Spatiotemporal Intermittency and Chaos in a Ginzburg Landau System for Oscillatory Instabilities in Anisotropic Systems

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Prototype Example for Anisotropic Systems

Electroconvection in Nematic Liquid Crystals

Symmetries in 2d axial anisotropic systems comprise:

Reflection invariances across and along a distinguished symmetry axis *(director axis in case of nematic electroconvection)*

In 2d (infinitely) extended anisotropic systems:

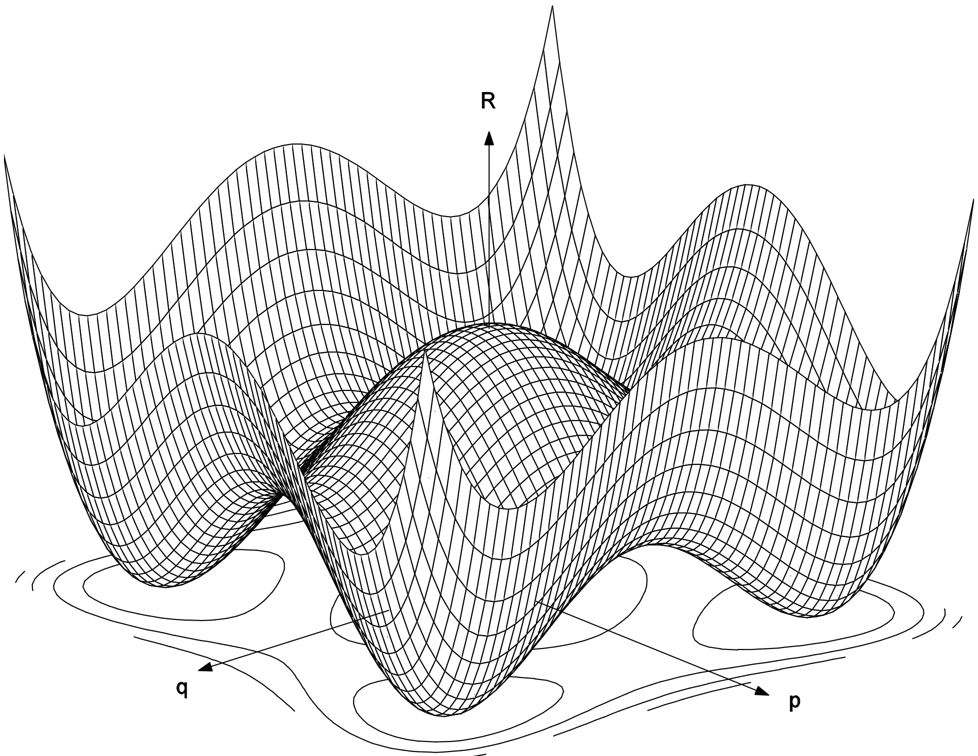
2 translation and 2 reflection symmetries: E(1) X E(1)

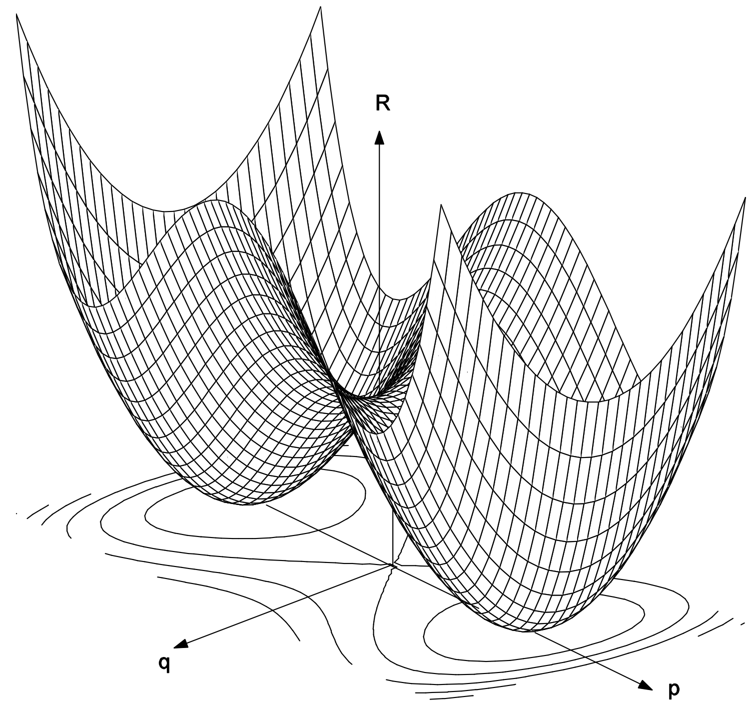
Since there is no Rotation Symmetry:

Instabilities of a uniform state involve only finitely many critical wave numbers

(in contrast to isotropic systems, where one has a circle of critical wave numbers)

