypical' GS-, GM-, etc. models belong to the special class of 'slowly linear' SP RDEs,

a significantly restricted (?) subclass of the general 'slowly nonlinear' SP RDEs.

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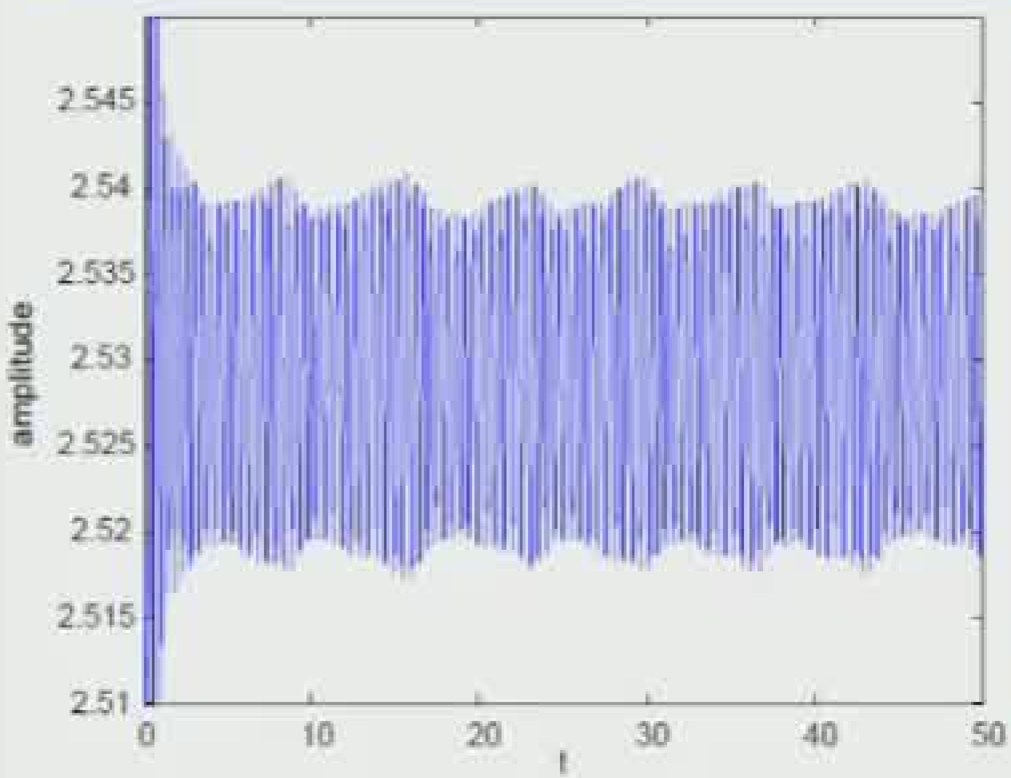

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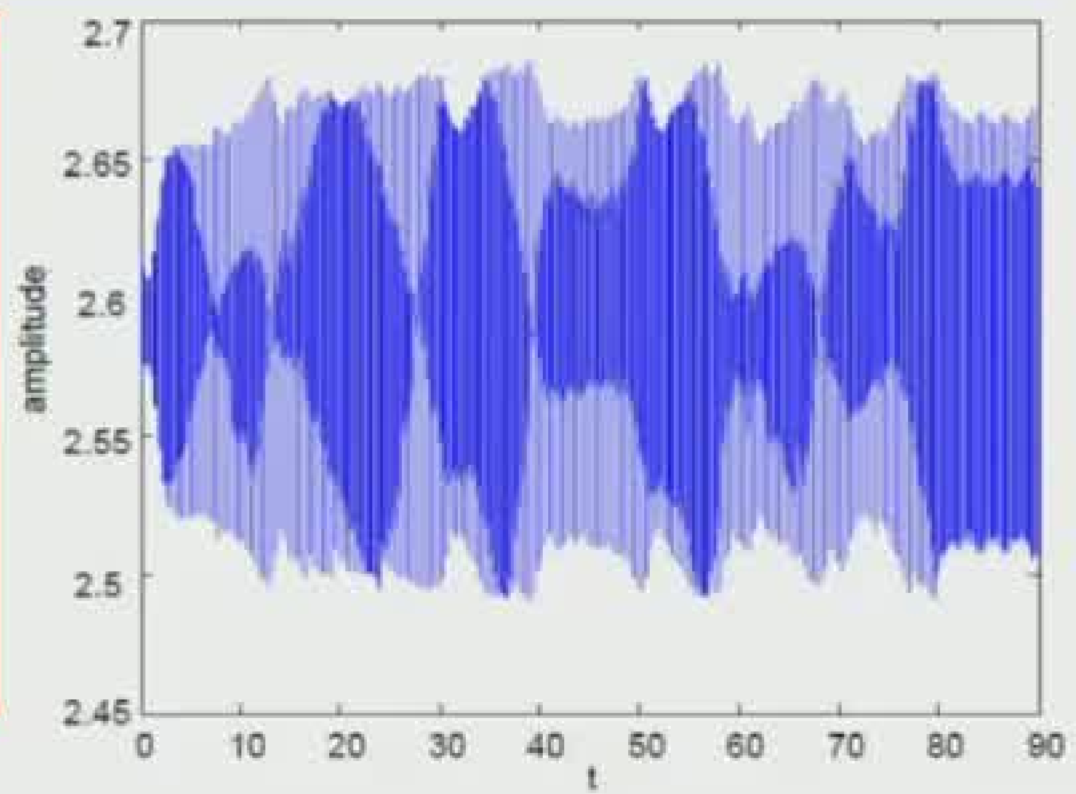
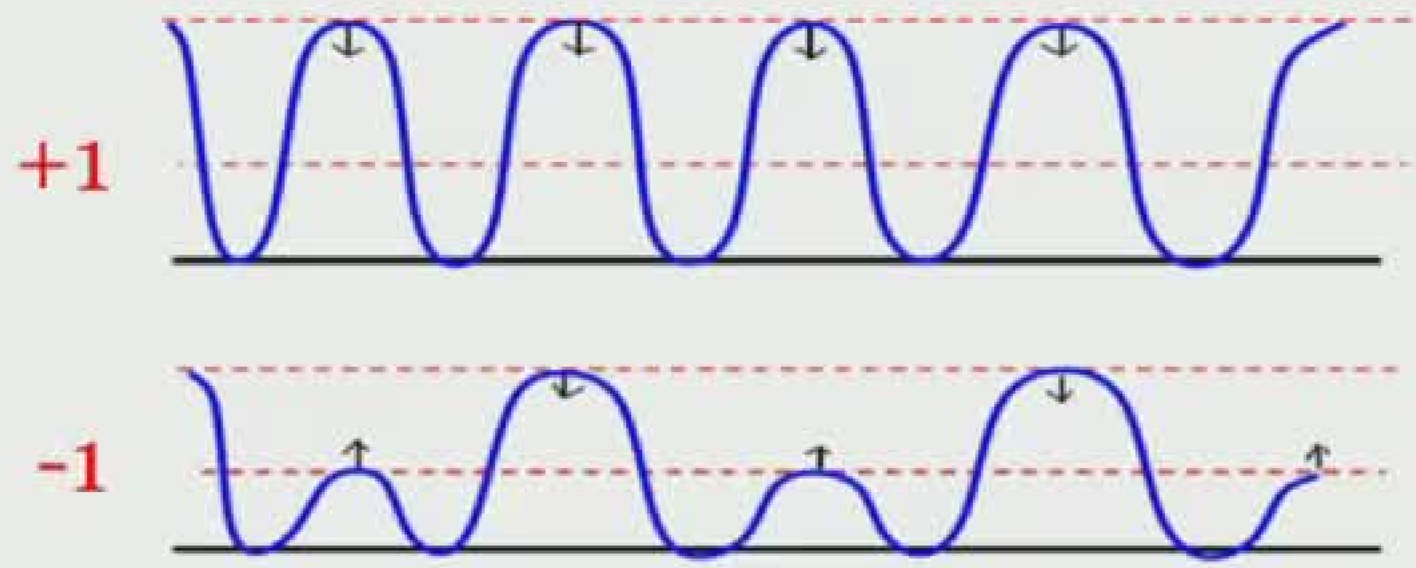
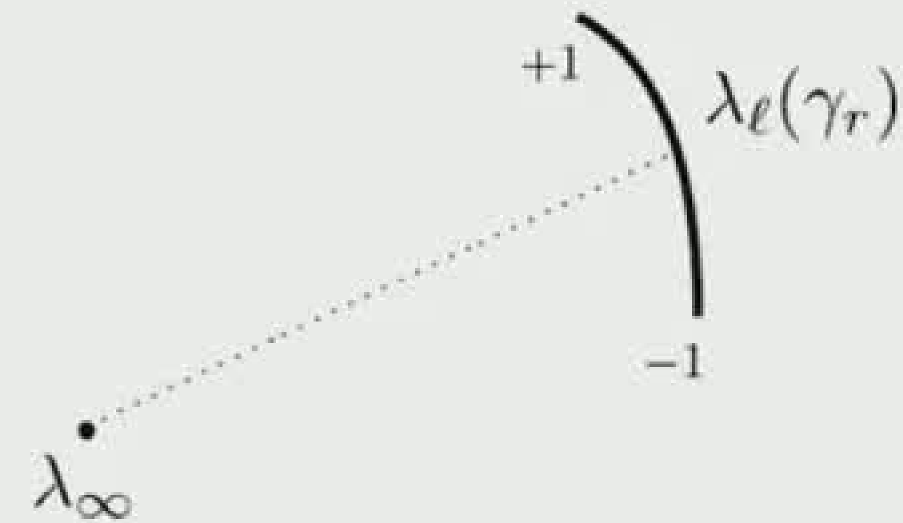


The spectrum associated to spatially periodic patterns

• The spectrum $\sigma(\Psi_\ell)$ associated to a spatially periodic pattern $\Psi_\ell(x)$ – with wavelength ℓ – consists of (up to countably many) 'closed loop images' $\lambda_\ell(\gamma)$ of S^1.

• The loops $\lambda_\ell(\gamma)$ degenerate to 'curved intervals' if $\lambda_\ell(\gamma)$ is stationary and (reversibly) symmetric, with $\gamma = \pm 1$ as endpoints.





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[Gardner, Sandstede & Scheel]

Near a homoclinic Hopf bifurcation: 3 critical configurations

In the semi-strong setting, $\sigma(\Psi_\ell)$ can be determined asymptotically:

- The branch $\lambda_\ell(\gamma)$ near λ_∞ by decomposing the Evans function.

  ([de Rijk, D. & Rademacher, '16]: a Riccatti transformation.)

- The asymptotically 'small spectrum' attached to $\lambda = 0$.

  ([de Rijk, '17]: Lin's method & exponential trichotomies.)

- $\lambda_\ell(\gamma)$ '**rotates**', '**straightens**' and shrinks as function of ℓ.

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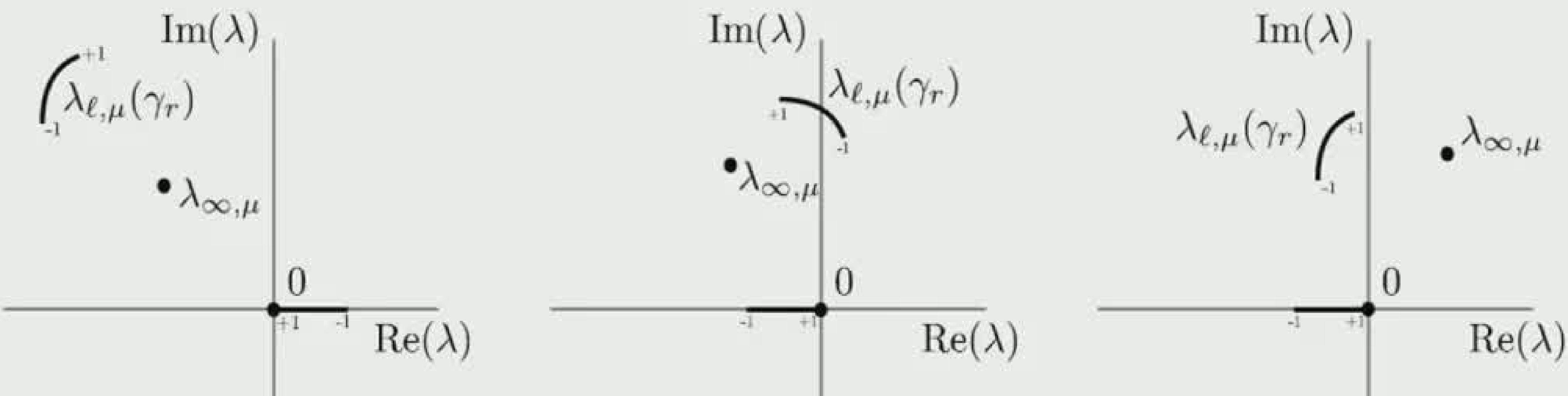

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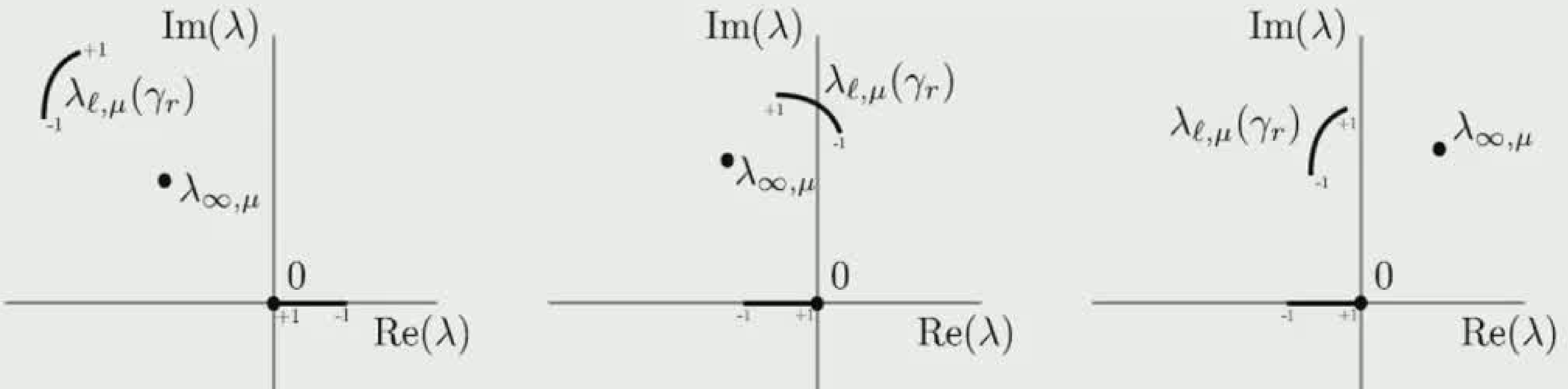
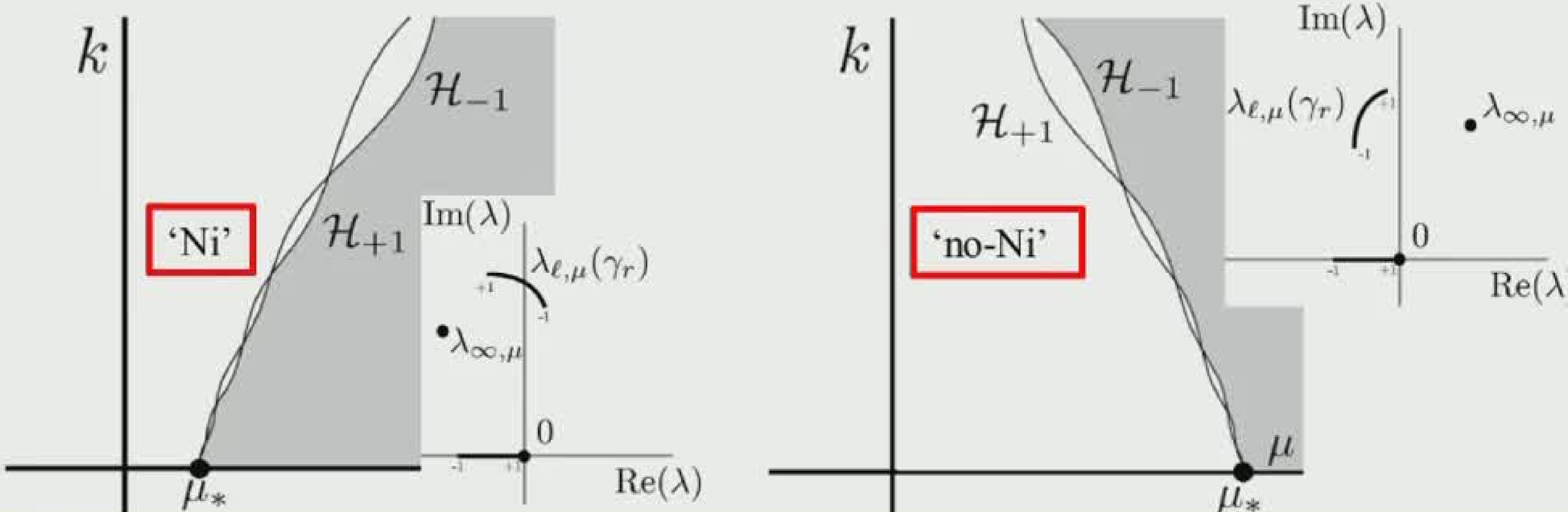


Ni's conjecture and the Hopf dance

Wei-Ming Ni (on GM,'98): The homoclinic limit pattern is the most stable pattern 'within' the family of spatially periodic patterns.



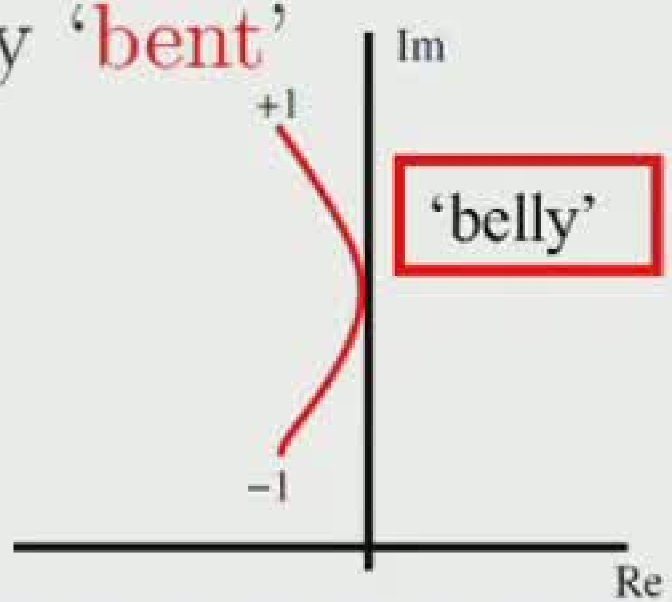
• In general slowly nonlinear SP RDEs, Ni's conjecture does not necessarily hold.

([D, Rademacher & vdStelt, '12]: 'Ni' holds for classical GM.)

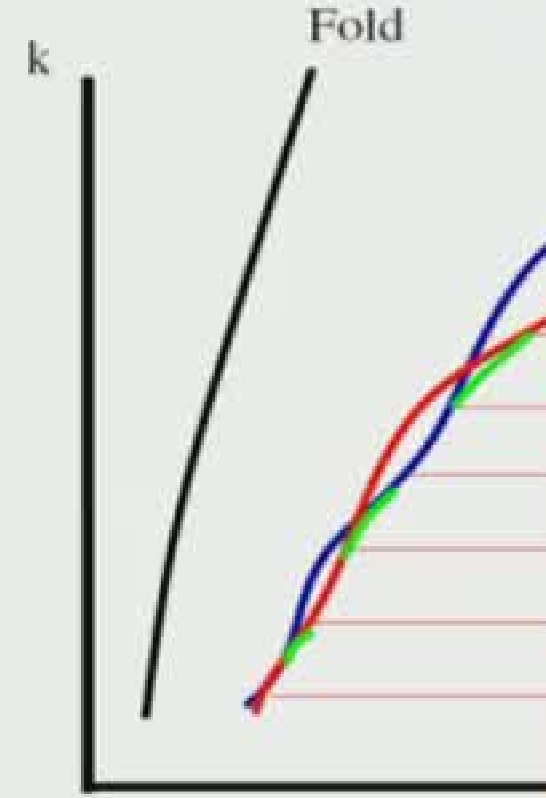
• The boundary of the Busse balloon has a fine-structure of 2 intertwining Hopf curves, $\mathcal{H}_{+1}$ and $\mathcal{H}_{-1}$: the Hopf dance.

The ‘belly dance’

- $\lambda_\ell(\gamma)$ is weakly ‘bent’



$\Rightarrow$



A boundary of 2 intertwining Hopf curves, without co-dimension 2 ‘corners’?? (**NO!**)

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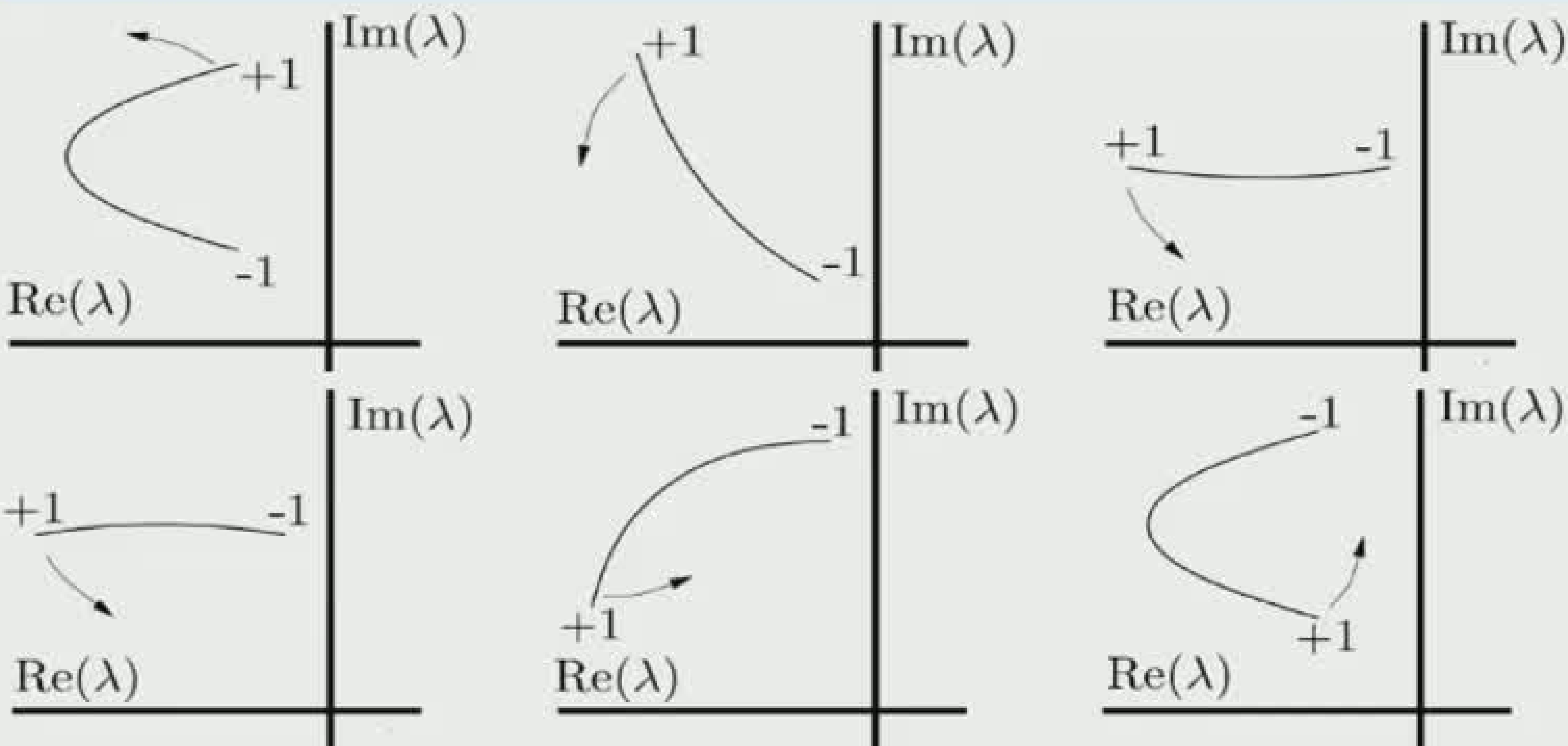
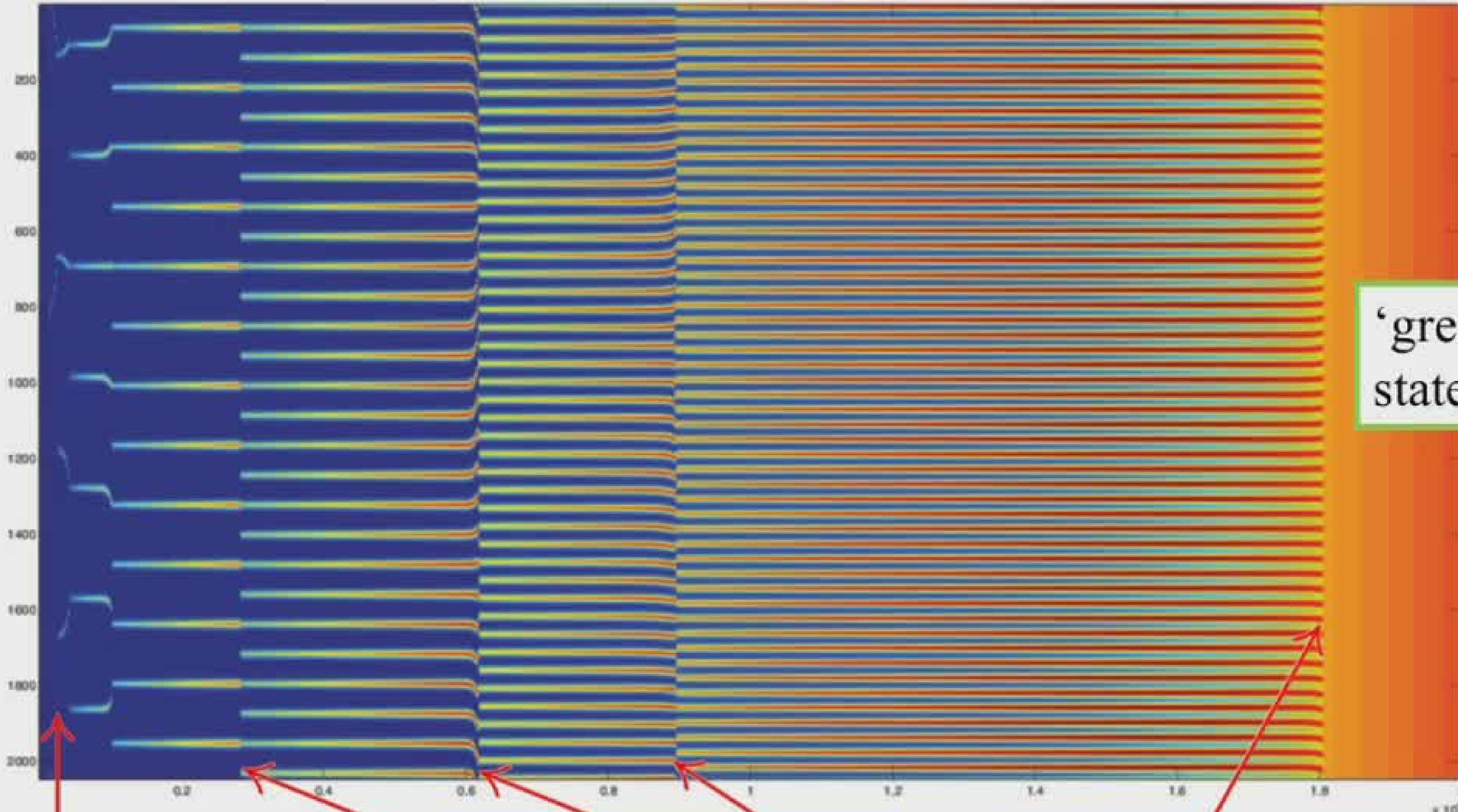


The belly ‘flips’ to the other side (w.r.t the connecting straight line between its endpoints) in half a rotation of the spectral curve.

