Spatiotemporal Intermittency and Chaos in a Ginzburg Landau System for Oscillatory Instabilities in Anisotropic Systems

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Prototype Example for Anisotropic Systems
Electroconvection in Nematic Liquid Crystals
Symmetries in 2d axial anisotropic systems comprise:

Reflection invariances across and along a distinguished symmetry axis (director axis in case of nematic electroconvection)

In 2d (infinitely) extended anisotropic systems:

2 translation and 2 reflection symmetries: $E(1) \times E(1)$

Since there is no Rotation Symmetry:

Instabilities of a uniform state involve only finitely many critical wave numbers

(in contrast to isotropic systems, where one has a circle of critical wave numbers)
Three Generic Possibilities for an Instability

**Oblique Case:**
4 critical wave numbers
$(\pm p_c, \pm q_c)$

**Normal Case:**
2 critical wave numbers
$(\pm p_c, 0)$

**Third Case:**
critical wave number $(0,0)$
Oblique Hopf Instability

Linearized system admits two counterpropagating pairs of traveling wave solutions

Extend Study to Nonlinear System through **Weakly Nonlinear Analysis**
Represent field variables through small and slowly varying amplitudes $\varepsilon A_1$, $\varepsilon A_2$, $\varepsilon A_3$, $\varepsilon A_4$:

$$ U(t, x, y, z) = \varepsilon \{ A_1 U_1(z)e^{i(p_c x + q_c y)} + A_2 U_2(z)e^{i(-p_c x + q_c y)} + A_3 U_3(z)e^{i(-p_c x - q_c y)} + A_4 U_4(z)e^{i(p_c x - q_c y)} \} e^{i\omega t} + cc + O(\varepsilon^2) $$

where $A_1 = A_1(T, X_+, Y_+)$, $A_2 = A_2(T, X_-, Y_+)$, $A_3 = A_3(T, X_-, Y_-)$, $A_4 = A_4(T, X_+, Y_-)$

with $X_\pm = \varepsilon(t \pm x/\nu_p)$, $Y_\pm = \varepsilon(t \pm y/\nu_q)$, $T = \varepsilon^2 t$, $R - R_c = \varepsilon^2$
Represent field variables through small and slowly varying amplitudes $\varepsilon A_1$, $\varepsilon A_2$, $\varepsilon A_3$, $\varepsilon A_4$:

$$U(t,x,y,z) = \varepsilon \{ A_1 U_1(z)e^{i(p_c x + q_c y)} + A_2 U_2(z)e^{i(-p_c x + q_c y)} +$$
$$A_3 U_3(z)e^{i(-p_c x - q_c y)} + A_4 U_4(z)e^{i(p_c x - q_c y)} \} e^{i \omega_c t} + cc + O(\varepsilon^2)$$

where $A_1 = A_1(T, X_+, Y_+), A_2 = A_2(T, X_-, Y_+), A_3 = A_3(T, X_-, Y_-), A_4 = A_4(T, X_+, Y_-)$

with $X_\pm = \varepsilon(t \pm x / v_p), \ Y_\pm = \varepsilon(t \pm y / v_q), \ T = \varepsilon^2 t, \ R - R_c = \varepsilon^2$

System of **Globally Coupled Ginzburg Landau Equations** for the $A_j$:

$$\partial_T A_1(T, X, Y) = \left( a_0 + D(\partial_X, \partial_Y) + a_1 |A_1|^2 + a_2 \langle |A_2(\chi,Y)|^2 \rangle +$$
$$a_3 \langle |A_3(X-\chi,Y-\chi)|^2 \rangle + a_4 \langle |A_4(X,\chi)|^2 \rangle \right) A_1 +$$
$$a_5 \langle A_2(\chi-X,Y) A_3(\chi-X,\chi-Y) A_4(X,\chi-Y) \rangle$$

where $\langle g(\chi) \rangle = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} g(\chi) \, d\chi = \text{average over } \chi$
Numerically:  \[ A_j(T, X, Y) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_j(m, n, T) e^{i\omega_0 X + \eta_0 Y} \]

Without spatial variations:  \[ A_j = A_j(T) = a_j(0, 0, T) \]
Numerically: \[ A_j(T, X, Y) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_j(m, n,T)e^{imp_0X+inq_0Y} \]

Without spatial variations: \( A_j = A_j(T) = a_j(0,0,T) \rightarrow \) Normal Form for a Hopf Bifurcation with \( O(2) \times O(2) \)-Symmetry:

\[
\frac{d}{dT} A_1 = (a_0 + a_1 |A_1|^2 + a_2 |A_2|^2 + a_3 |A_3|^2 + a_4 |A_4|^2) A_1 + a_5 A_2 \bar{A}_3 A_4
\]
Numerically: \[ A_j(T, X, Y) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_j(m, n, T) e^{im\mu + in\nu} \]

Without spatial variations: \( A_j = A_j(T) = a_j(0,0,T) \rightarrow \text{Normal Form for a Hopf Bifurcation with } O(2) \times O(2)-\text{Symmetry} \):

\[
\frac{d}{dT} A_1 = (a_0 + a_1|A_1|^2 + a_2|A_2|^2 + a_3|A_3|^2 + a_4|A_4|^2)A_1 + a_5A_2\bar{A}_3A_4
\]

**Basic wave solutions:**

- **TW:** Traveling waves \((A,0,0,0)\)
- **SW:** Standing waves \((A,0,A,0)\)
- **TR_x:** Traveling rectangles in \(x\) \((A,0,0,A)\)
- **TR_y:** Traveling rectangles in \(y\) \((A,A,0,0)\)
- **SR:** Standing rectangles \((A,A,A,A)\)
- **AW:** Alternating waves \((A,IA,A,iA)\)

**Other solutions:** Quasiperiodic waves in 4d

Various heteroclinic cycles
Examples of Patterns shown by the GL System

Stable TW

Unstable TW
NF has attracting heteroclinic cycle
$TR_x$-SR-AW
For certain parameters, NF shows a chaotic attractor created through a period doubling cascade; no stable basic waves.

Iterates (variable of a suitable Poincare map) vs $-a_3$
Chaotic NF-Attractors at $-a_{3r}=0.6442$: 4 symmetry-conjugated copies ($R_j=|A_j|$)

Nearby Symmetrized Attractors (2 copies)
Corresponding GL-Dynamics
Corresponding GL-Dynamics
Associated Pattern Dynamics
Low-dimensional model for switches (PhD-work of Zou)

Perturbed NF: Breaking of y->y+\phi

\[
\frac{d}{dt} A_1 = (a_4 + a_5) b^2 A_4 + (a_0 + a_1 |A_1|^2 + a_2 |A_2|^2 + a_3 |A_3|^2 + a_4 |A_4|^2) A_1 + a_5 A_2 \overline{A}_3 A_4
\]

\[
b=0.2
\]
Transition occurs at $b_c \approx 0.199$.

Mean switch time:

$$T_{\text{switch}} \sim |b - b_c|^{-\gamma}, \quad \gamma \approx 3.9235$$
Further increase of $b$: Transition to a periodic orbit via In-Out-Intermittency at $b_c \approx 0.308$

Probability of mean time between bursts:

$$P(T_{\text{mean}}) \sim \alpha n^{-3/2}e^{-\beta n}+\gamma e^{-\delta n} \text{ (Ashwin’s model), where}$$

$$n=0.1T_{\text{mean}}, \quad \alpha \approx 0.3321, \quad \beta \approx 0.0524, \quad \gamma \approx 0.0156, \quad \delta \approx 0.0506$$
Thank You