The dynamics of patterns under slowly varying conditions



'green state'

'mini-catastrophes'

Turing/morphogenesis

DESERTIFICATION: the final collapse/morphothanatos

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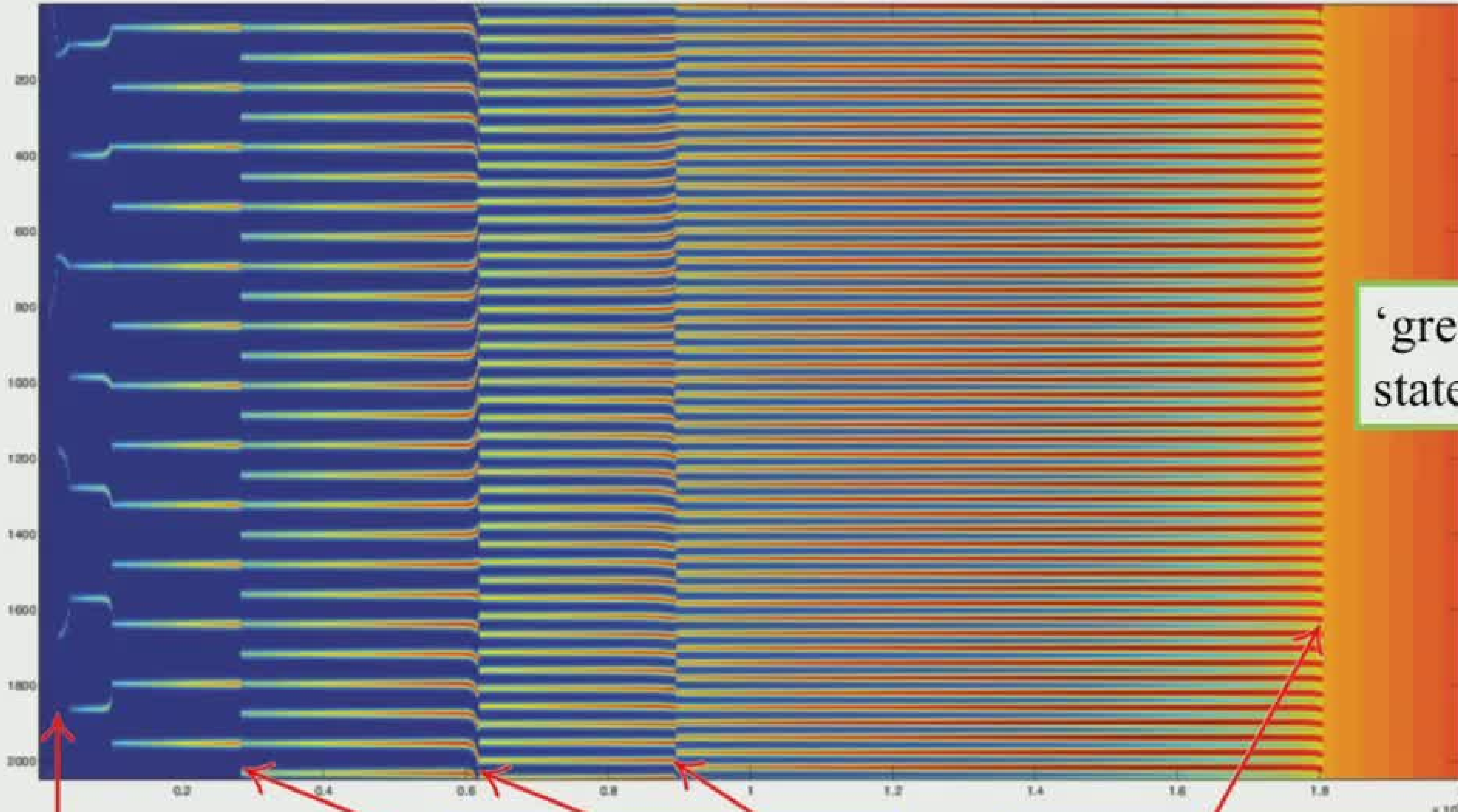

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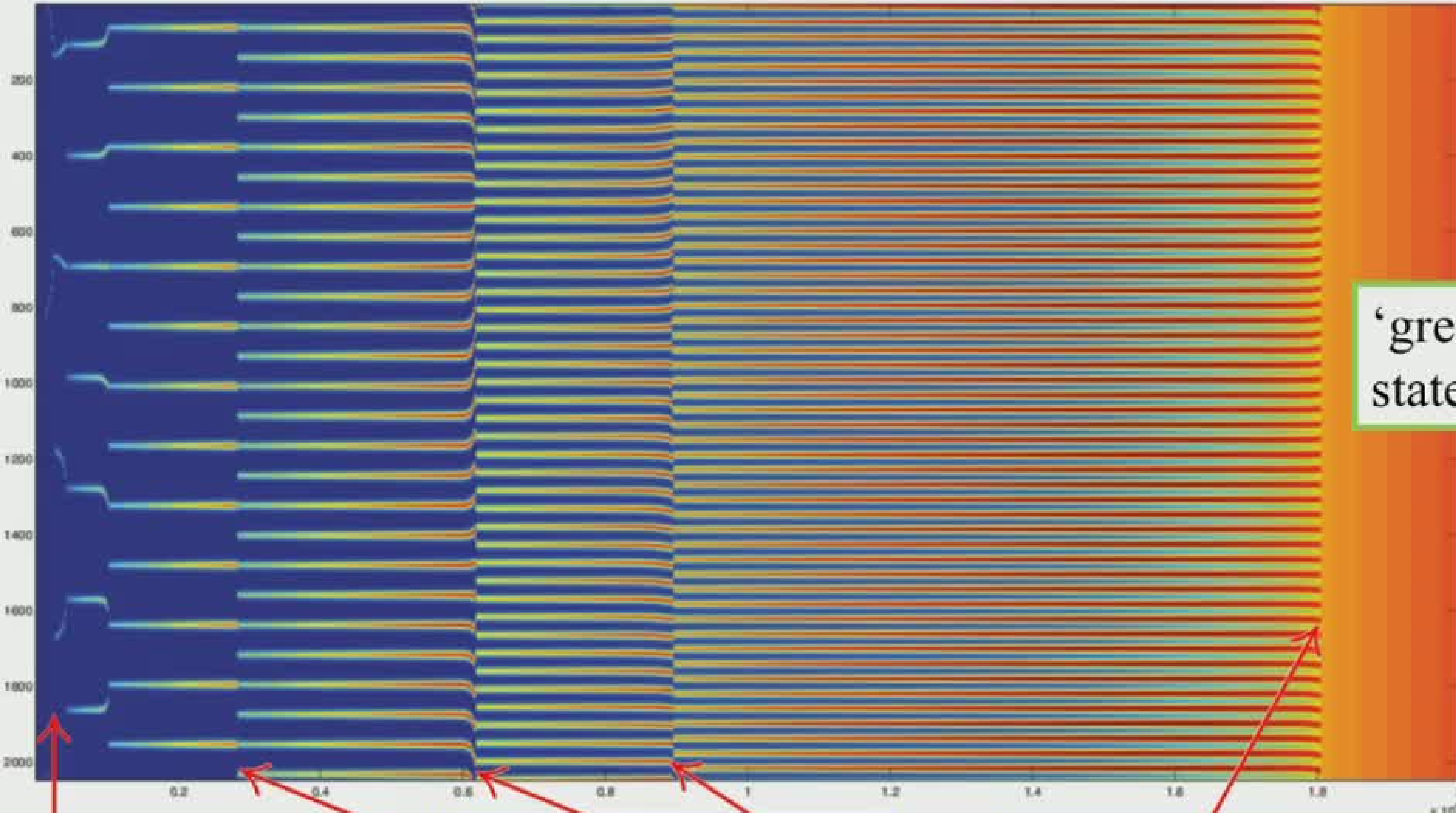

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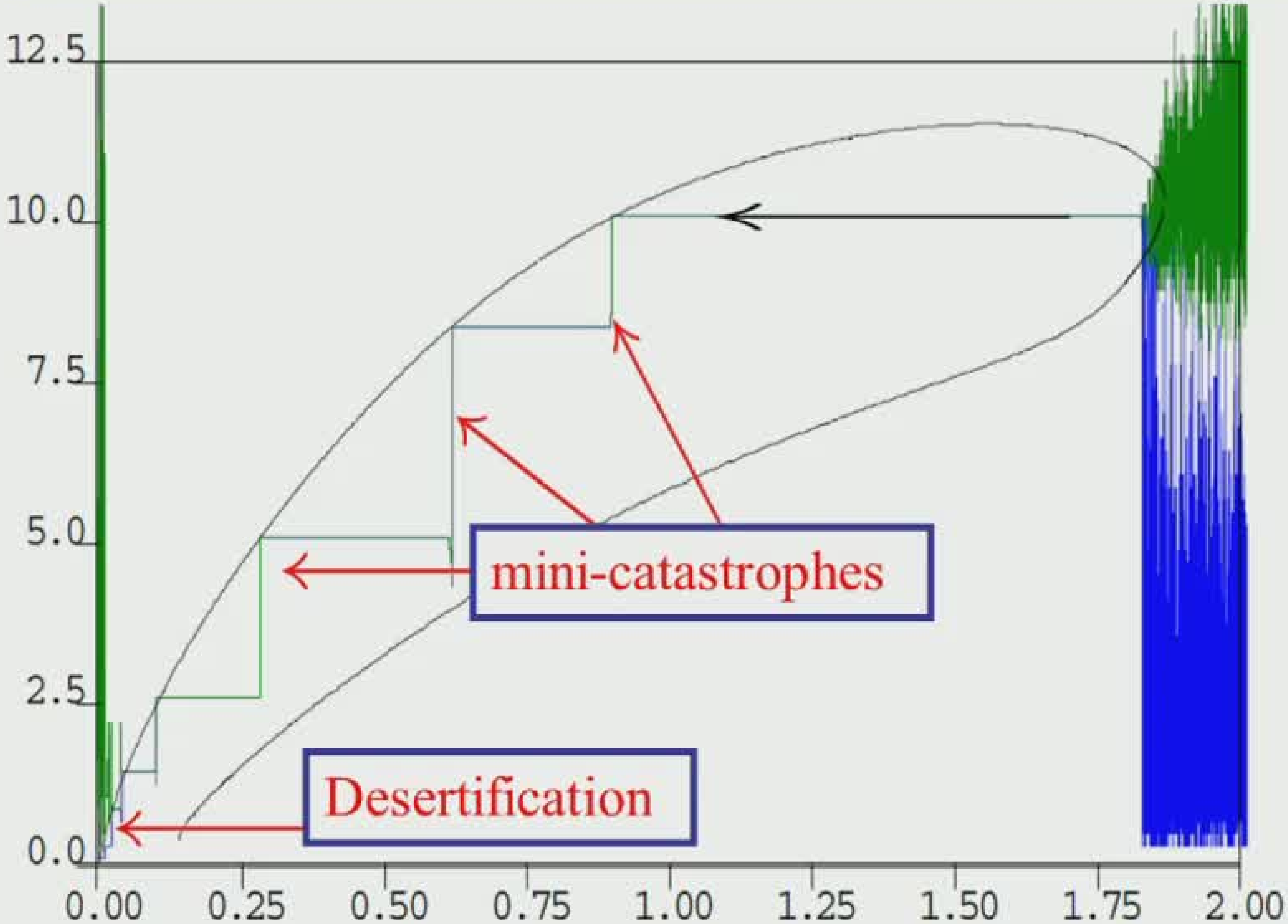

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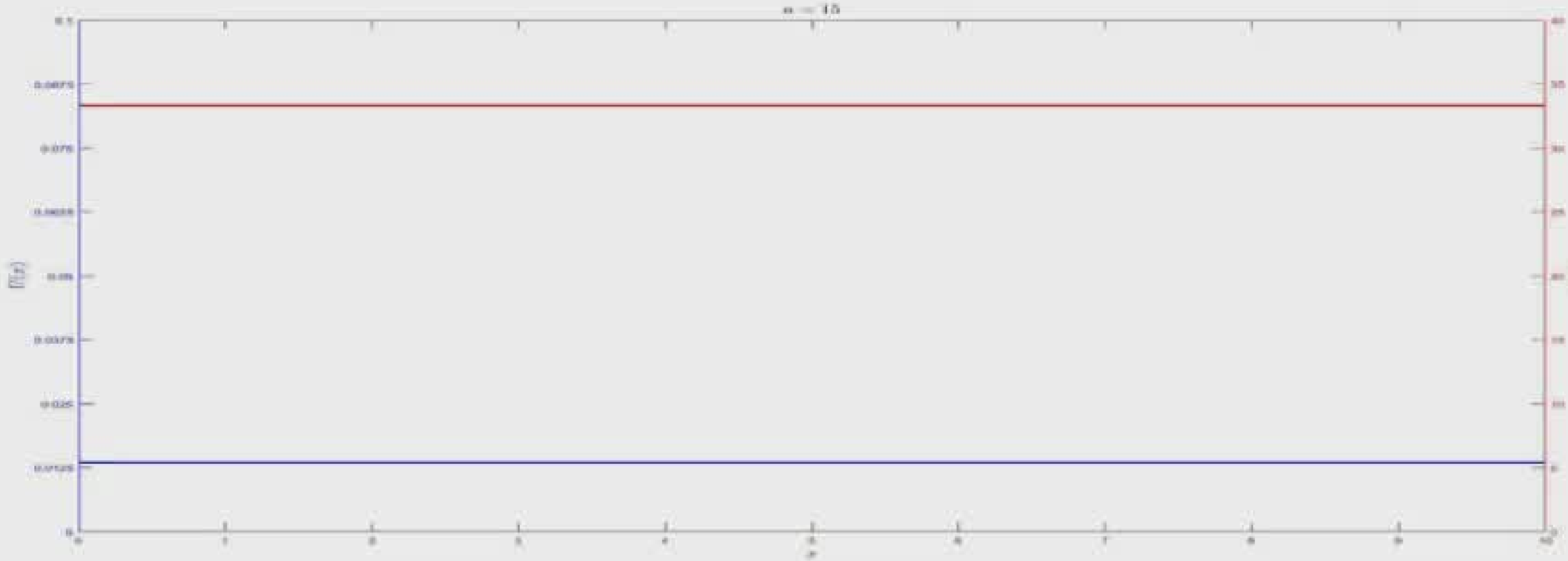
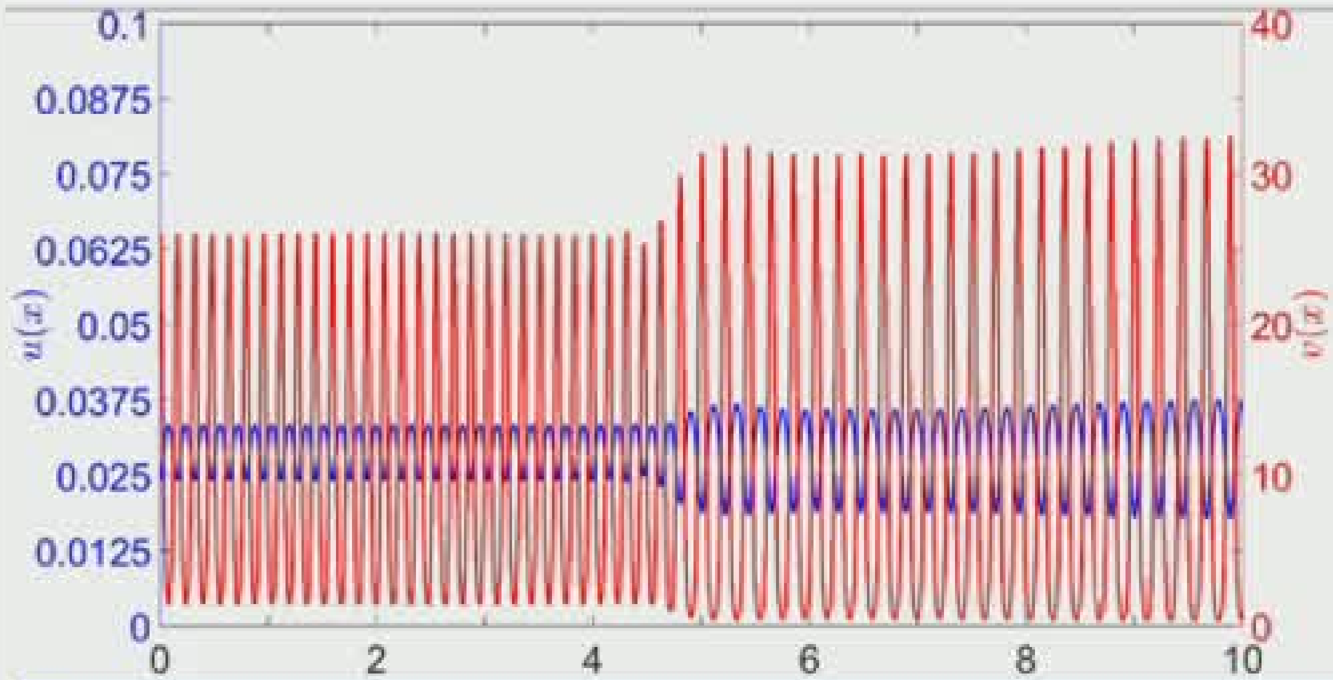
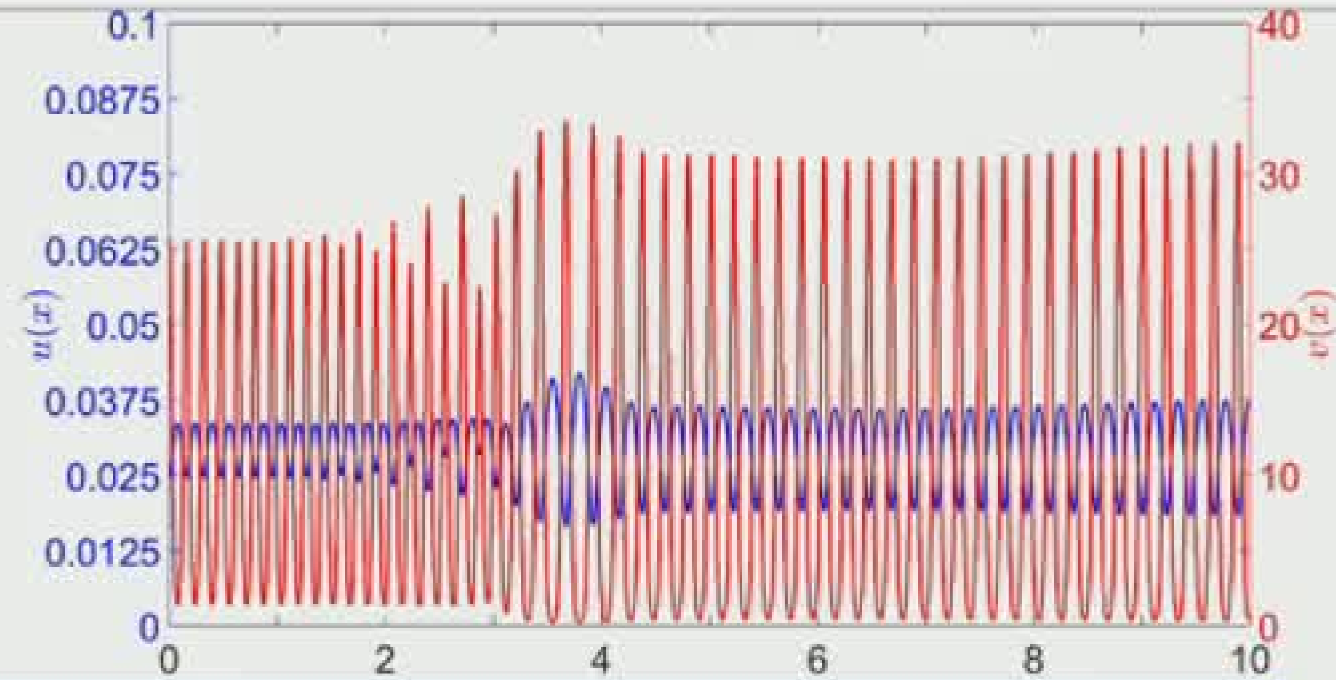
Which rules are driving the reflection process?

Intermezzo: mini-catastrophe ↔ a traveling invasion front?

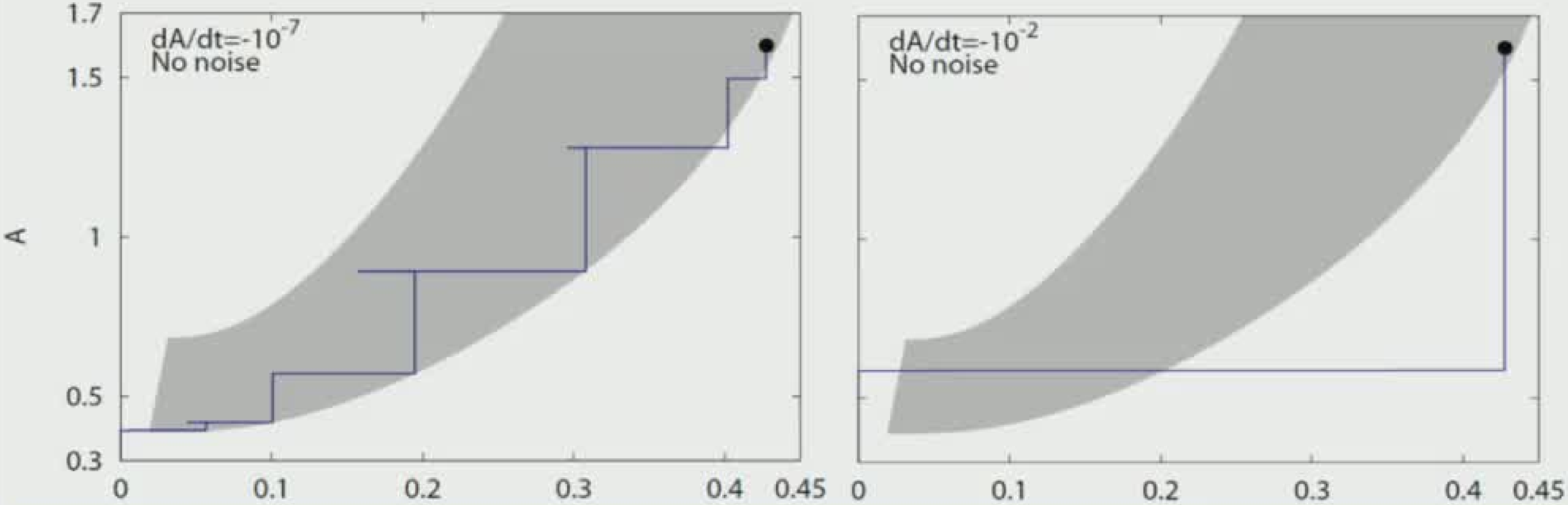






The fast dynamical 'jump' from $\mathcal{M}^{\varepsilon}_{J}$ to $\mathcal{M}^{\varepsilon}_{J'}$, $(J' < J)$.

The impact of the rate of change *dA/dt*

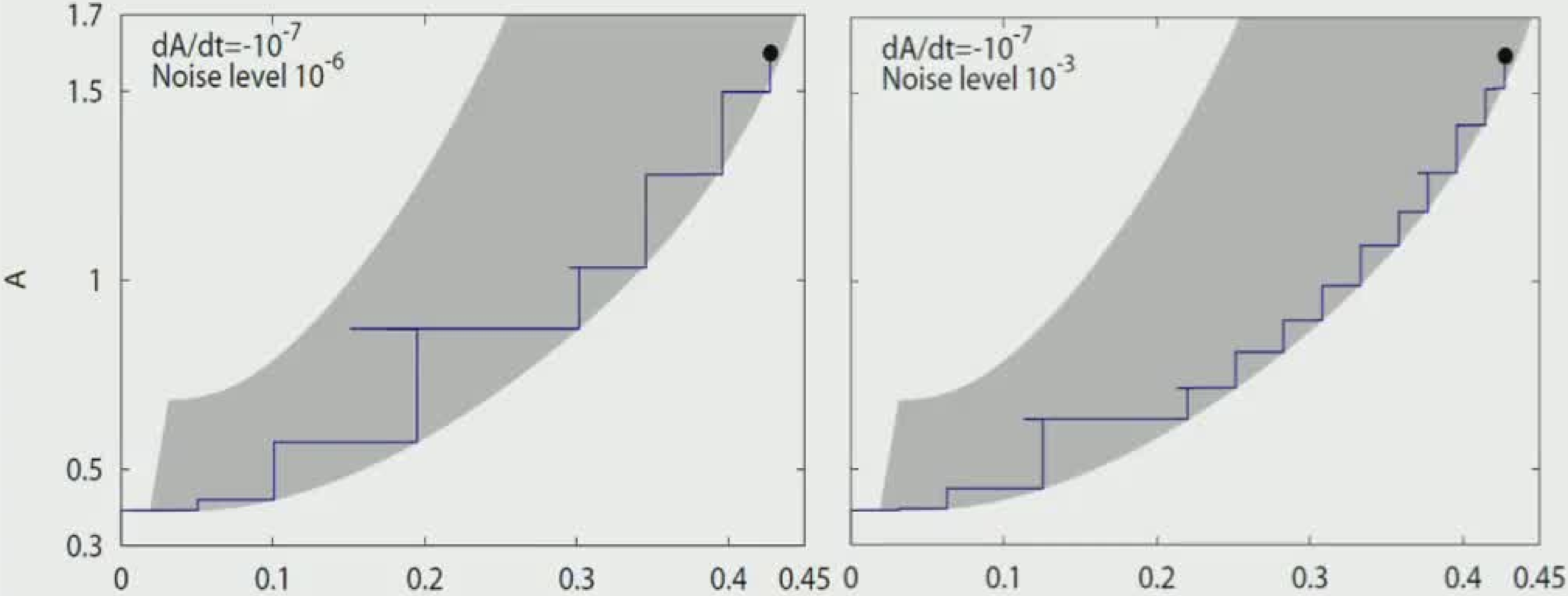


- If *dA/dt* is 'too large', then the 'internal dynamics' of the system cannot adapt.
- The first (& only) (mini-)catastrophe is delayed. Desertification occurs at the initial (Turing) wavenumber. However, 'morphotanatos' sets in for a higher value of *A*.

(↔ Collapse to bare soil with 'sufficient' rainfall.)

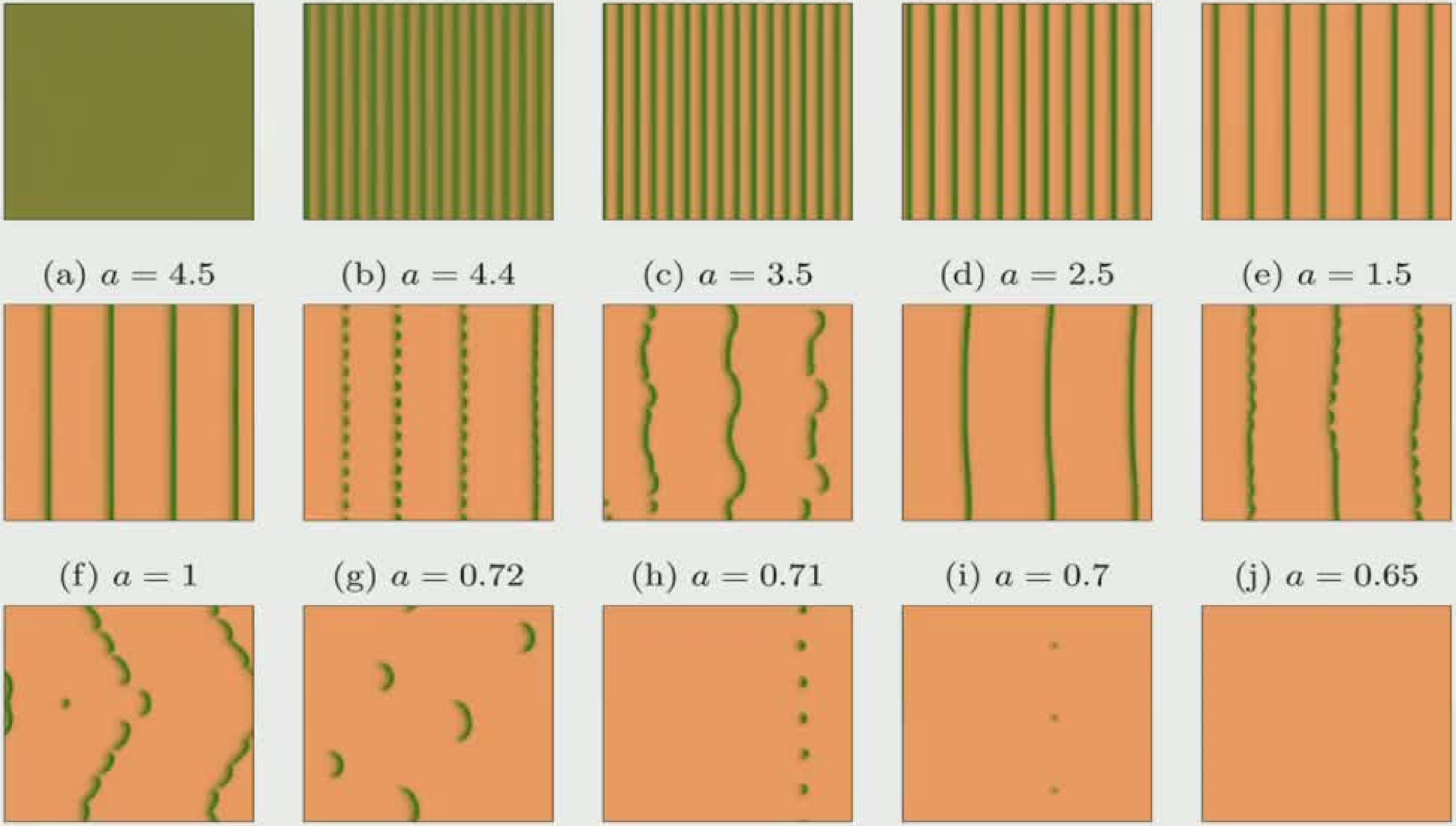
[Siteur, Siero, D., Eppinga, Rademacher & Rietkerk, '14 & '16]

The impact of noise



- As expected from ODE-insights, noise reduces the delay effects: the boundary of the Busse balloon becomes a(n even) better 'predictor'.
- Moreover, by increasing the noise level, the pattern sticks closer & closer to the boundary of the Busse balloon.

Intermezzo: the dynamics of stripes



(a) $a = 4.5$ (b) $a = 4.4$ (c) $a = 3.5$ (d) $a = 2.5$ (e) $a = 1.5$

(f) $a = 1$ (g) $a = 0.72$ (h) $a = 0.71$ (i) $a = 0.7$ (j) $a = 0.65$

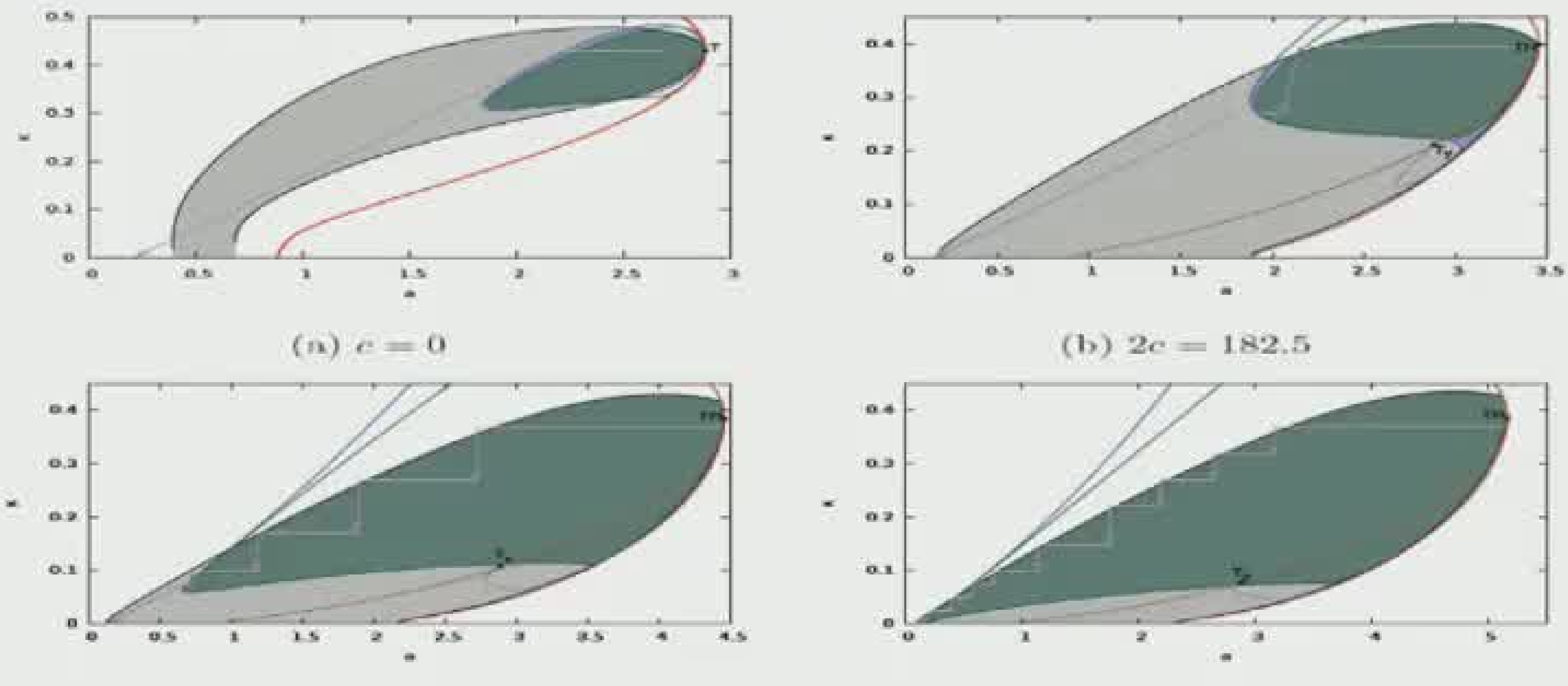
(k) $a = 0.55$ (l) $a = 0.3$ (m) $a = 0.18$ (n) $a = 0.13$ (o) $a = 0.12$

[Siero, D., Eppinga, Rademacher, Rietkerk, Siteur, '15]

The Busse balloon perspective

Busse balloons for systems on hillsides with increasing slopes.

'Green' areas: stable stripes.



(a) $c = 0$ (b) $2c = 182.5$ (c) $2c = 365$ (d) $2c = 500$

[Sewalt & D, '17] on (gKGS): Homoclinic and spatially periodic stripes are unstable for 'bounded advection'.

Conjecture: Homoclinic stripes may be stable in (two-component) SP reaction-diffusion-advection equations with sufficiently large advection.

Observations

- flat terrains ↔ spots
- on slopes ↔ stripes

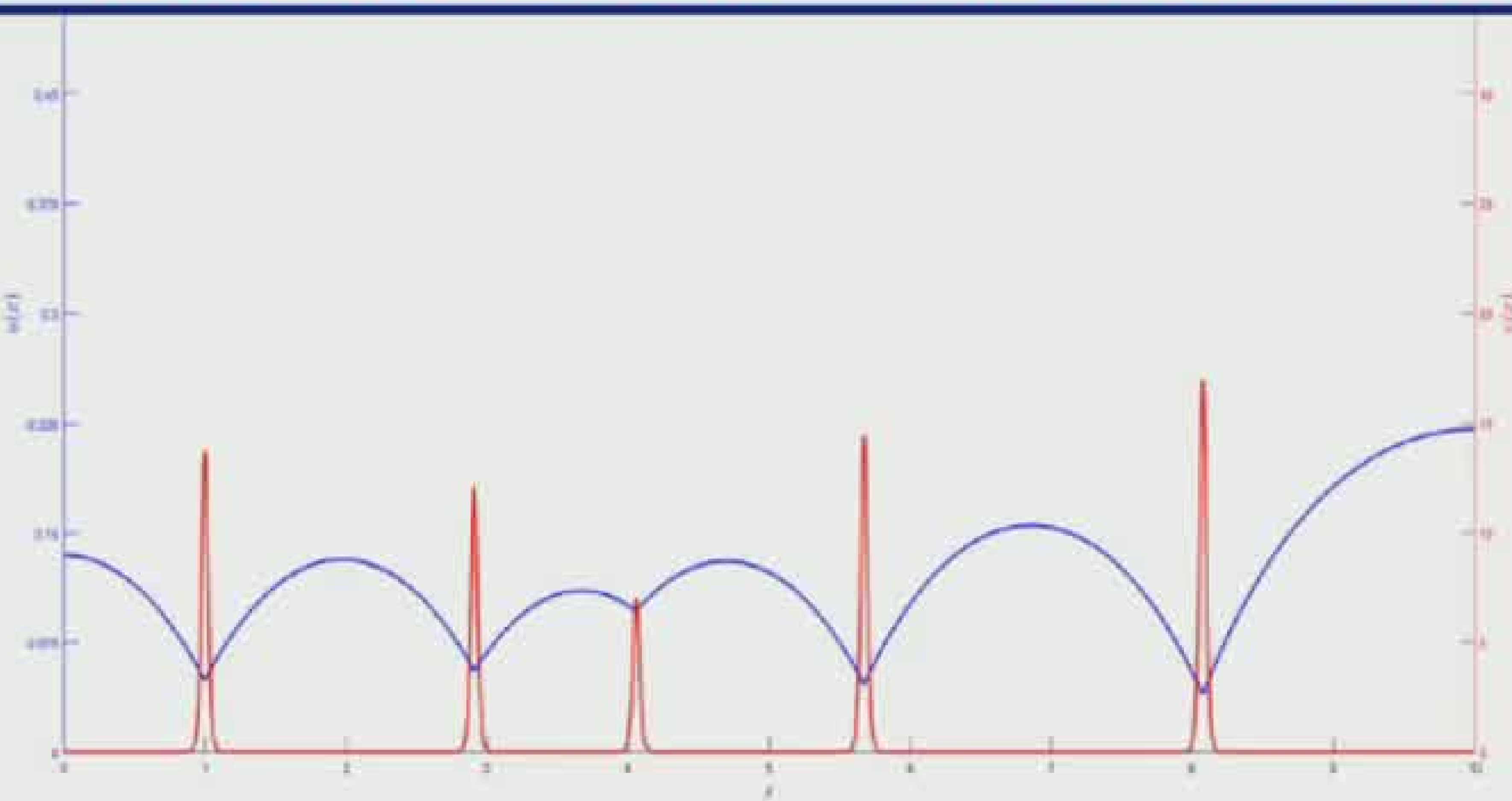
Catastrophic ↔ gradual?

Some conjectures

[Bastiaansen & D.,’17] – in progress

- (Mini-)catastrophes occur in regular patterns.
- Irregular patterns follow a more gradual course.
- Systems naturally evolve towards regularity.

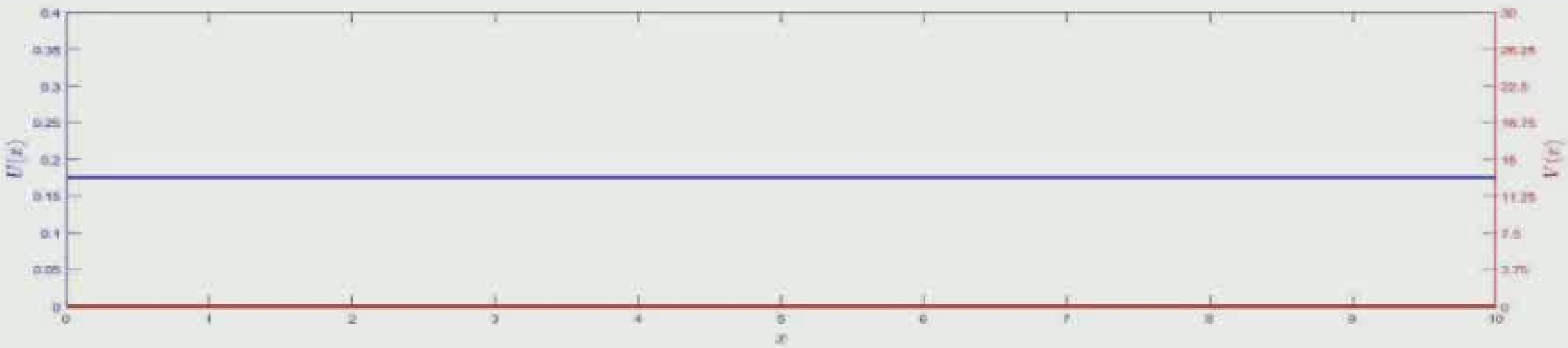
Internal time scale ↔ Time scale environmental change

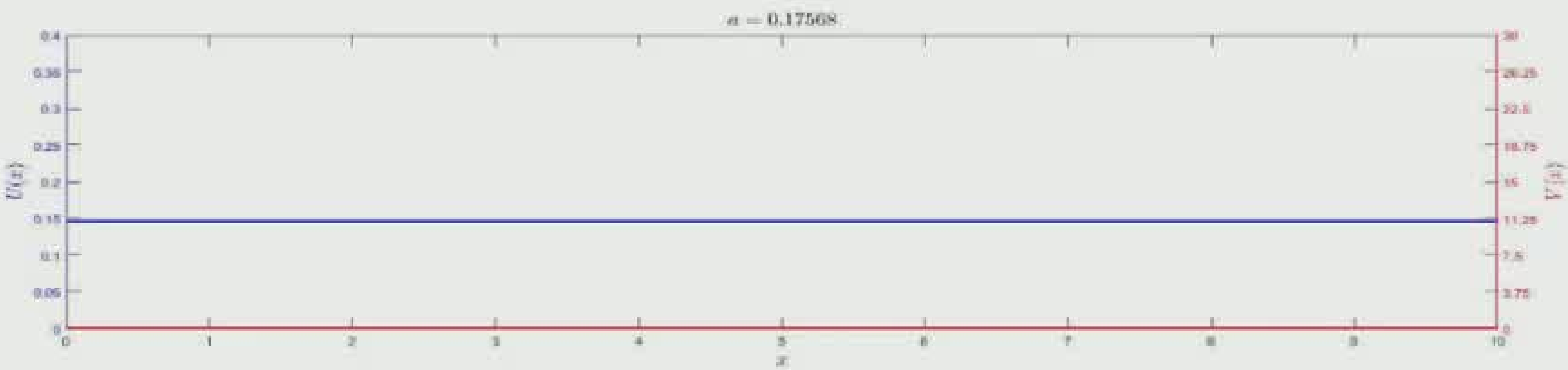


Follow the reduced J– ⟶ J-k–dim. ODE dynamics as the pattern ‘jumps’/’falls’ over the edge of its invariant manifold.

A simulation

Start out with identical initial conditions, but different (external) environmental time scales & synchronize w.r.t. the external evolution.





- Regular patterns are more resilient.
- Vegetation 'survives' with less rain if 'the climate' changes slowly.

Conclusions & Discussion

A STRONG CROSS-FERTILIZATION BETWEEN 'APPLICATIONS' AND MATHEMATICS

- Singular patterns are both realistic & suitable for analysis.
  - Semi-strong interactions.
  - Finite-dimensional reductions.
- The Busse balloon as central 'concept'.
  - Provides an ecological framework.
  - Novel mathematical questions and insights.
- Slowly varying parameters.
  - Ecologically naturally & obvious.
  - Mathematically new & challenging.

End of slide show, click to exit.

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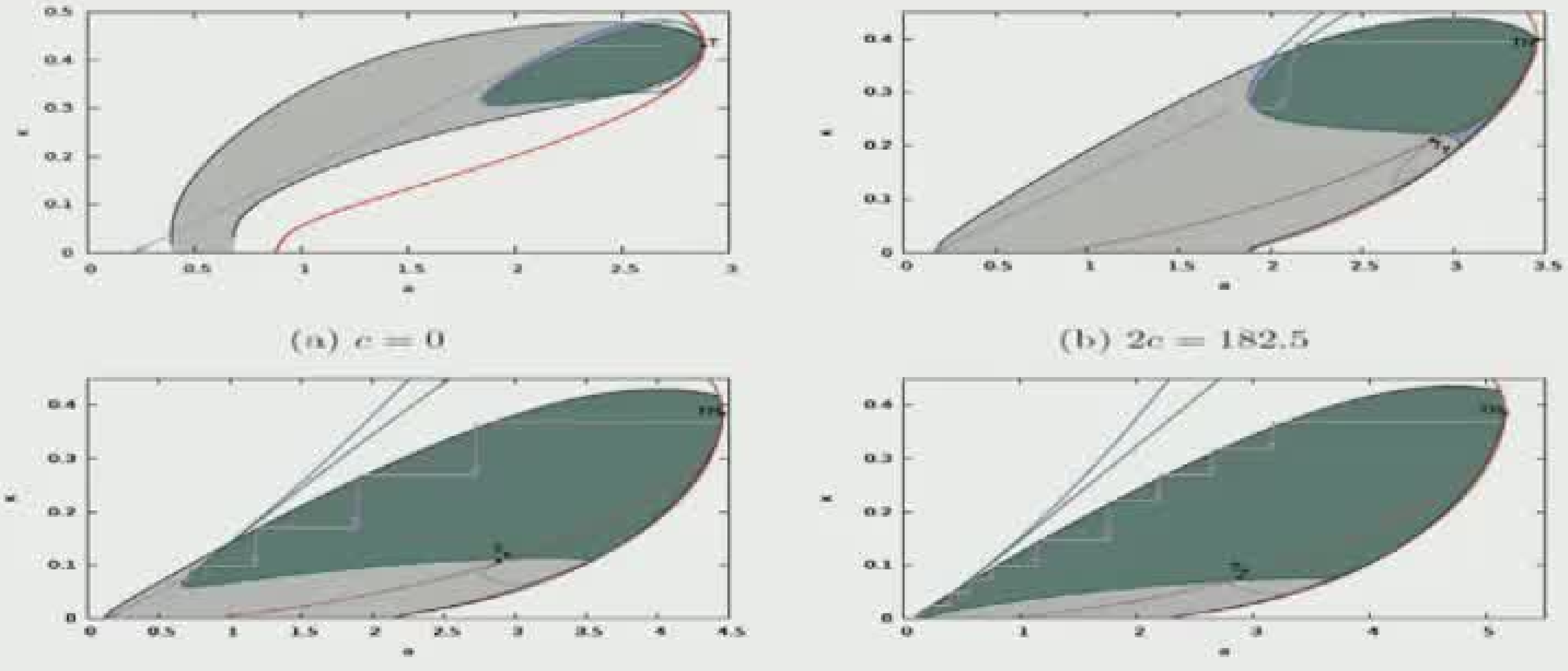

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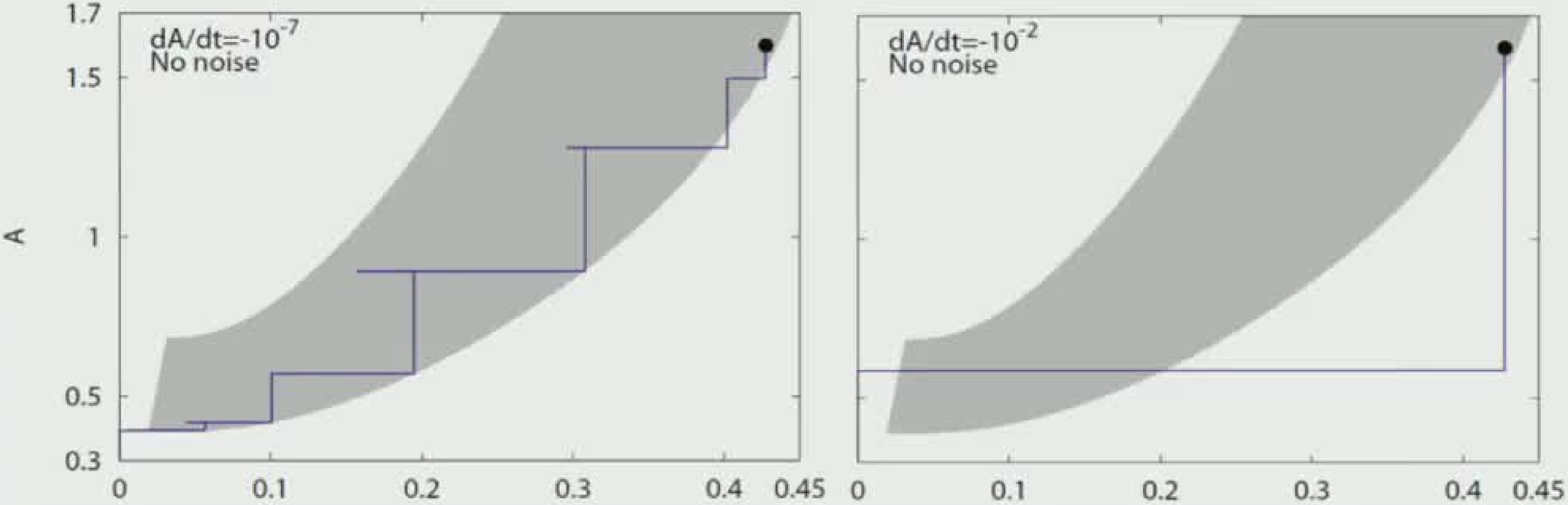

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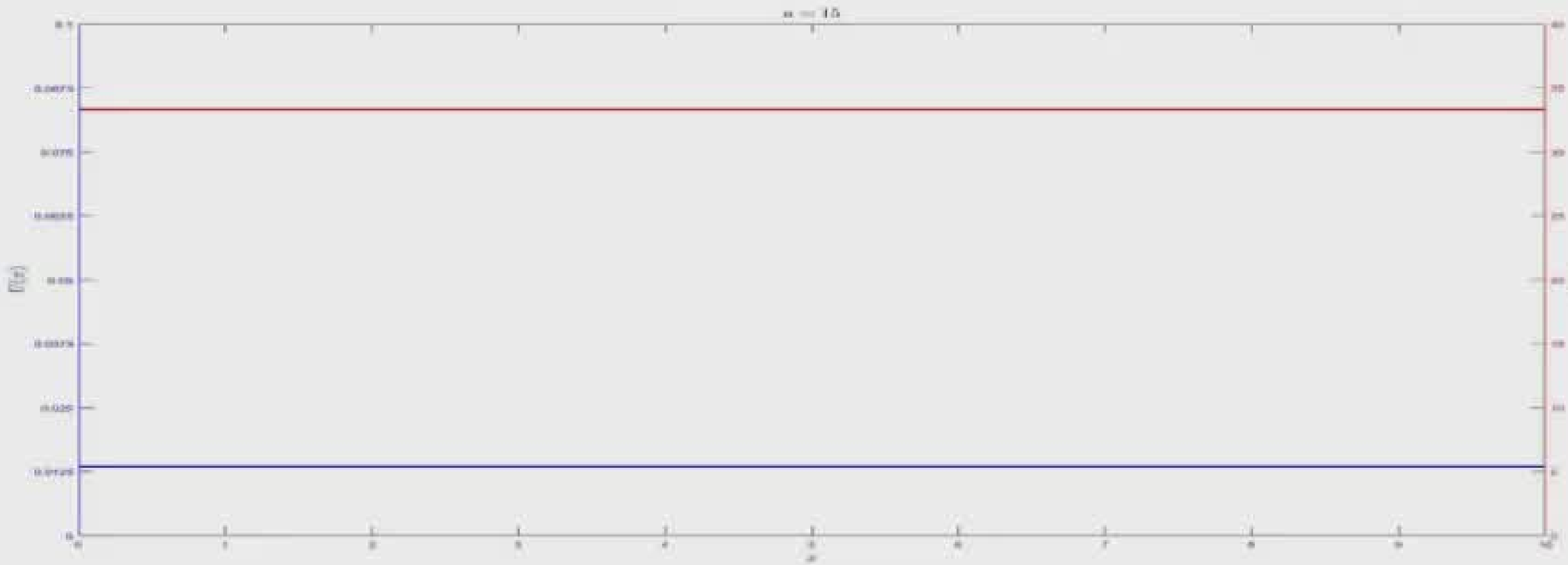

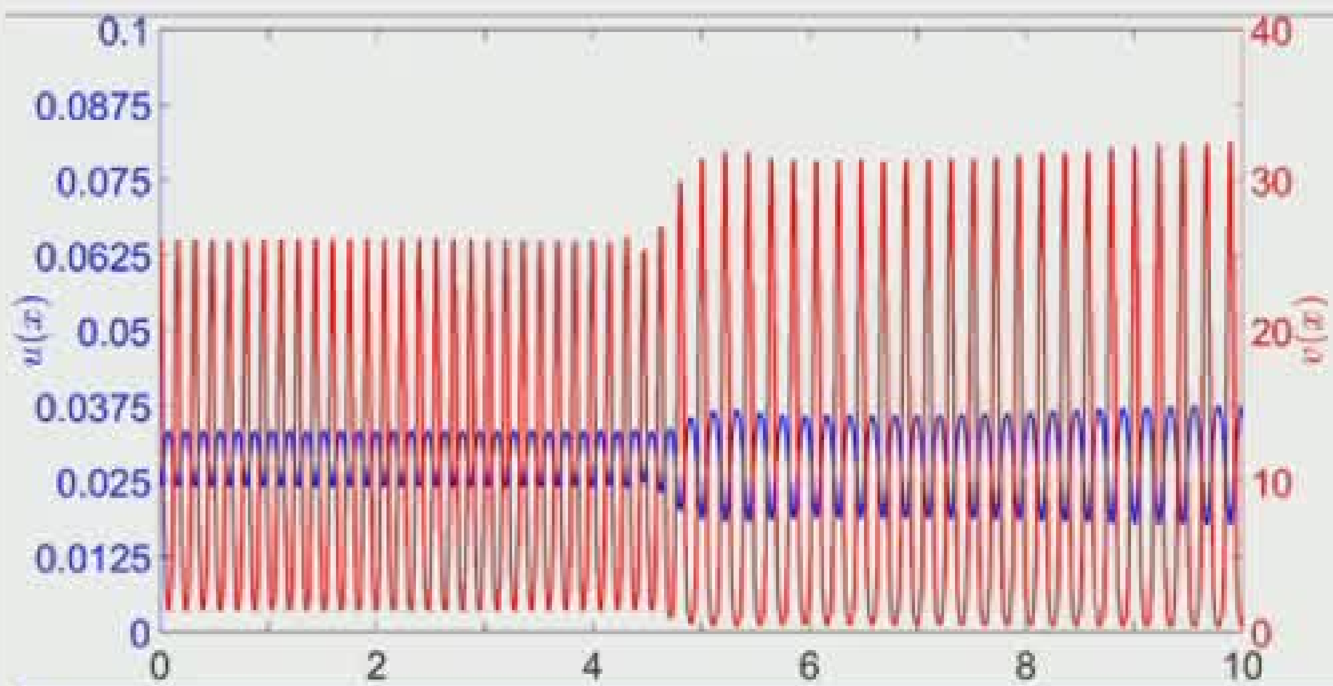

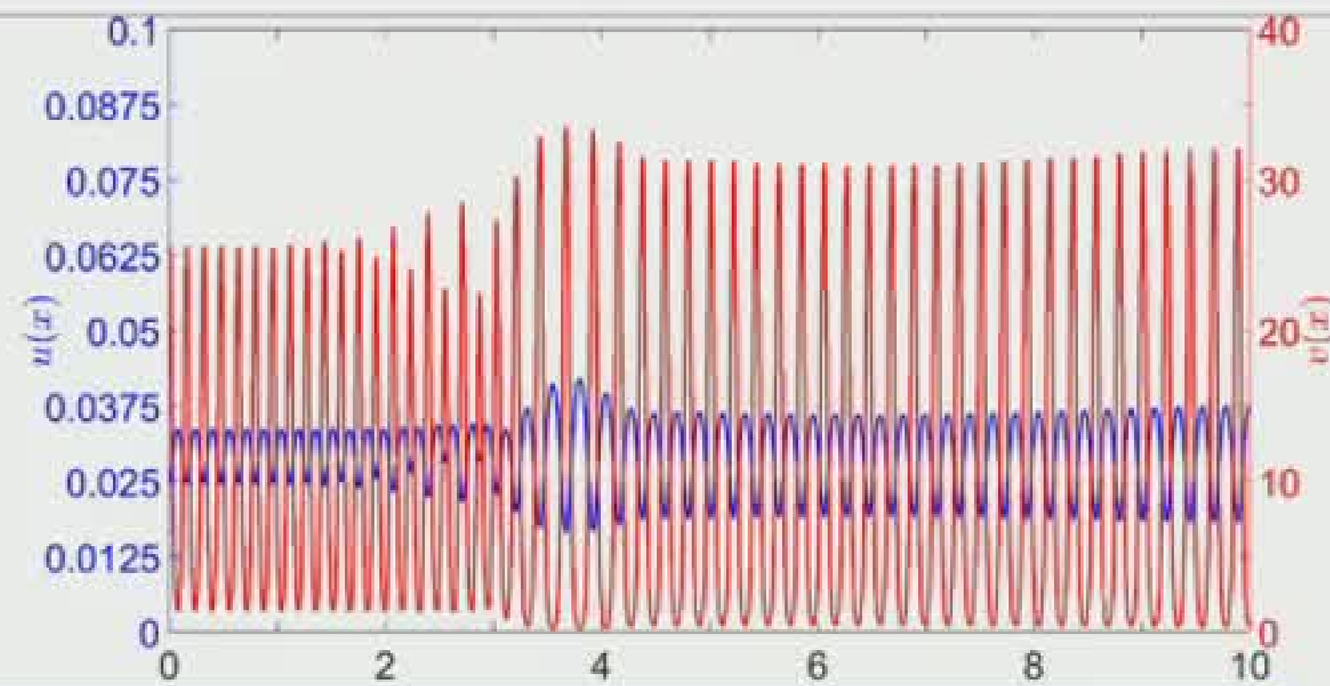

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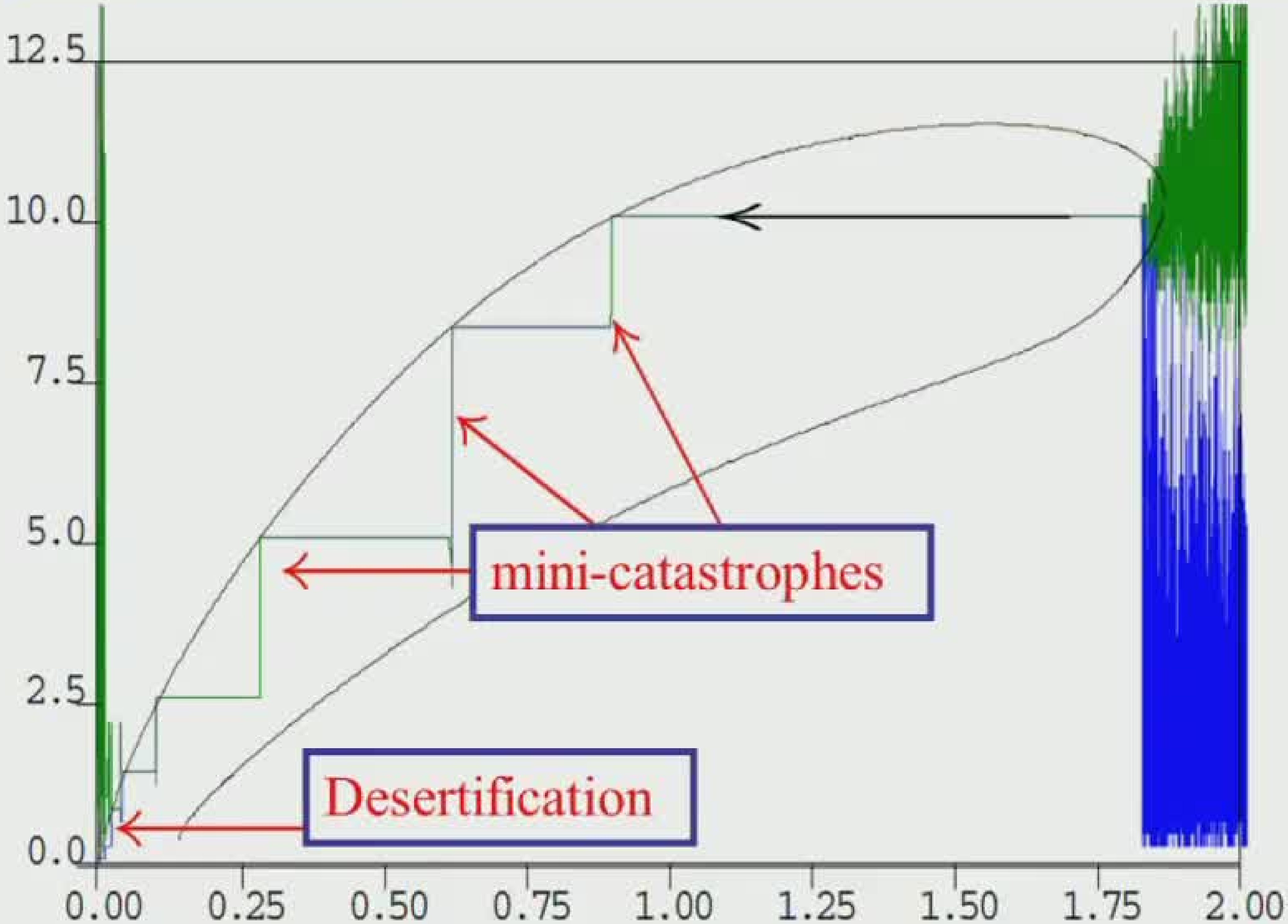

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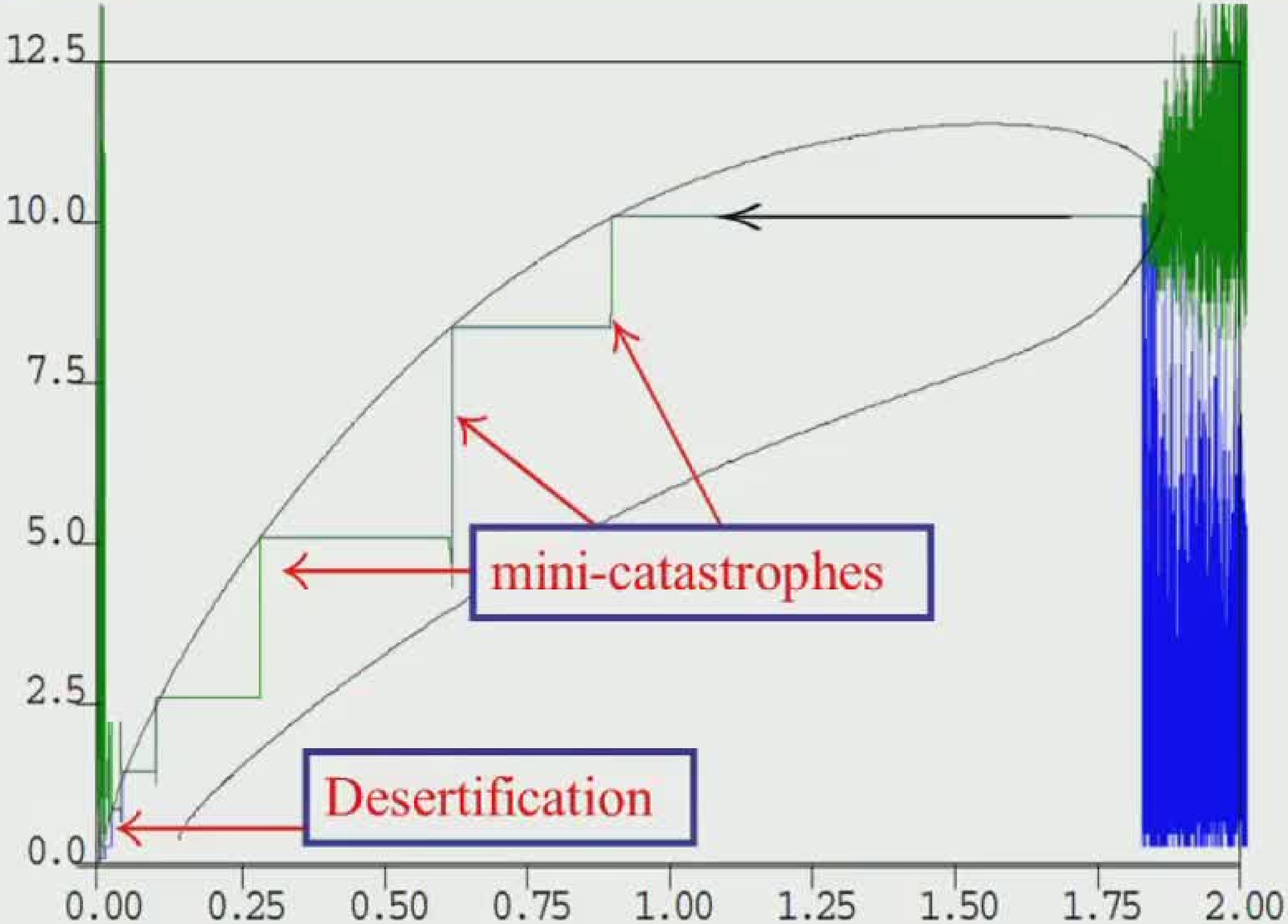

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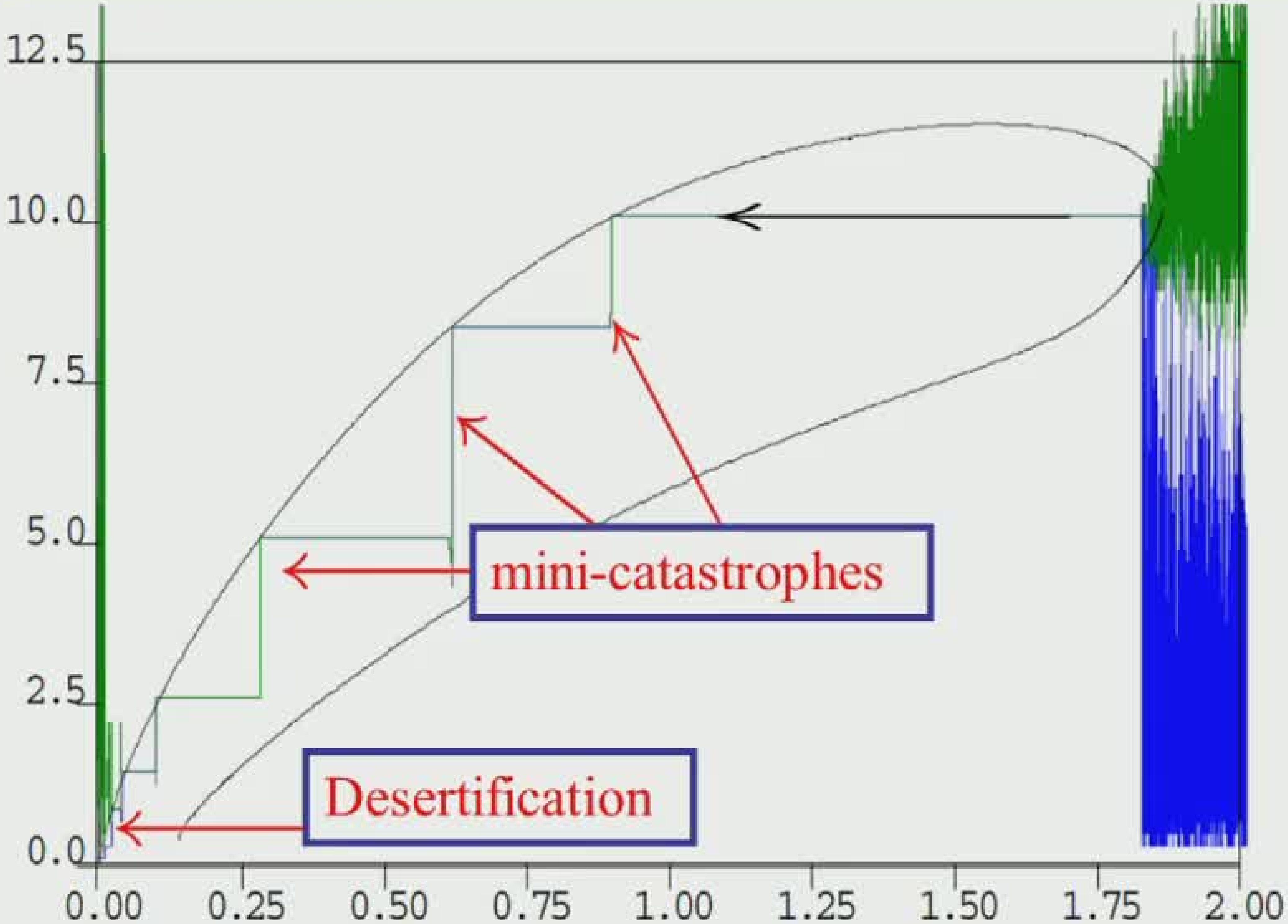

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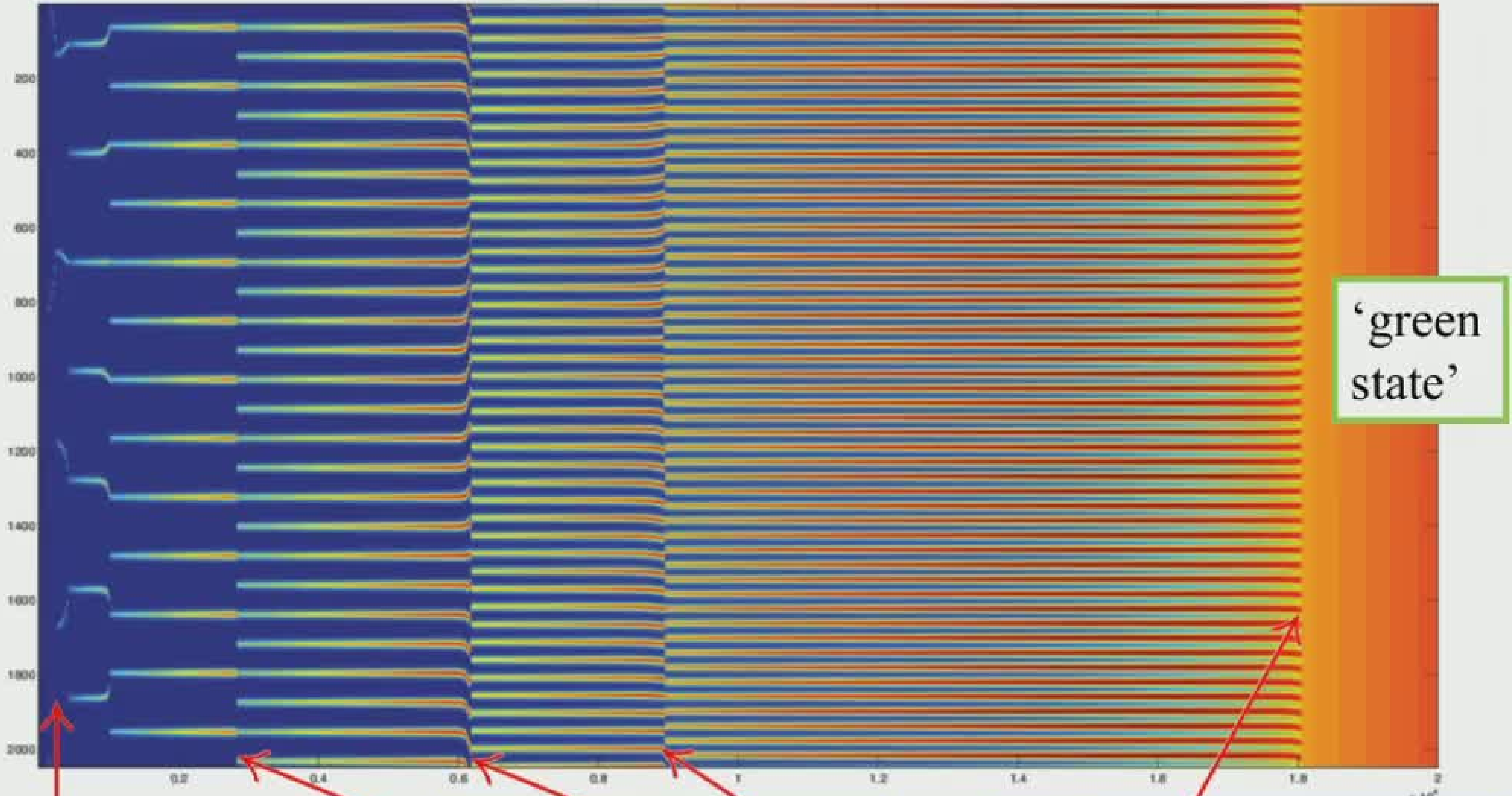

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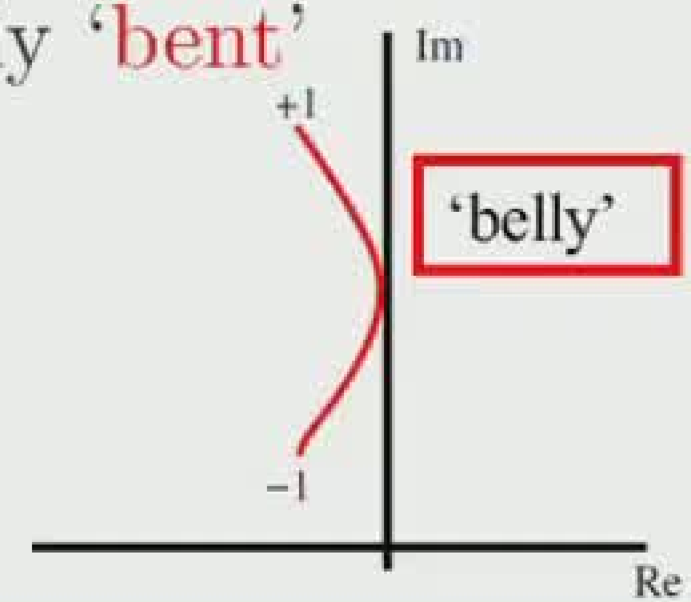

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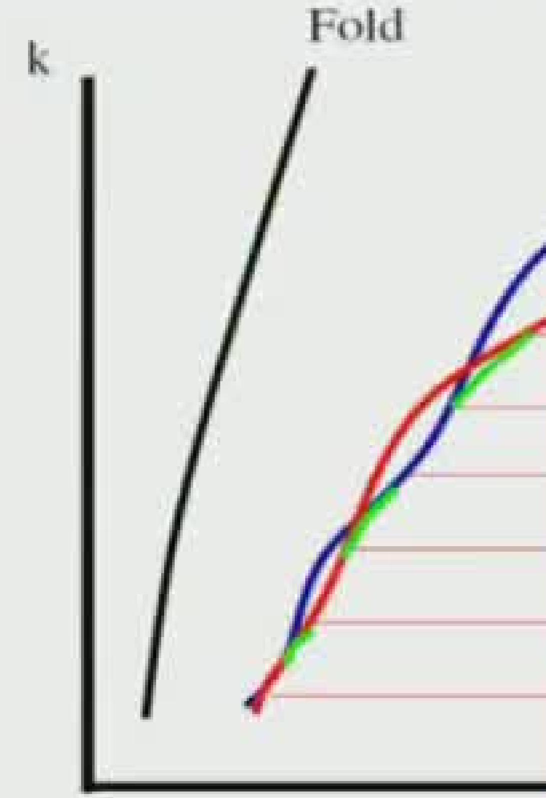

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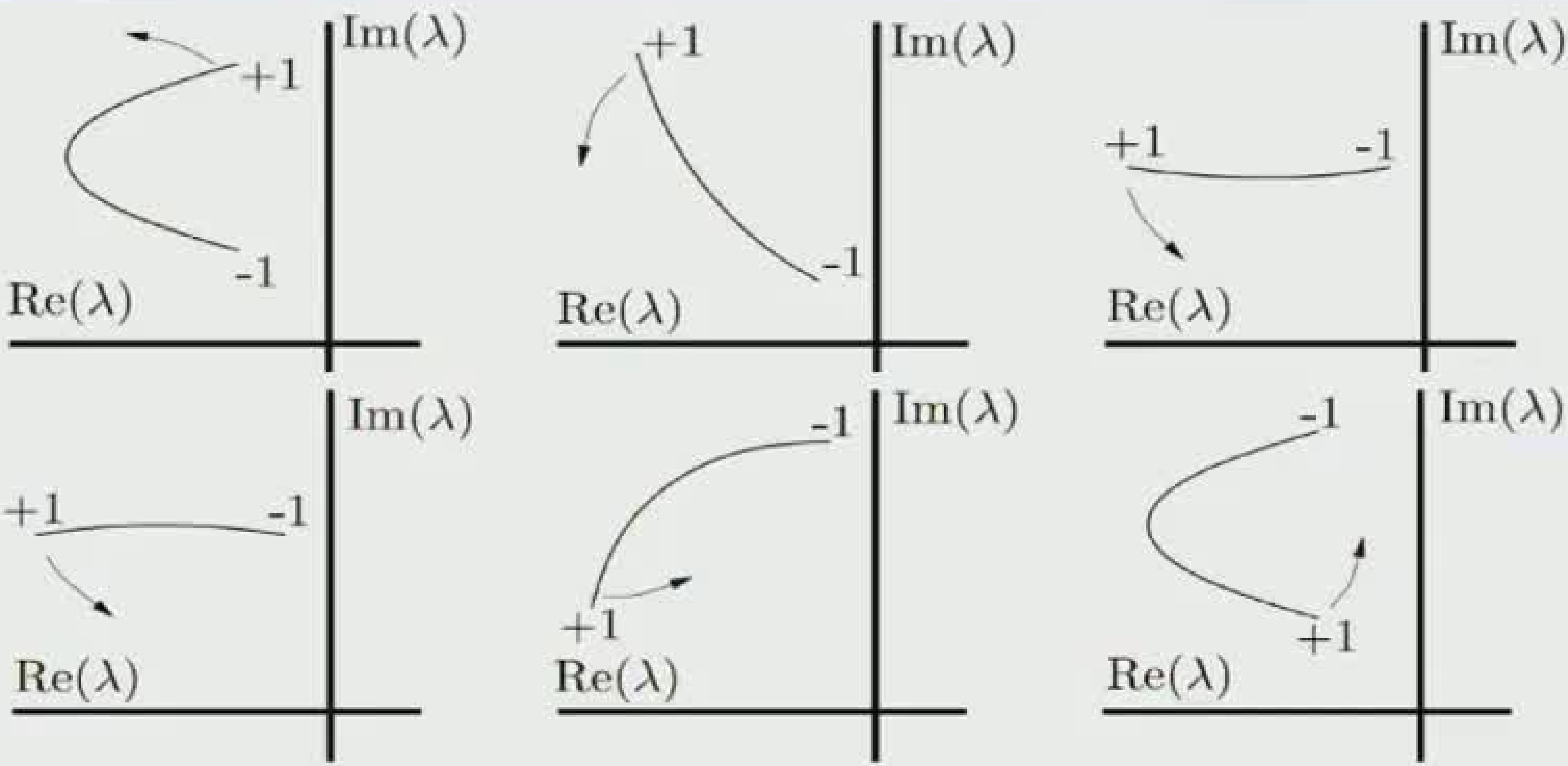

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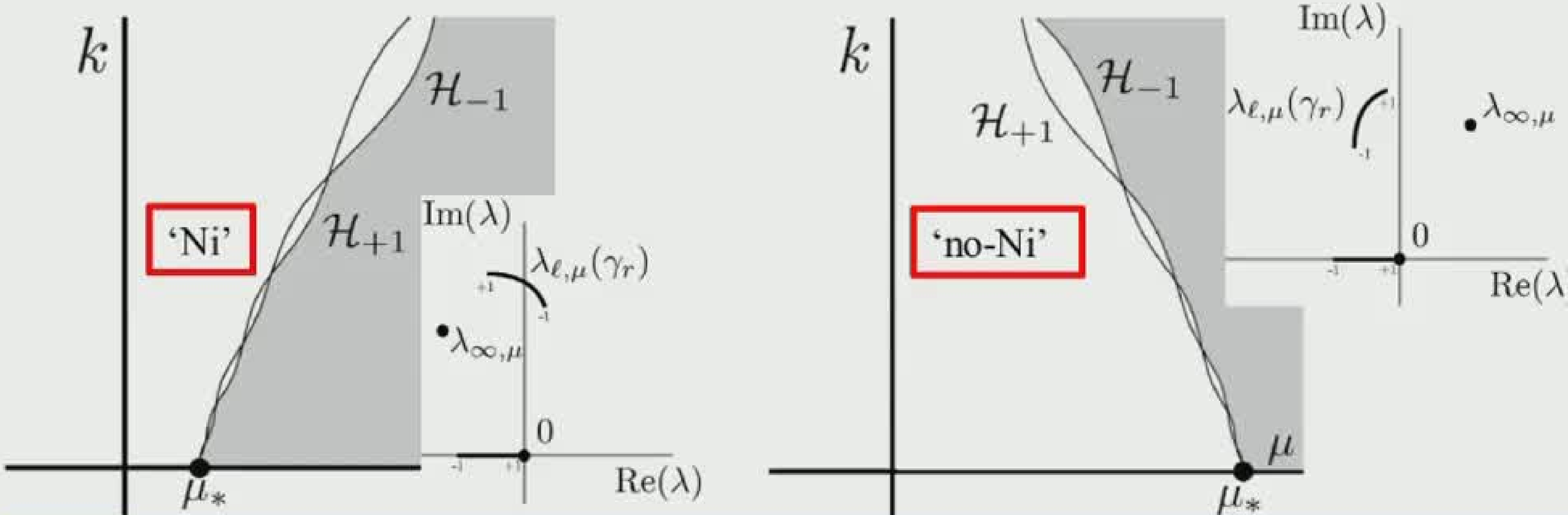

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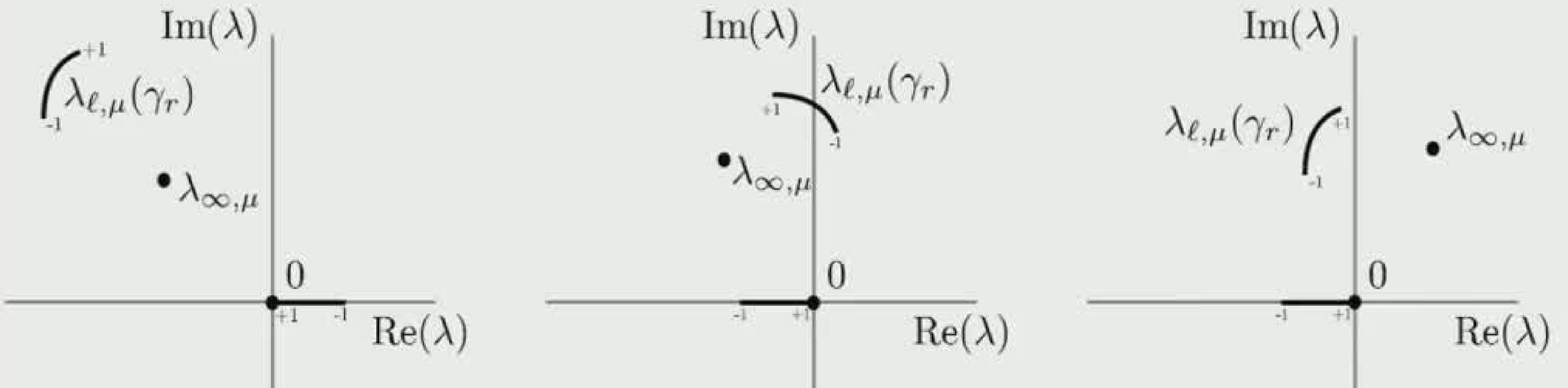

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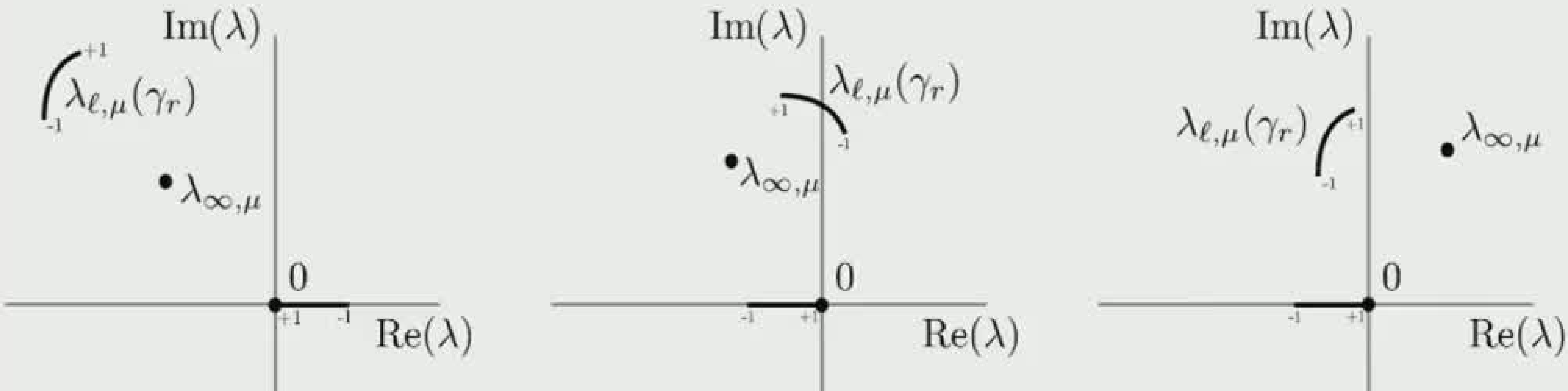

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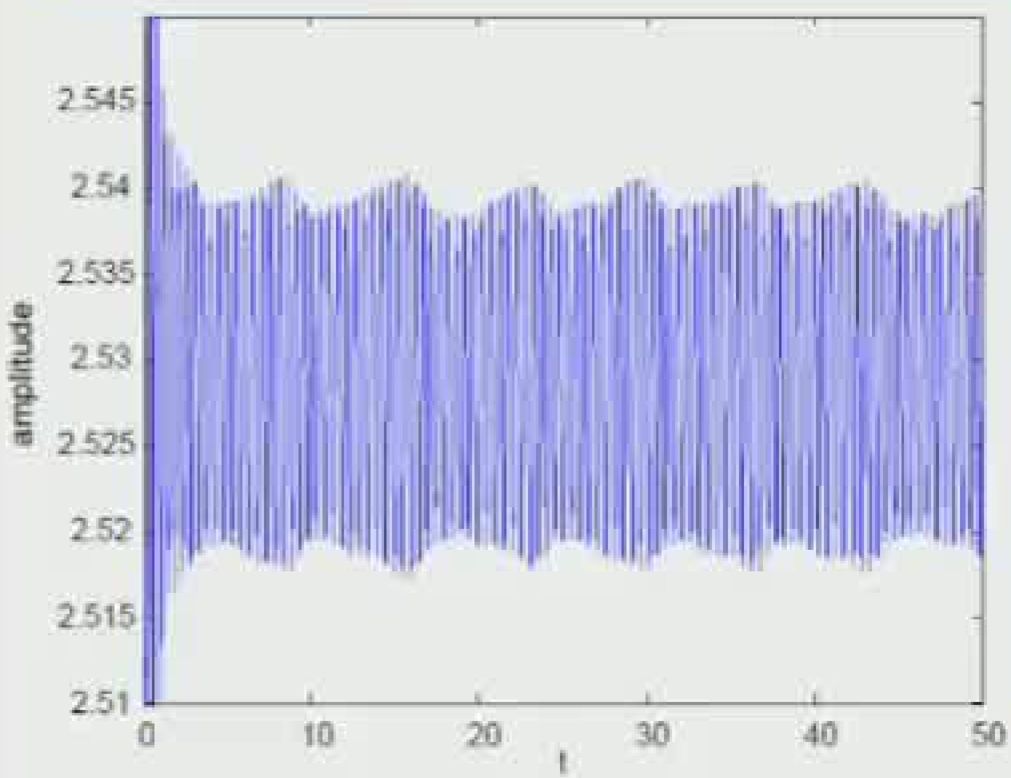

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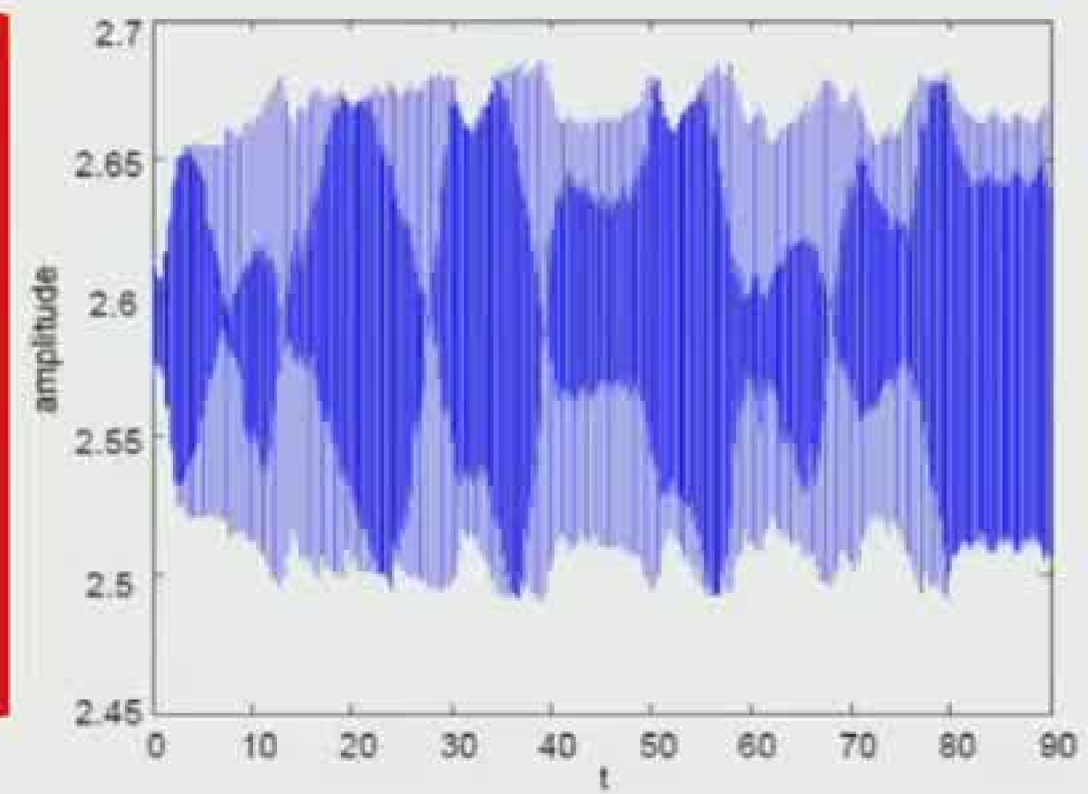

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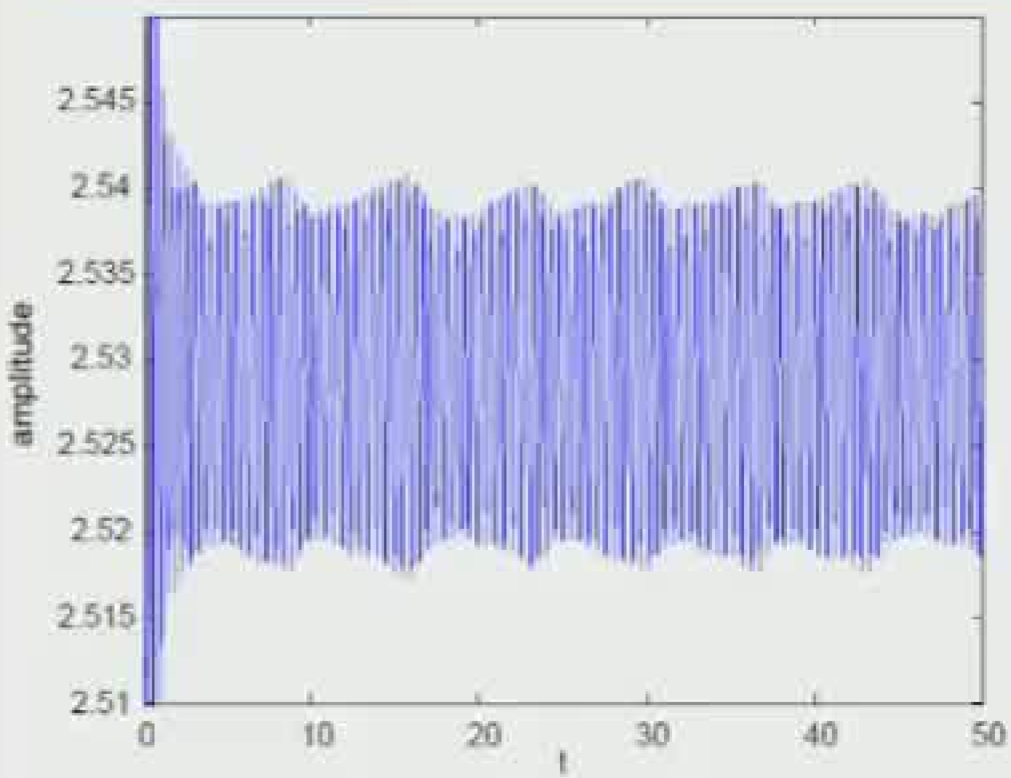

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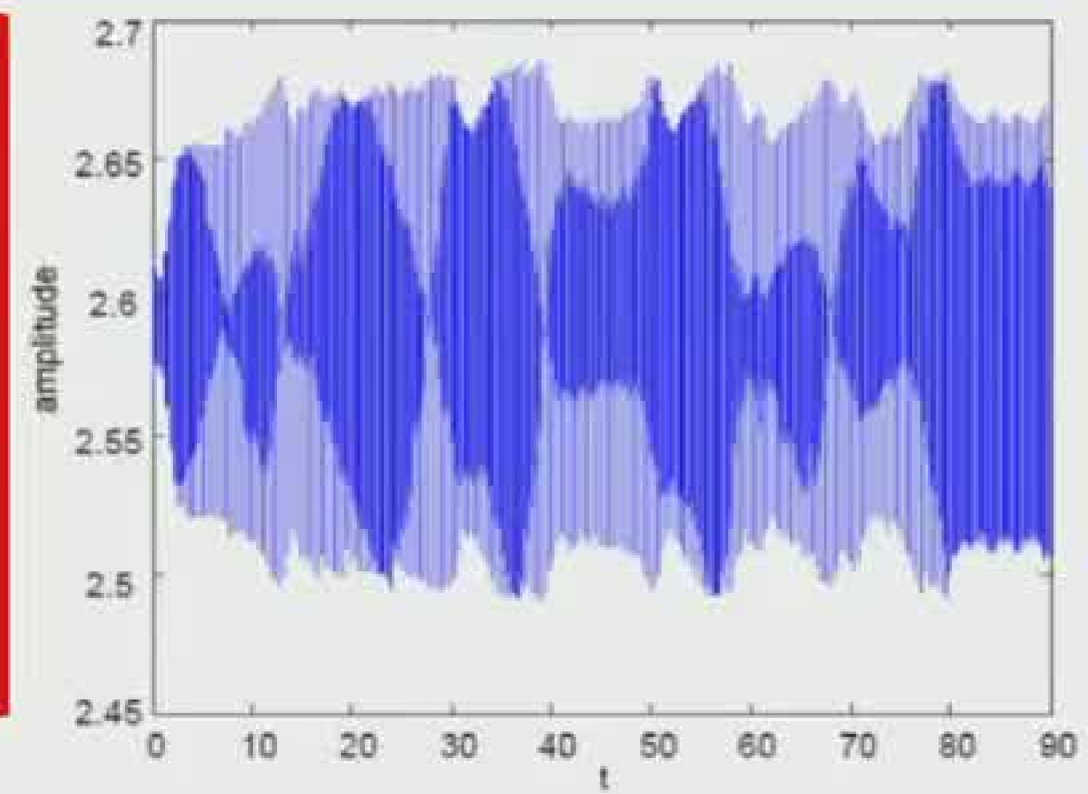

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- Unlike in fluid dynamics, there are no stable patterns of increasing complexity bifurcating from (the boundary of) the Busse balloon?

  [From morphogenesis to morphothanatos & nothing else??]

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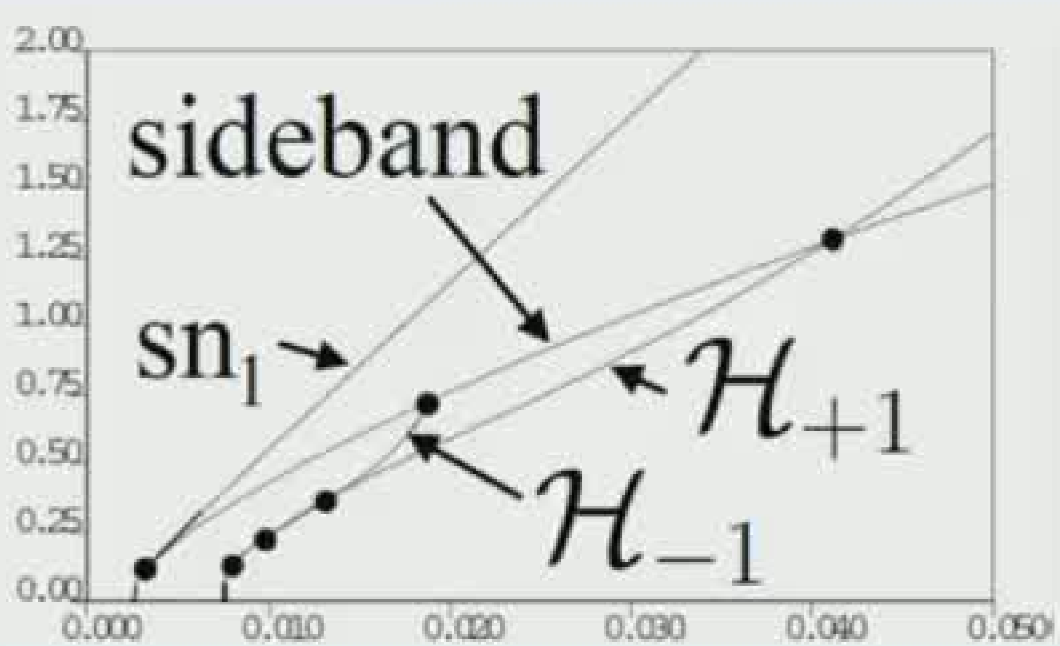
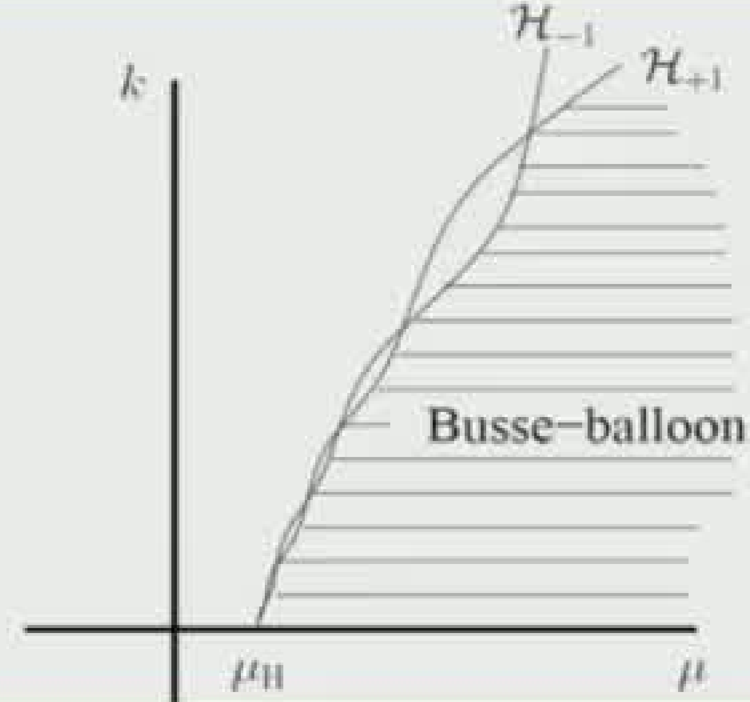
[The same behavior is exhibited by the Gierer-Meinhardt model!]



↔



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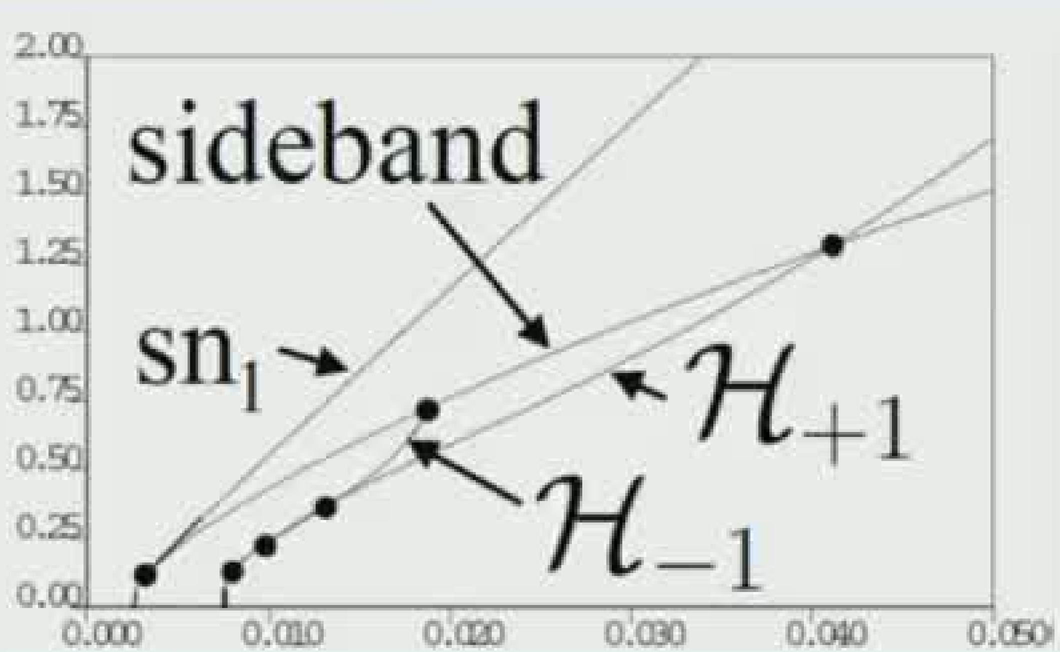

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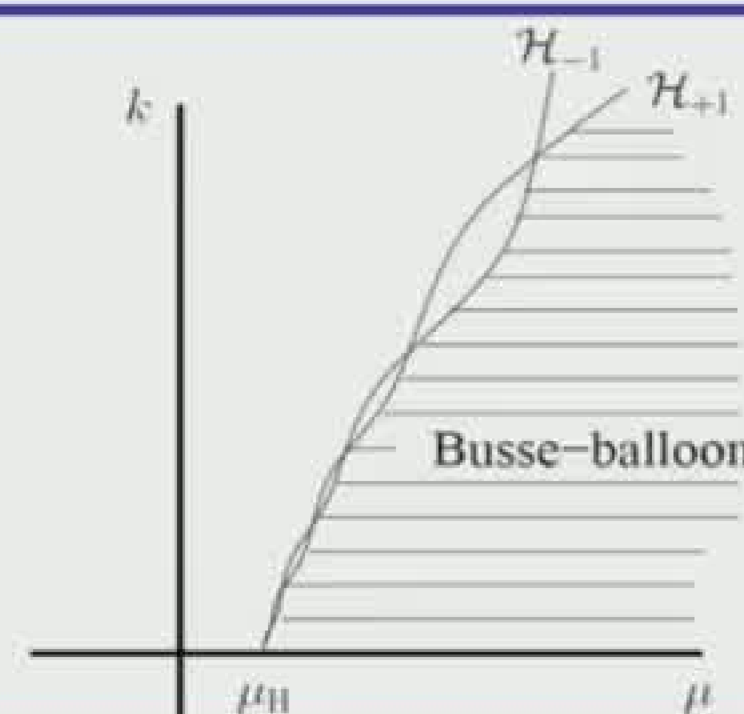

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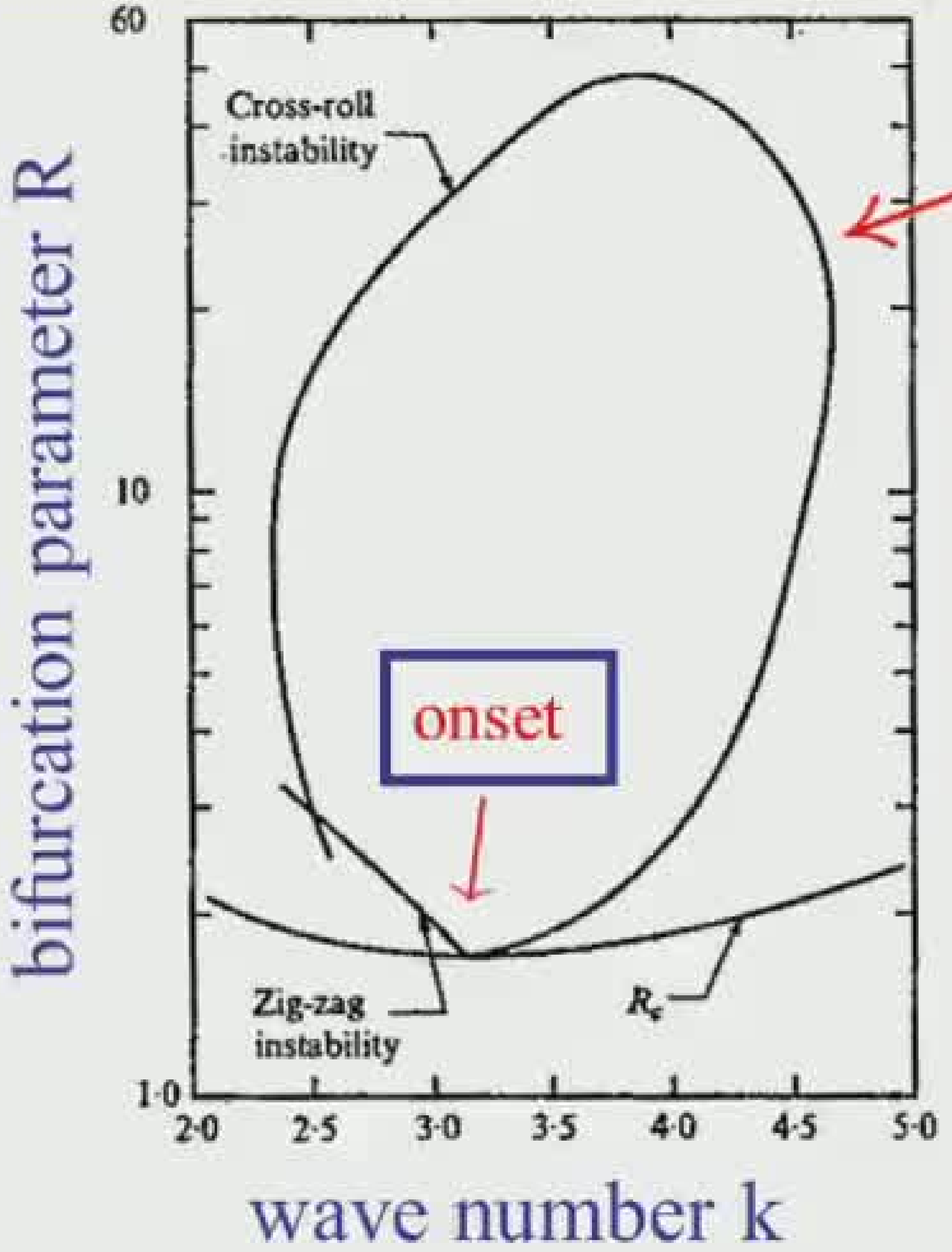

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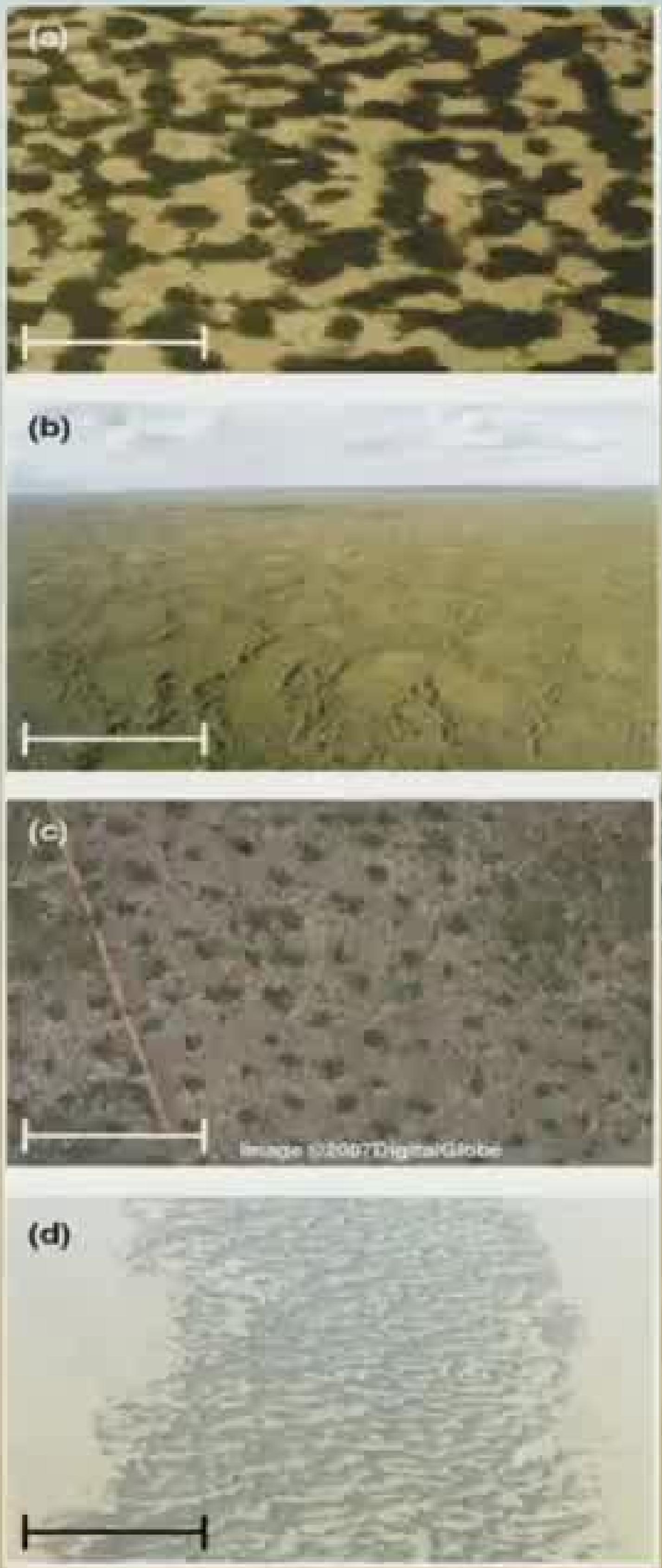
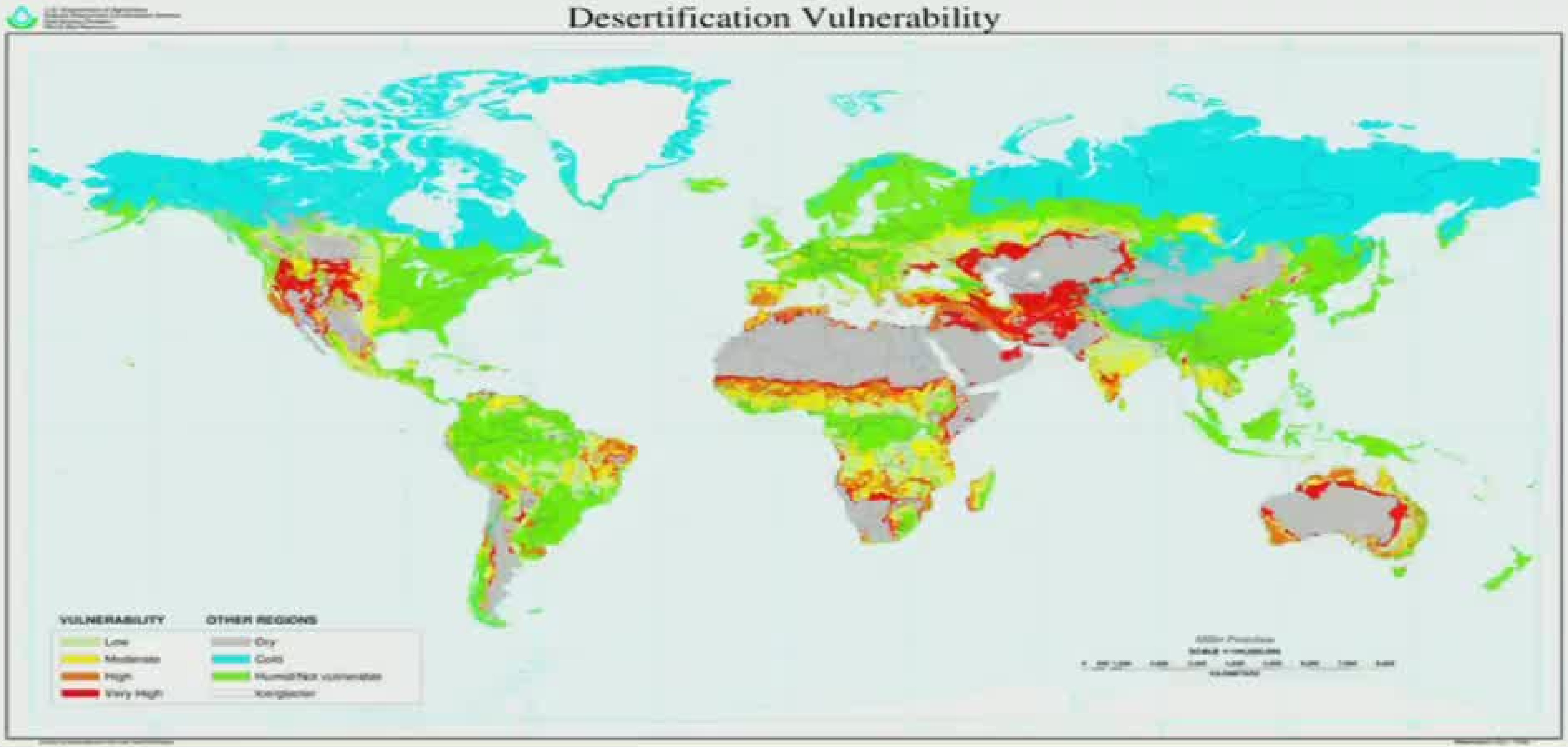
Intermezzo: spatial ecology & singular patterns



[Rietkerk & van de Koppel, '08]: Pattern formation in ecological systems is driven by counteracting feedback mechanisms on widely different spatial scales.

Mathematics:
The dynamics of singular patterns.

SPATIAL ECOLOGY: vegetation patterns & desertification



"Drylands occupy approximately 40–41% of Earth's land area and are home to more than 2 billion people. It has been estimated that 10–20% of drylands are already degraded, and that a billion people are under threat from further desertification."

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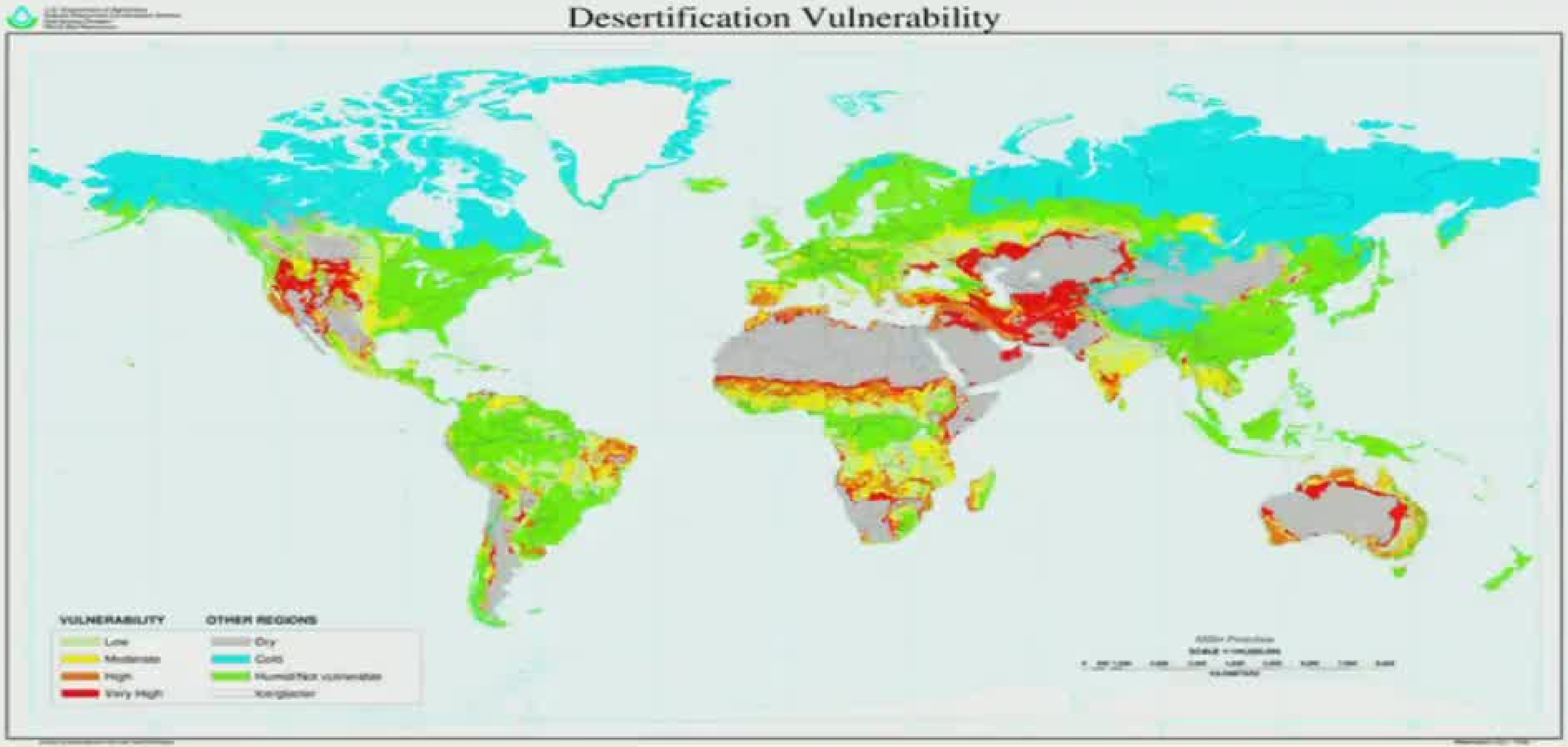

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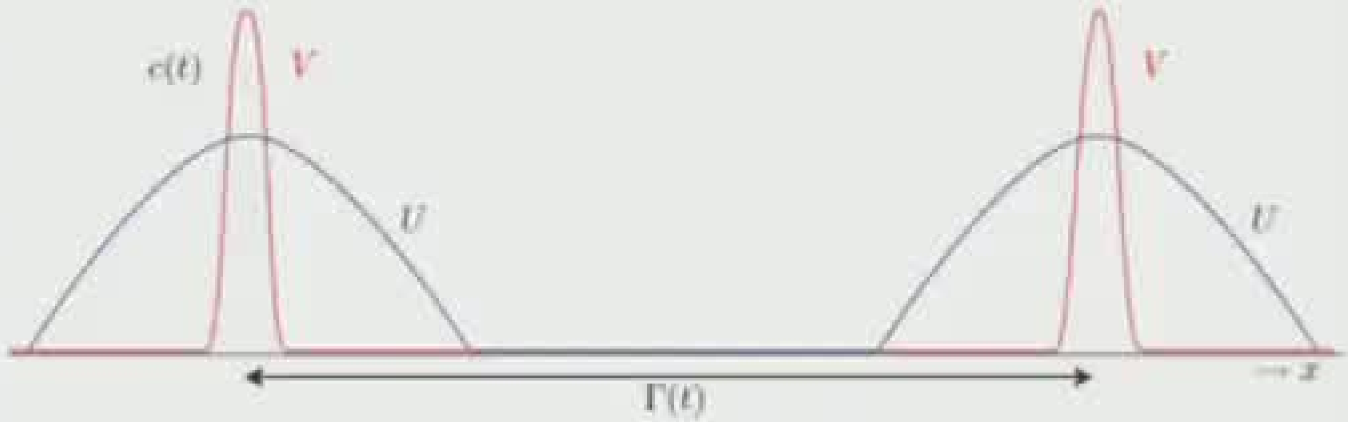
Weak ↔ Strong interactions

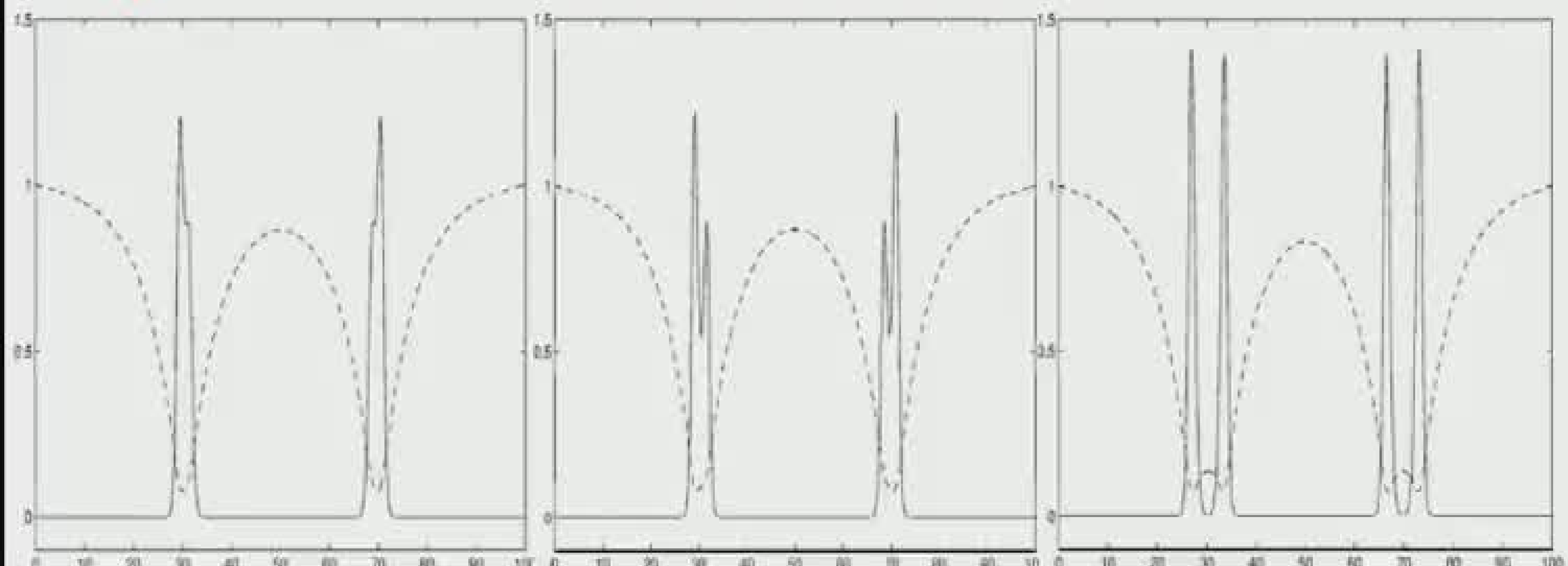


Assumption: $|\Gamma(t)| > C \gg 1$

WEAK interactions:
pulses are so far apart that they only 'communicate' through exponentially small 'tail-tail' interactions → pulses behave as 'particles'.

[Ei, Mimura, Promislow, Sandstede, Zelik, …]

'Exploit' the singular structure: SEMI-STRONG interactions.



STRONG interactions: the pulse splitting process → all components change.

What is a 'singular pattern'?

A singular pattern is a(n evolving) solution of a multiple scale system – typically a singularly perturbed PDE – where $0 < \varepsilon \ll 1$ 'measures' the perturbation.

Why would one study singular patterns?

- Multiple scale systems – or SP PDEs – appear naturally in 'applications'.
- While exhibiting behavior of a richness comparable to general, non-SP systems, the SP nature of these systems provides a framework – based on various $\varepsilon \to 0$ limits – by which the behavior of the patterns can be unraveled.

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Structure of the talk

- Autocatalytic reactions & Gray-Scott dynamics.

→ Semi-strong pulse interactions.

- Spatial ecology, vegetation patterns and desertification.

→ Pattern dynamics of under slowly varying circumstances.

- The Busse balloon: turbulence ↔ desertification.

→ Slowly nonlinear singularly perturbed equations.

→ A fine-structure of the boundary of the Busse balloon.

- Pattern dynamics under slowly varying conditions.

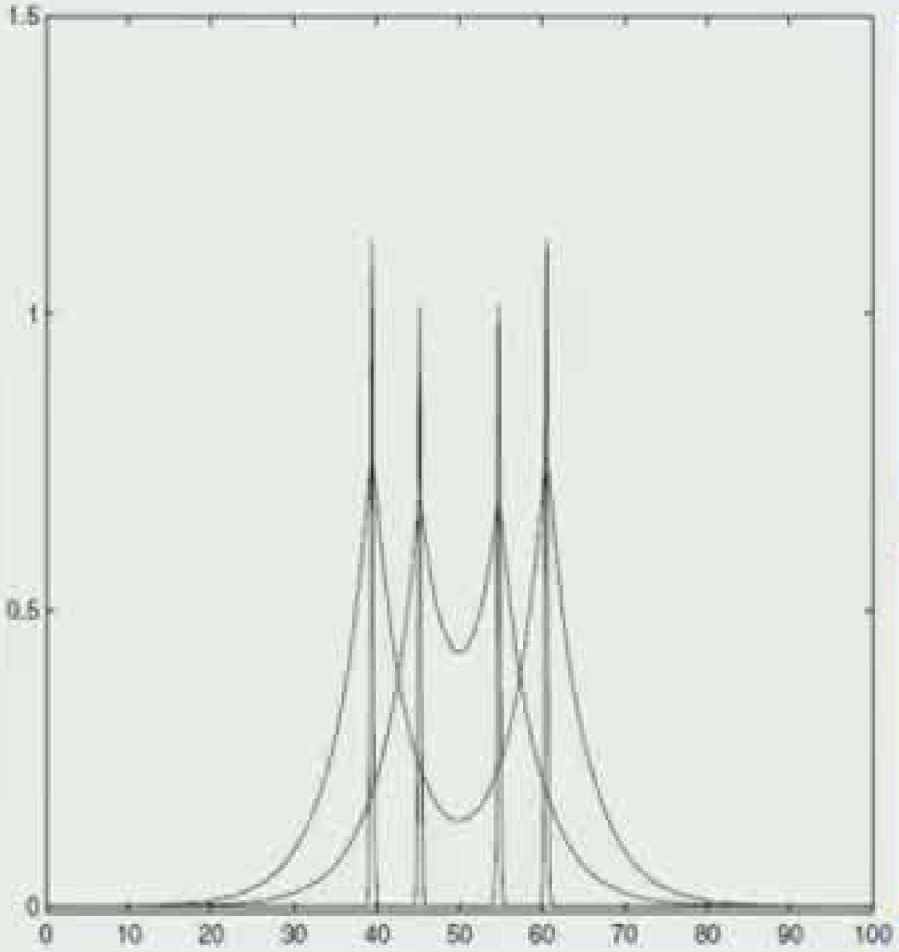
← The impact of the speed of change.

← Catastrophic ↔ gradual decline.

- Conclusions & Discussion.

+ some intermezzos

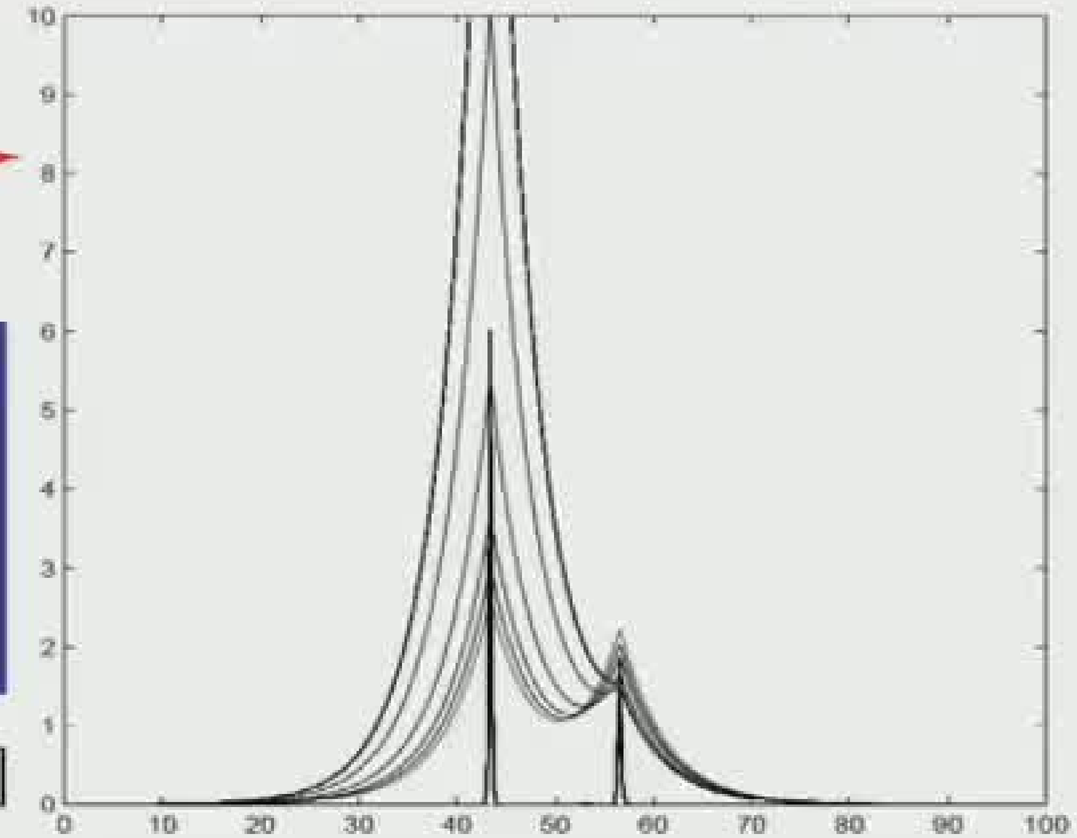
Semi-strong interactions (of singular patterns!)



A pair of pulses in semi-strong interaction: the 'fast' component is exponentially small between pulses, the 'slow' component varies 'at leading order'.

→ Pulses change in amplitude & shape during the evolution/interaction, they may even 'push' each other through a 'dynamic bifurcation'.

Finite-time blow-up induced by semi-strong pulse interactions. →



Biology! Methods developed in context of (the GS and) the Gierer-Meinhardt (GM) model for morphogenesis.

[Ward, Kolokolnikov, Nishiura, …, D., Kaper & students]

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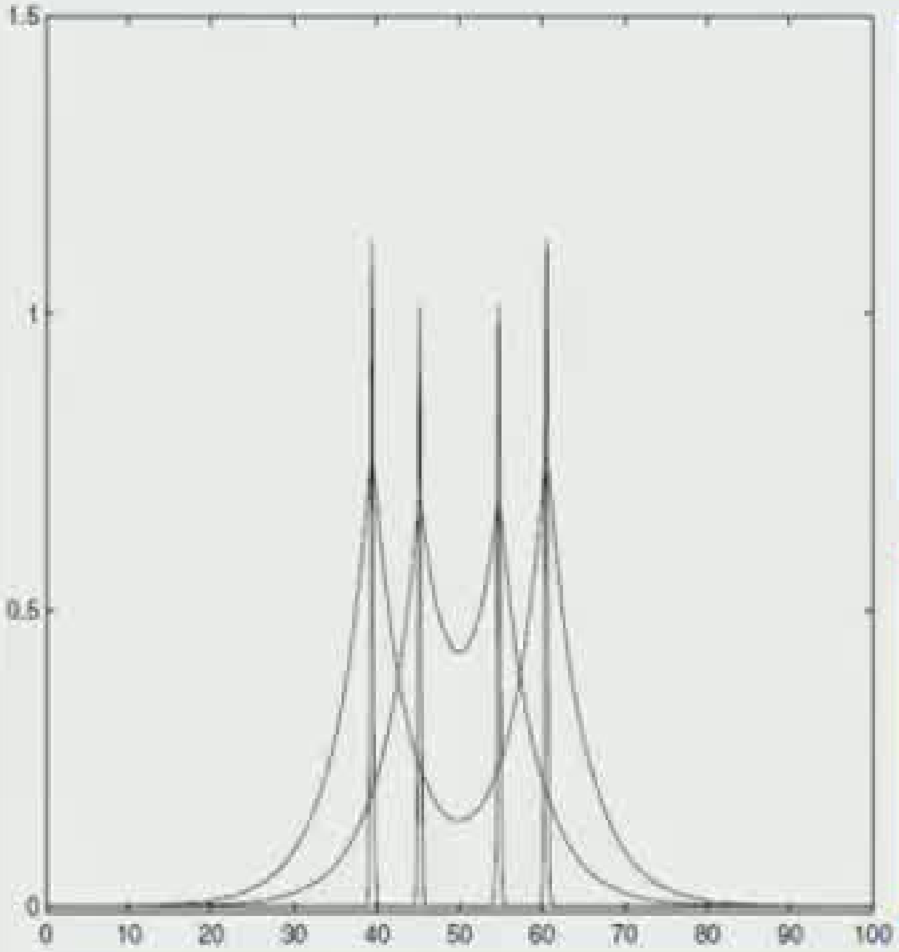

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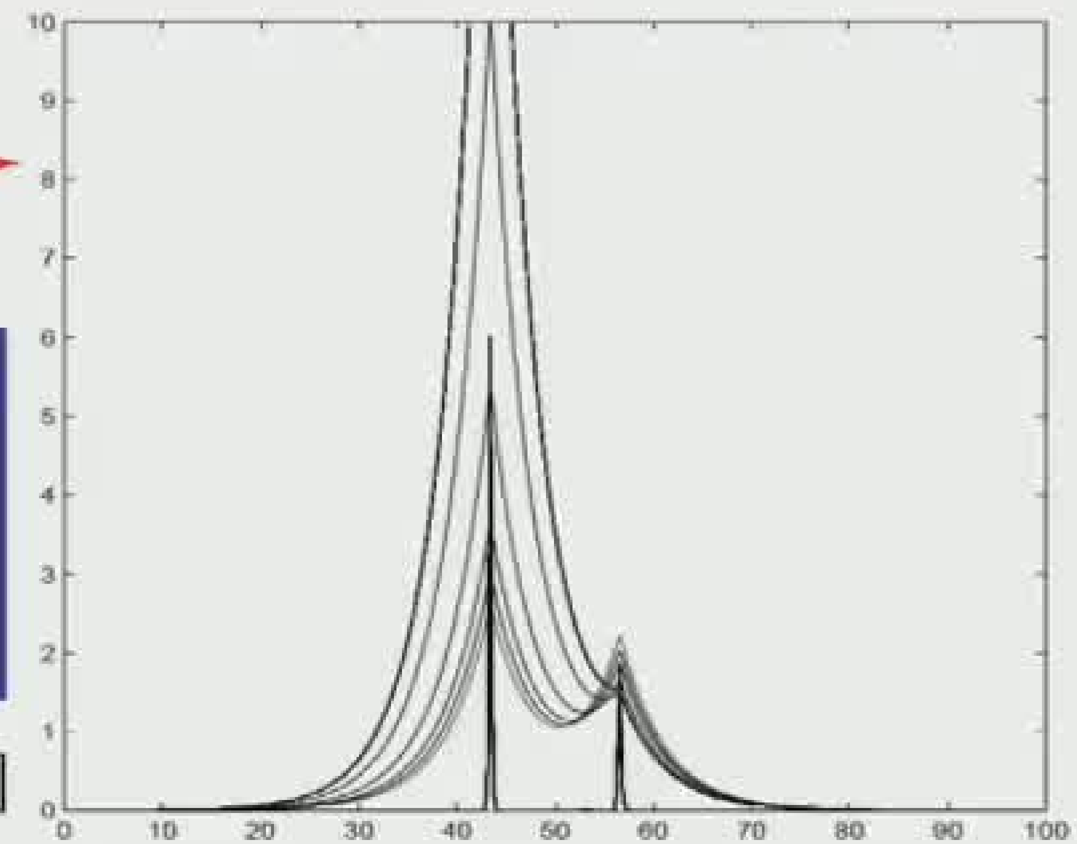

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- Partly discontinuous/fast – catastrophic – partly gradual.
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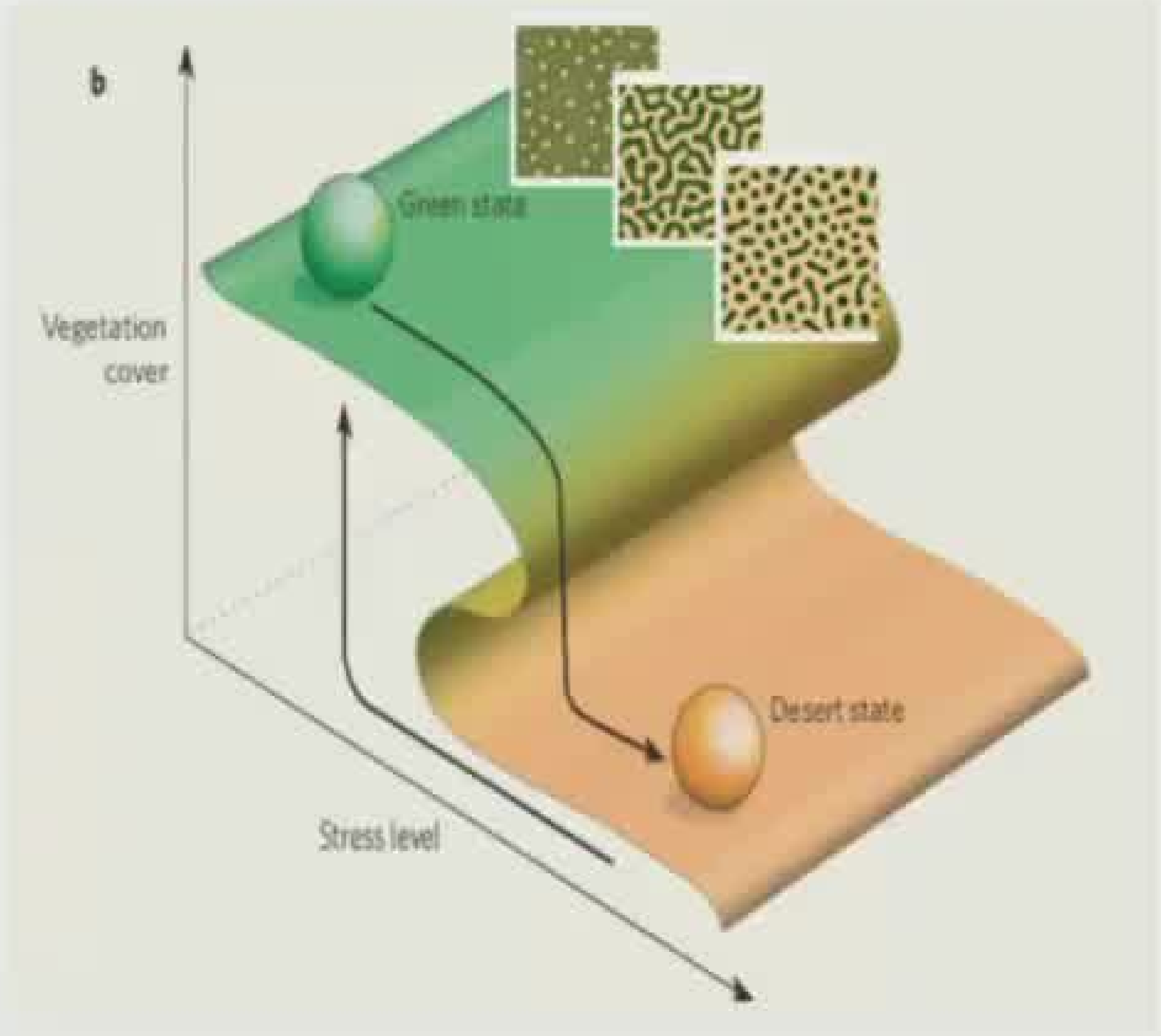

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The mathematical perspective:
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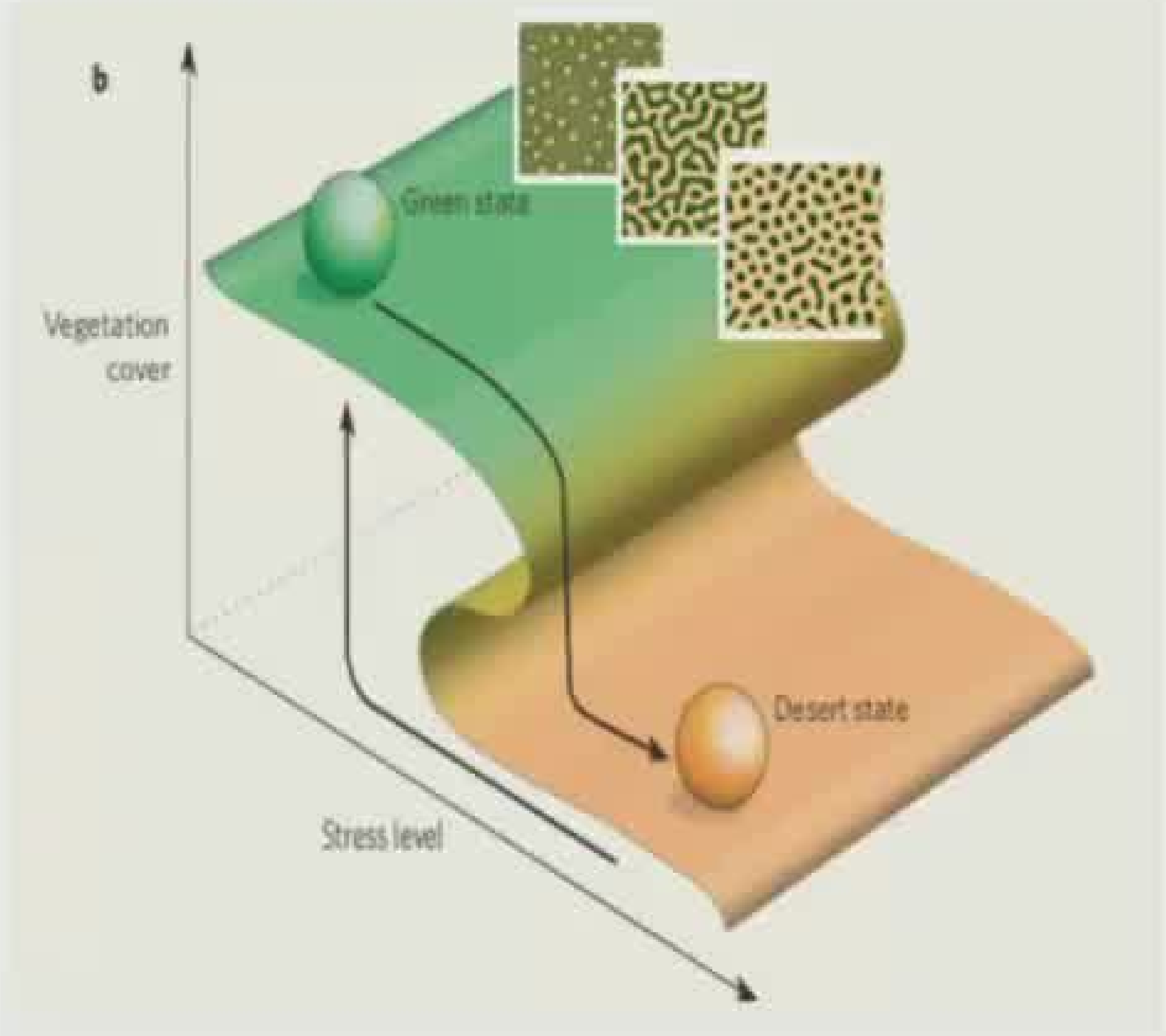

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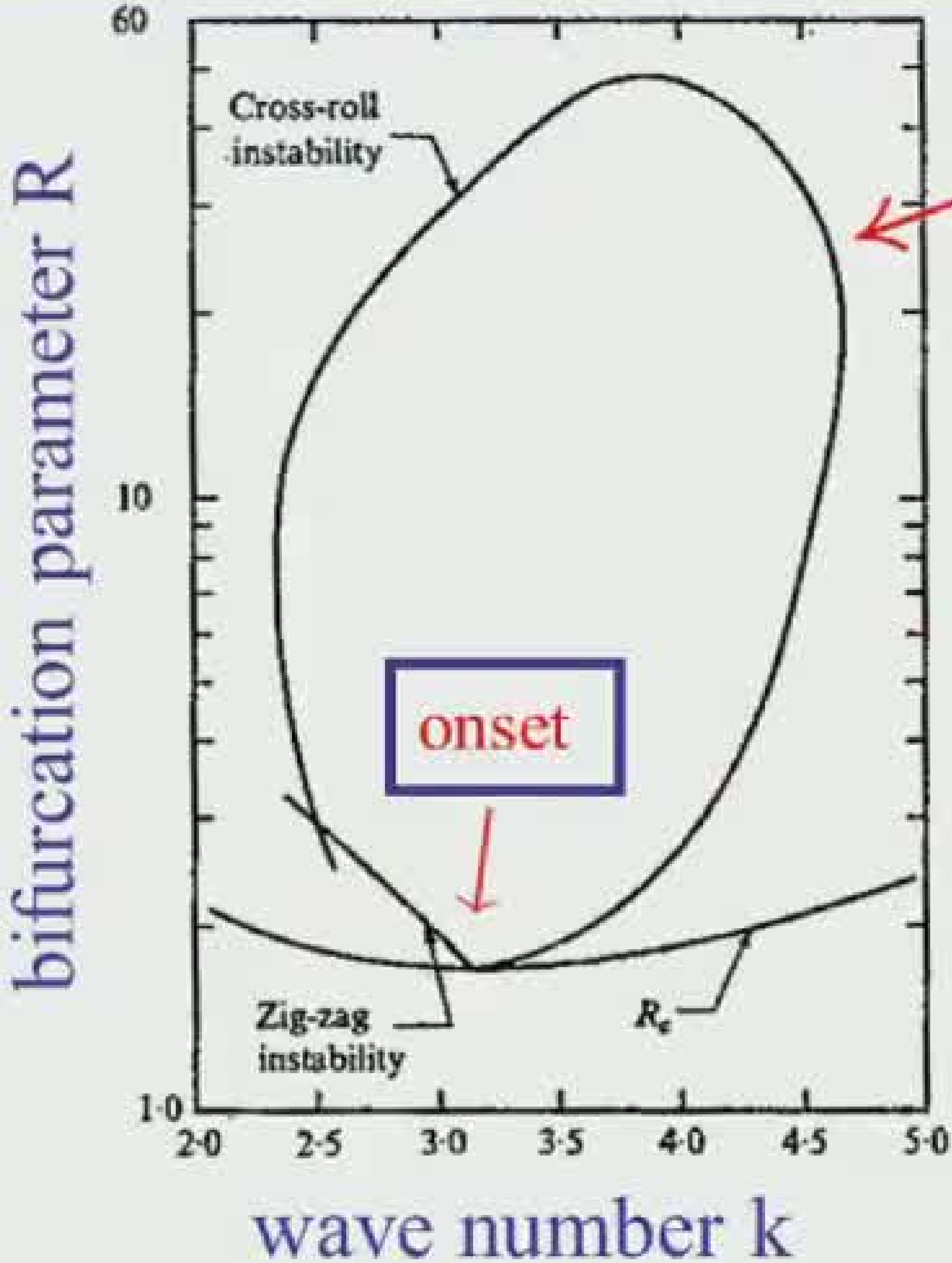

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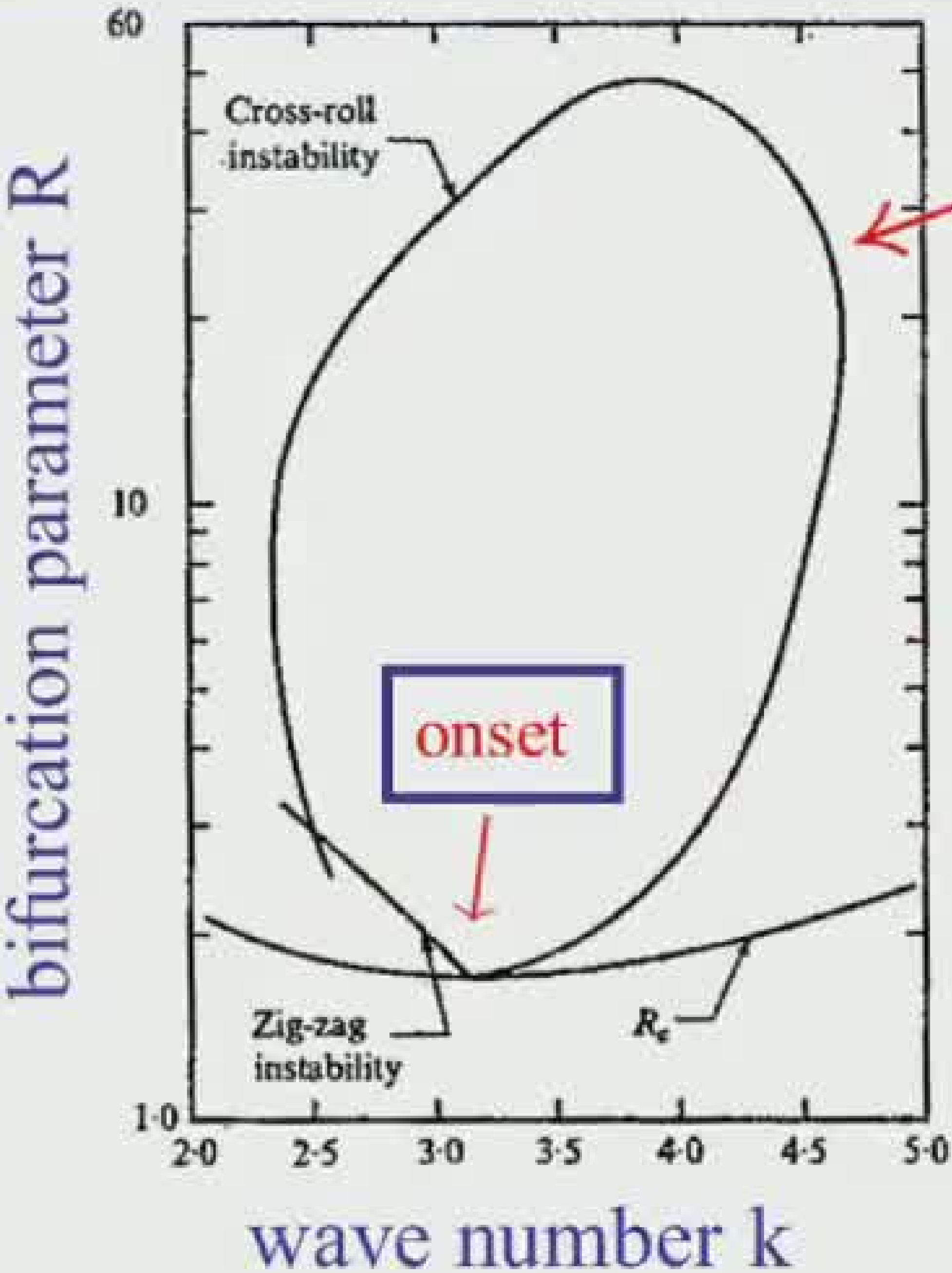

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Slowly linear & slowly nonlinear SP RDEs

General SP RDE (for simplicity in 2 components),

$$\begin{cases} U_t &= \quad U_{xx} &+ \quad F(U,V;\varepsilon) \\ V_t &= \varepsilon^2 V_{xx} &+ \quad G(U,V;\varepsilon) \end{cases}$$

Outside the 'fast' V-pulses (or fronts) – where, by translation, $V = 0$ (+ exp. small) – the dynamics are governed by,

$$U_t = U_{xx} + F(U,0;\varepsilon)$$

$F(U,0;\varepsilon)$ is linear in U for all (??) explicit models in the literature (GS, GM, gKGS, FH-N, Schnakenberg).

The 'prototypical' GS-, GM-, etc. models belong to the special class of 'slowly linear' SP RDEs,

a significantly restricted (?) subclass of the general 'slowly nonlinear' SP RDEs.

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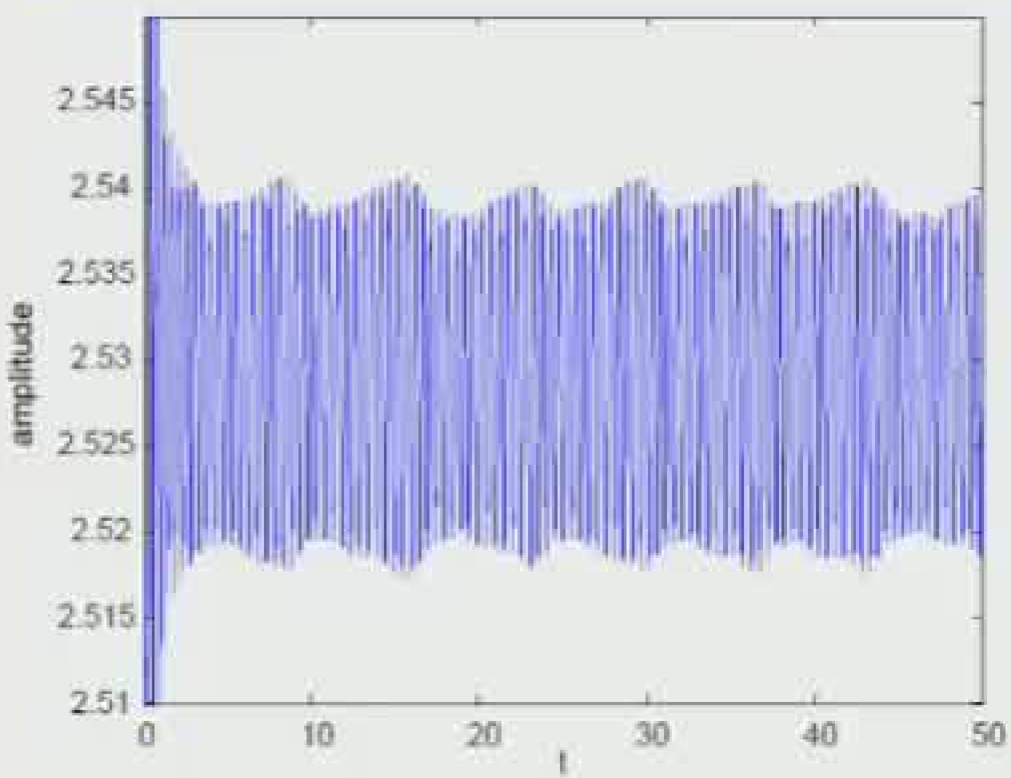

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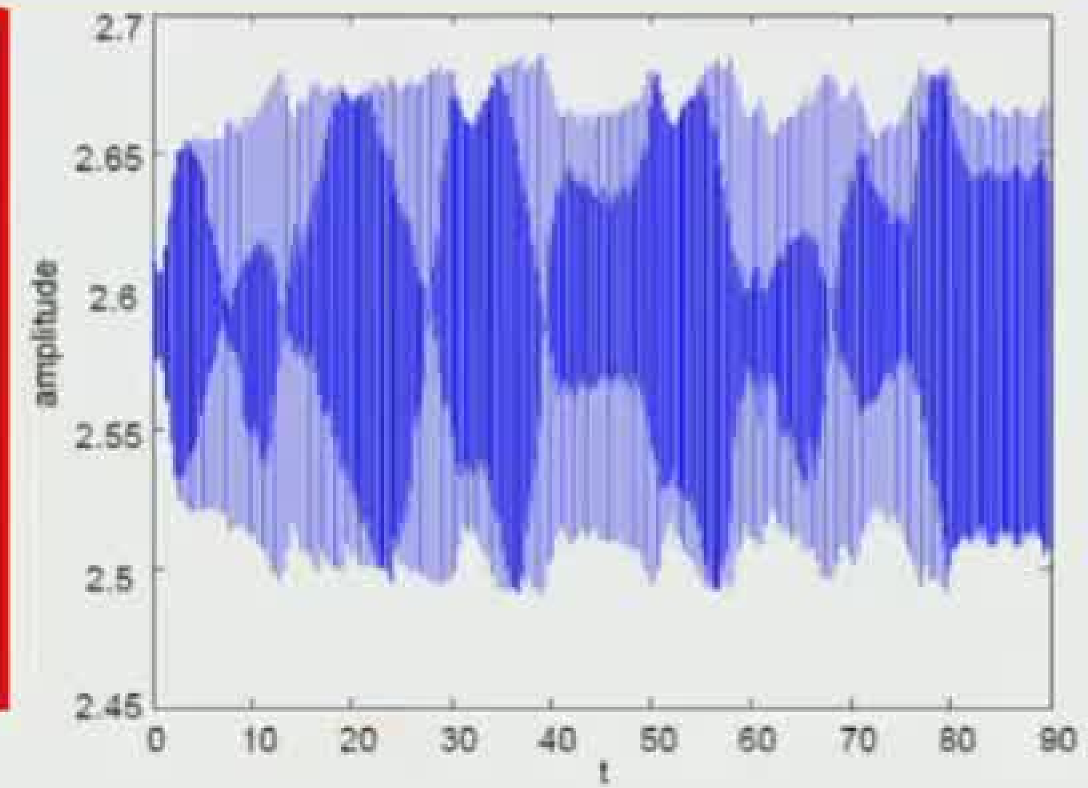

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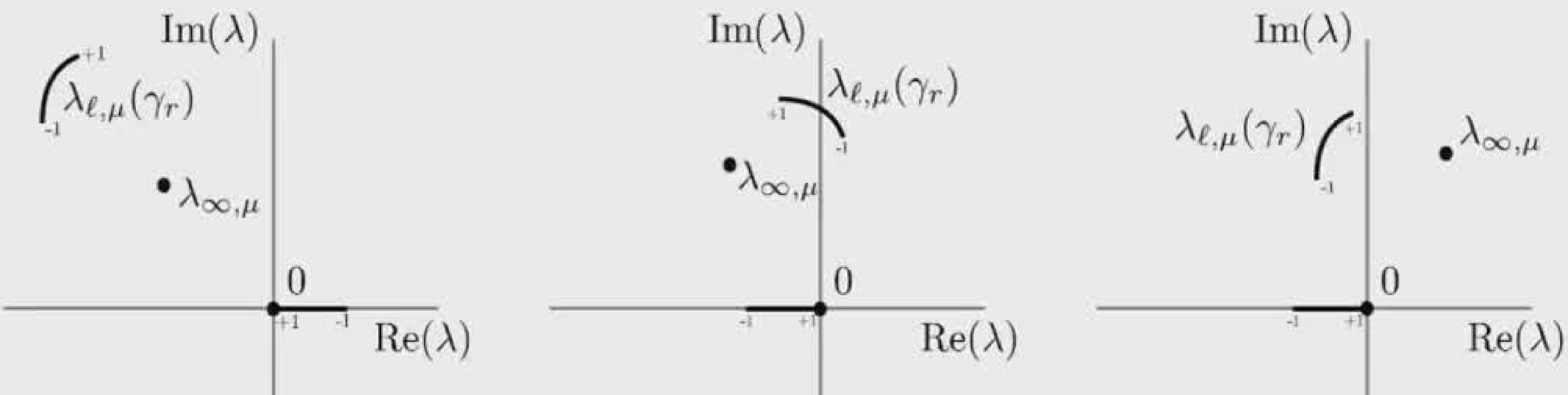

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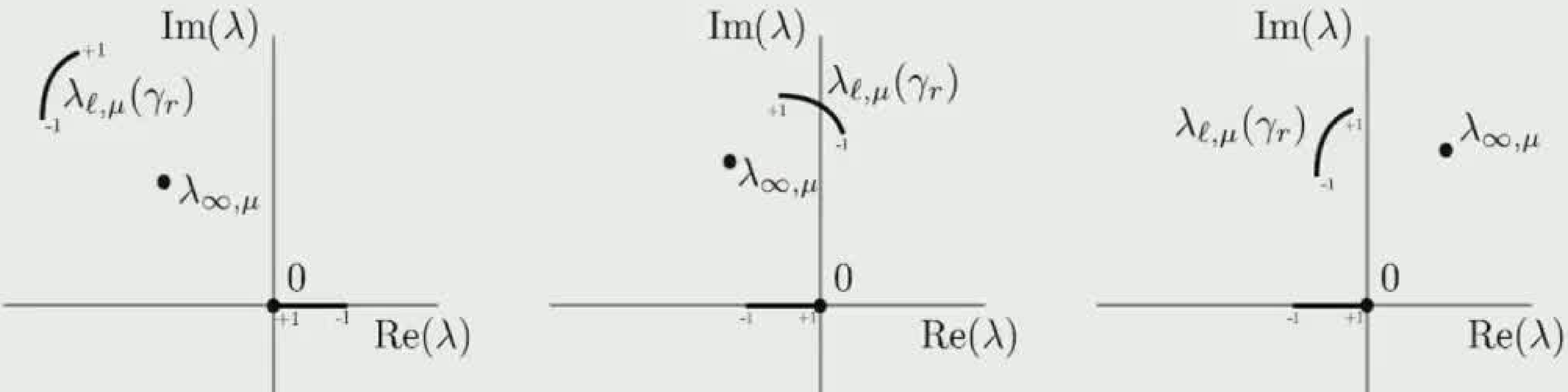
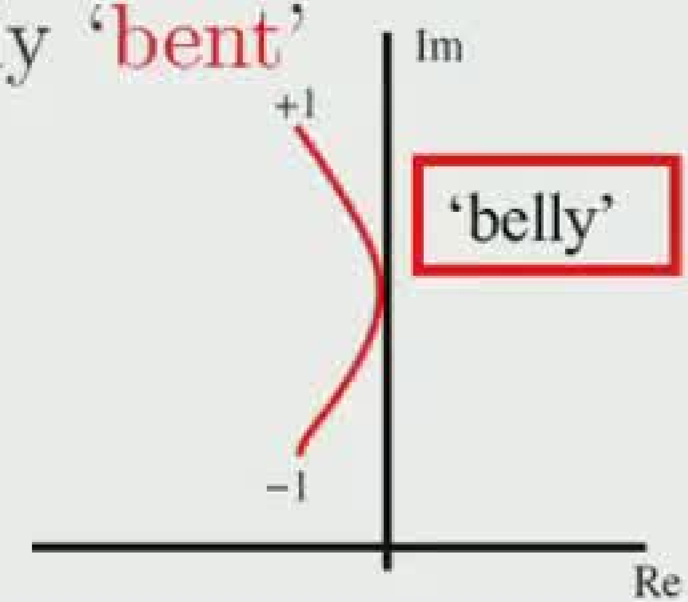


The ‘belly dance’

- $\lambda_\ell(\gamma)$ is weakly ‘bent’



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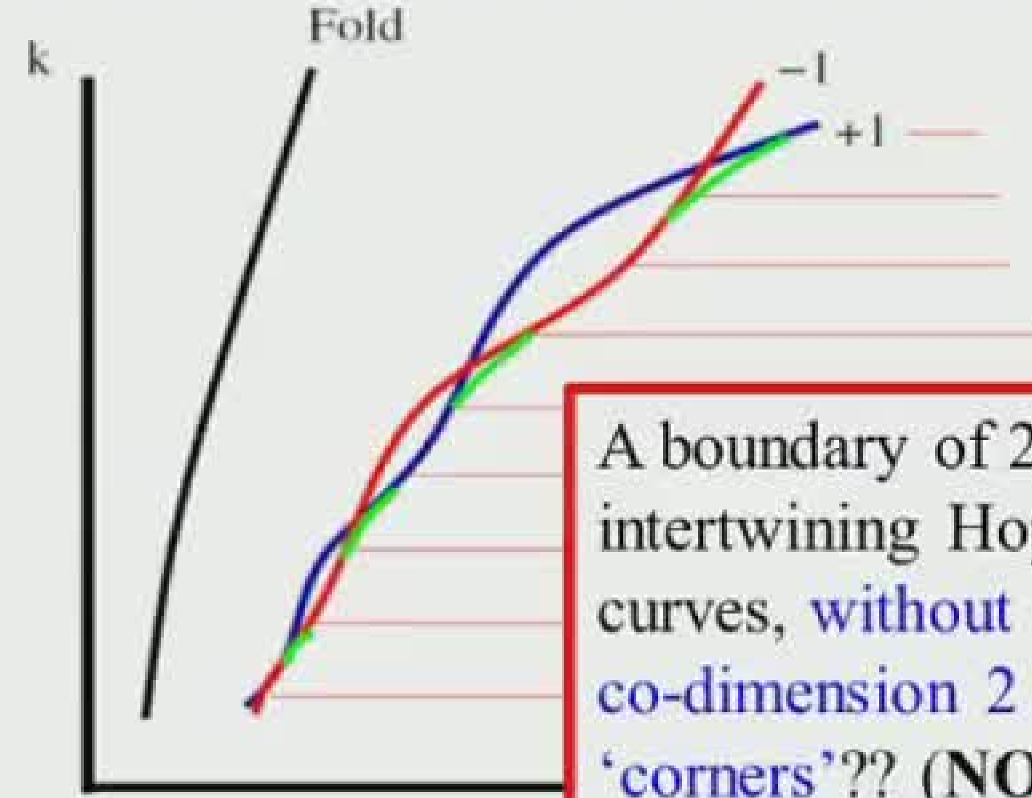
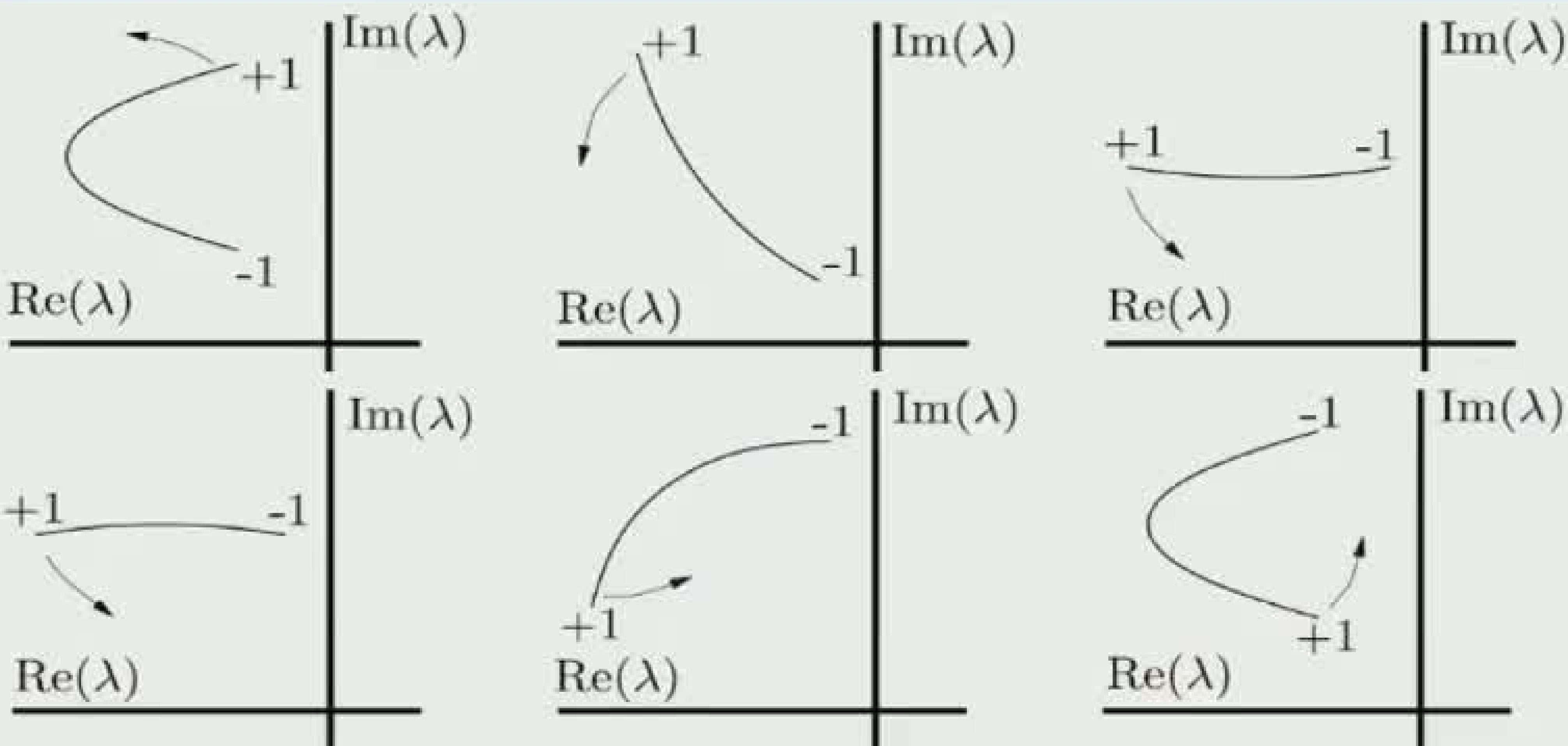


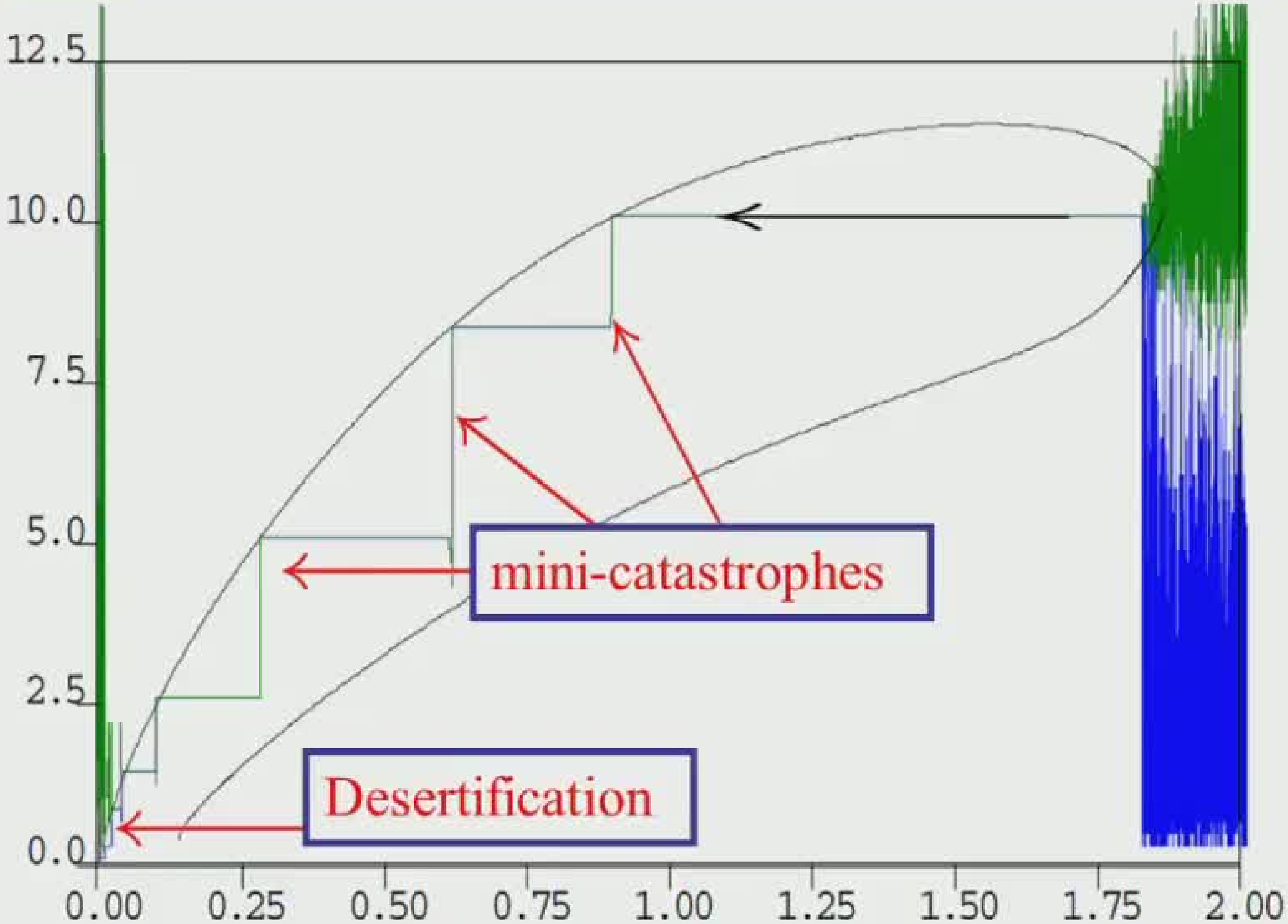
A boundary of 2 intertwining Hopf curves, without co-dimension 2 ‘corners’?? (**NO!**)

- In general slowly nonlinear SP RDEs, a ‘belly dance’ takes place.



The belly ‘flips’ to the other side (w.r.t the connecting straight line between its endpoints) in half a rotation of the spectral curve.

‘A game of billiards’ inside a Busse balloon (??)



‘Dynamics under slowly varying parameters’ is a well-studied – but also quite recent – subject in finite-dim. ODEs.

It is a novel subject of study in PDEs.

Which rules are driving the reflection process?

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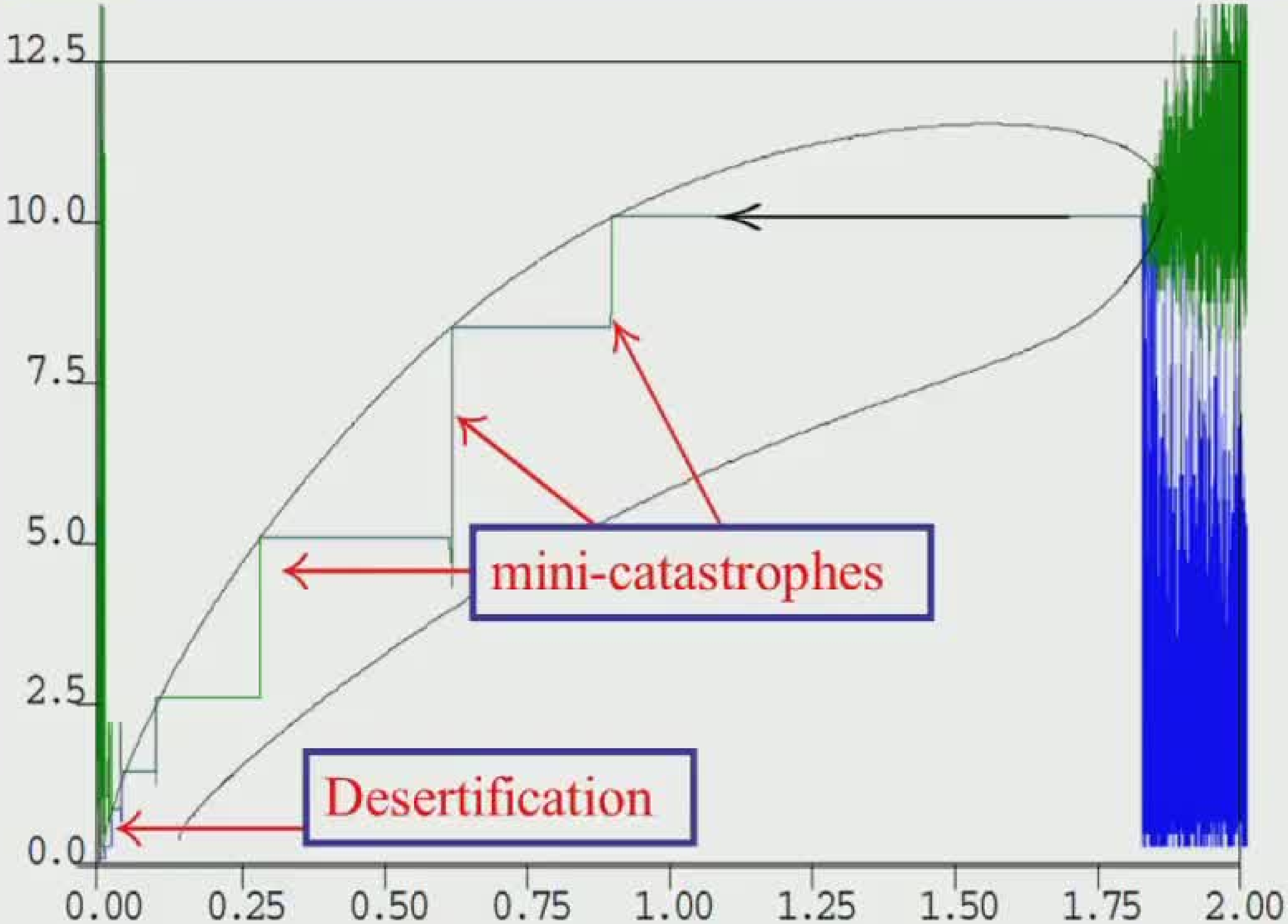

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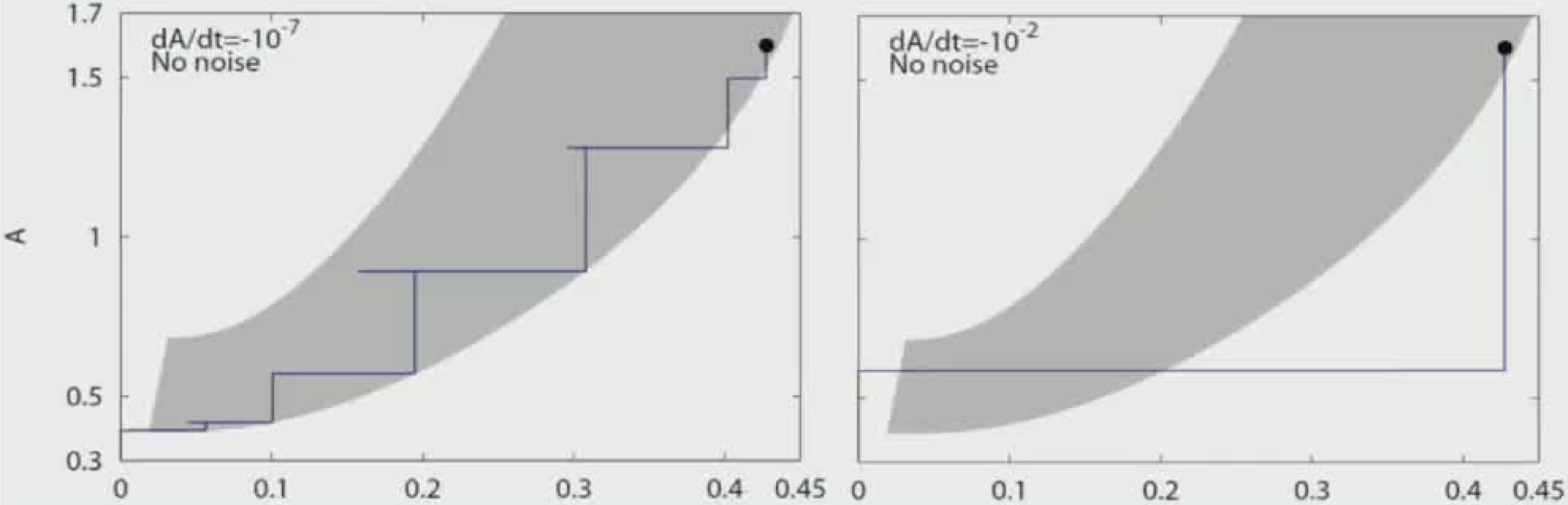
The impact of the rate of change *dA/dt*



- If *dA/dt* is 'too large', then the 'internal dynamics' of the system cannot adapt.
- The first (& only) (mini-)catastrophe is delayed. Desertification occurs at the initial (Turing) wavenumber. However, 'morphotanatos' sets in for a higher value of *A*.

(↔ Collapse to bare soil with 'sufficient' rainfall.)

[Siteur, Siero, D., Eppinga, Rademacher & Rietkerk, '14 & '16]

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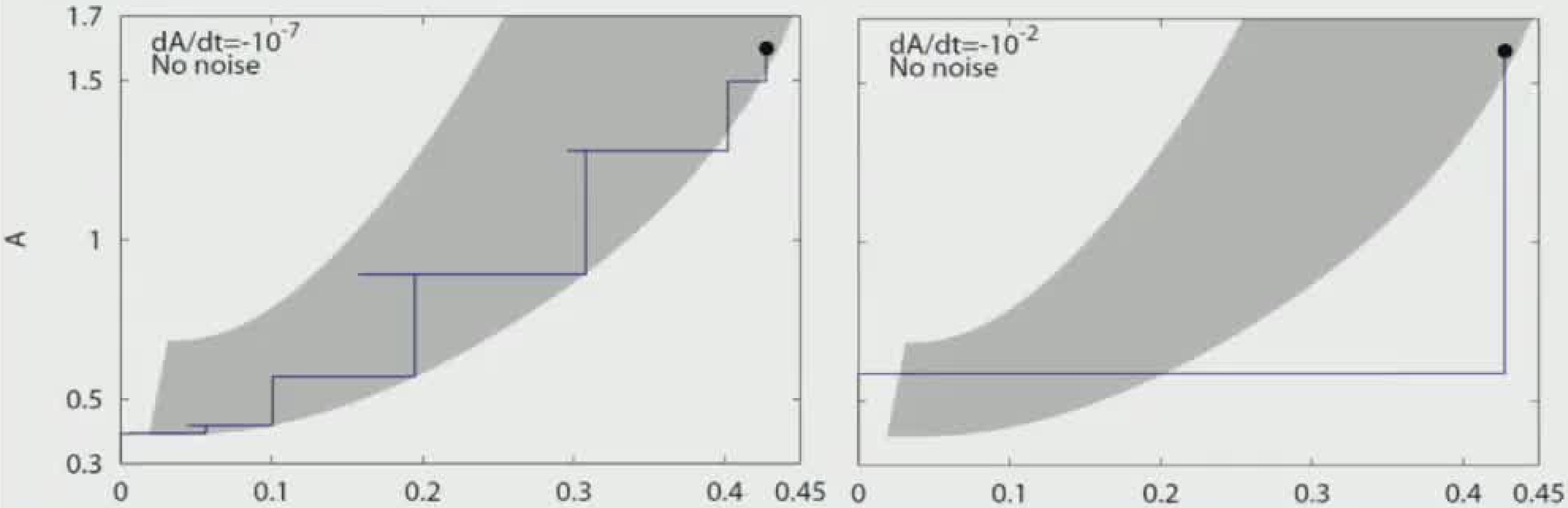

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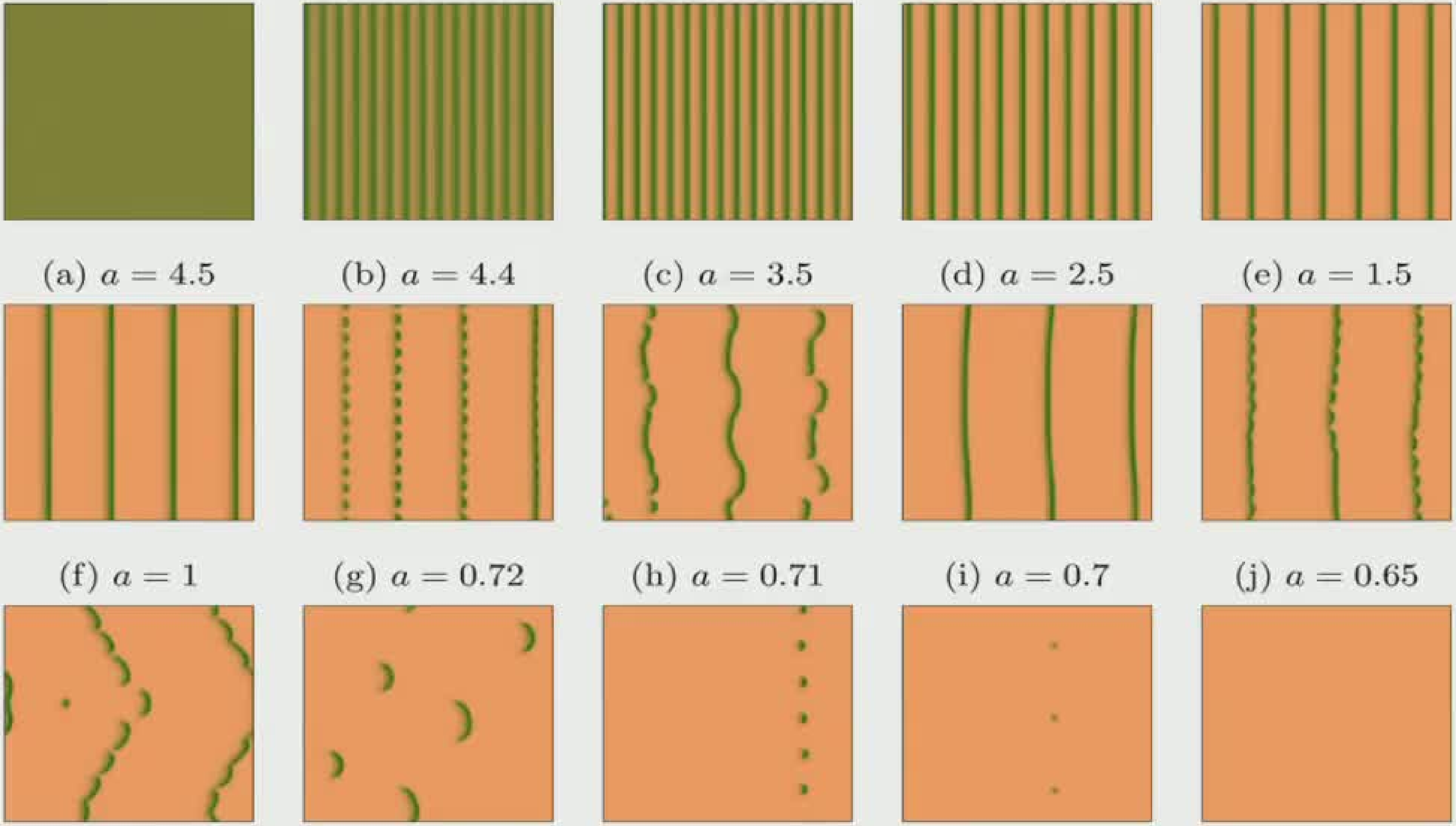
Intermezzo: the dynamics of stripes



(a) $a = 4.5$ (b) $a = 4.4$ (c) $a = 3.5$ (d) $a = 2.5$ (e) $a = 1.5$

(f) $a = 1$ (g) $a = 0.72$ (h) $a = 0.71$ (i) $a = 0.7$ (j) $a = 0.65$

(k) $a = 0.55$ (l) $a = 0.3$ (m) $a = 0.18$ (n) $a = 0.13$ (o) $a = 0.12$

[Siero, D., Eppinga, Rademacher, Rietkerk, Siteur, '15]

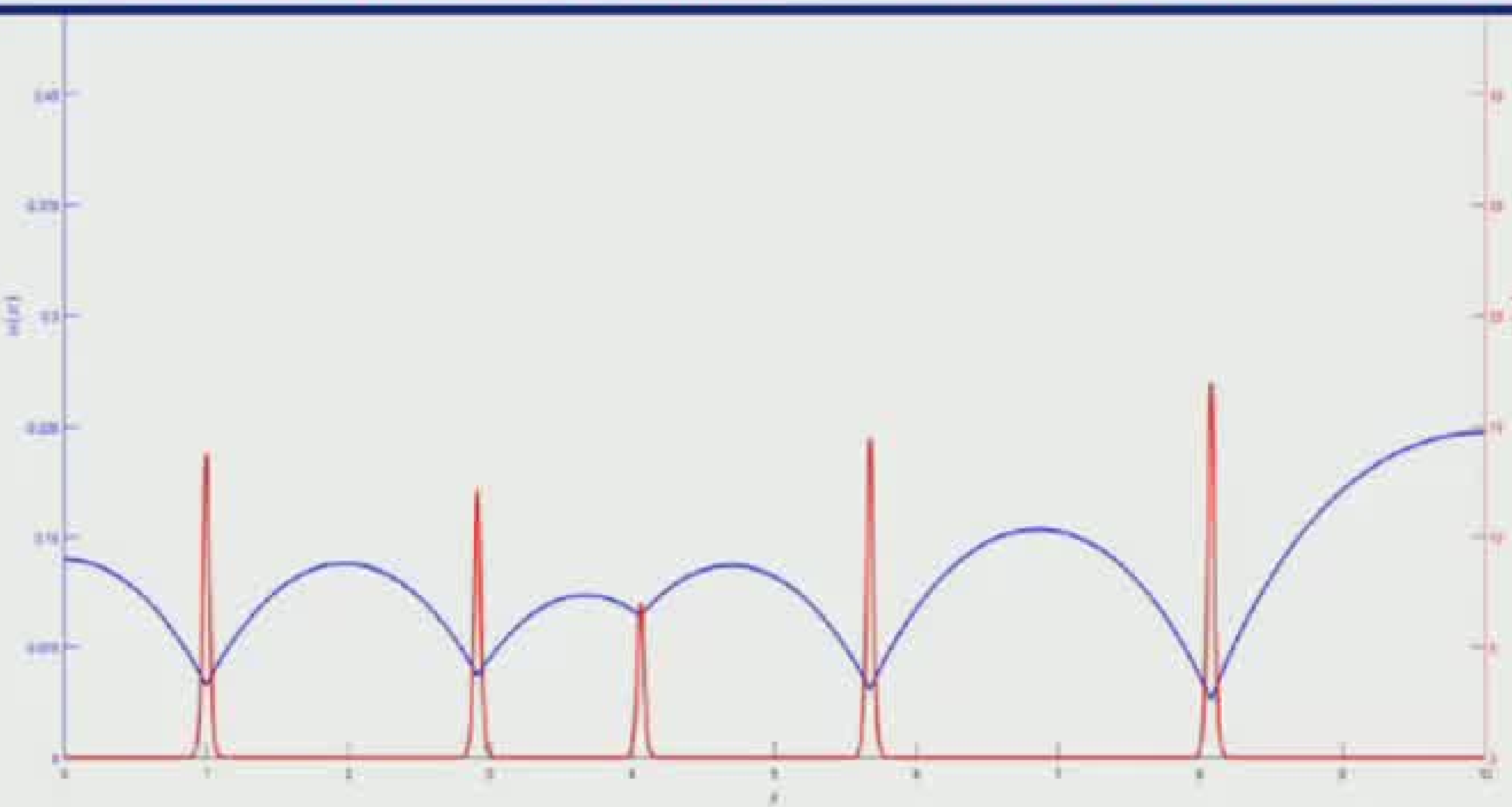
Catastrophic ↔ gradual?

<u>Some conjectures</u>

[Bastiaansen & D.,'17] – in progress

- (Mini-)catastrophes occur in regular patterns.
- Irregular patterns follow a more gradual course.
- Systems naturally evolve towards regularity.

Internal time scale ↔ Time scale environmental change



Follow the reduced J– ⟶ J-k–dim. ODE dynamics as the pattern 'jumps'/'falls' over the edge of its invariant manifold.