Three Generic Possibilities for an Instability





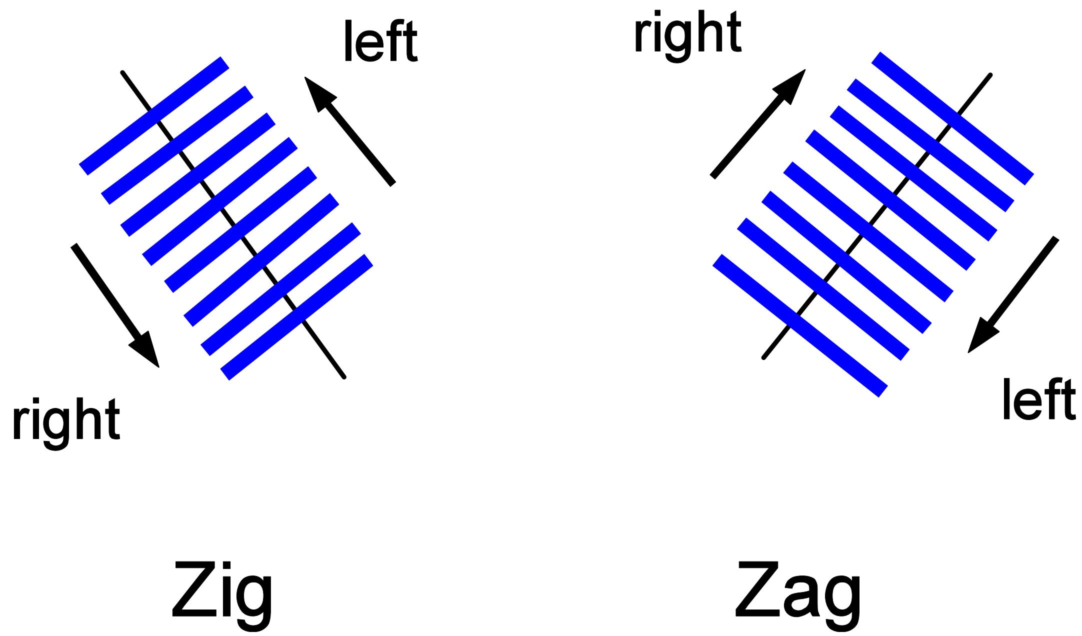
Oblique Case:
4 critical wave numbers
$(\pm p_c, \pm q_c)$

Normal Case:
2 critical wave numbers
$(\pm p_c, 0)$

Third Case:
critical wave number (0,0)

Oblique Hopf Instability

Linearized system admits two counterpropagating pairs of traveling wave solutions



Extend Study to Nonlinear System through
Weakly Nonlinear Analysis

Represent field variables through small and slowly varying amplitudes εA_1, εA_2, εA_3, εA_4:

$$\begin{aligned} U(t,x,y,z) \quad = \quad & \varepsilon\{A_1 U_1(z) e^{i(p_c x + q_c y)} + A_2 U_2(z) e^{i(-p_c x + q_c y)} + \\ & A_3 U_3(z) e^{i(-p_c x - q_c y)} + A_4 U_4(z) e^{i(p_c x - q_c y)}\} e^{i\omega_c t} + cc + O(\varepsilon^2) \end{aligned}$$

where $A_1 = A_1(T, X_+, Y_+),\ A_2 = A_2(T, X_-, Y_+),\ A_3 = A_3(T, X_-, Y_-),\ A_4 = A_4(T, X_+, Y_-)$

with $\quad X_\pm = \varepsilon(t \pm x / v_p), \quad Y_\pm = \varepsilon(t \pm y / v_q), \quad T = \varepsilon^2 t, \quad R - R_c = \varepsilon^2$

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System of **Globally Coupled Ginzburg Landau Equations** for the A_j :

$$\begin{aligned} \partial_T A_1(T, X, Y) \quad = \quad & \Big(a_0 + D(\partial_X, \partial_Y) + a_1 |A_1|^2 + a_2 \langle |A_2(\chi, Y)|^2 \rangle + \\ & a_3 \langle |A_3(X-\chi, Y-\chi)|^2 \rangle + a_4 \langle |A_4(X, \chi)|^2 \rangle \Big) A_1 + \\ & a_5 \langle A_2(\chi - X, Y) \overline{A}_3(\chi - X, \chi - Y) A_4(X, \chi - Y) \rangle \end{aligned}$$

where $\langle g(\chi) \rangle = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} g(\chi)\, d\chi$ = average over χ

≤

Numerically: $$A_j(T,X,Y) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} a_j(m,n,T) e^{imp_0 X + inq_0 Y}$$

Without spatial variations: $A_j = A_j(T) = a_j(0,0,T)$

≤

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Without spatial variations: $A_j=A_j(T)=a_j(0,0,T)$ → Normal Form for a **Hopf Bifurcation with O(2) X O(2)-Symmetry**:

$$\frac{d}{dT}A_1 = (a_0 + a_1|A_1|^2 + a_2|A_2|^2 + a_3|A_3|^2 + a_4|A_4|^2)A_1 + a_5 A_2\bar{A}_3 A_4$$

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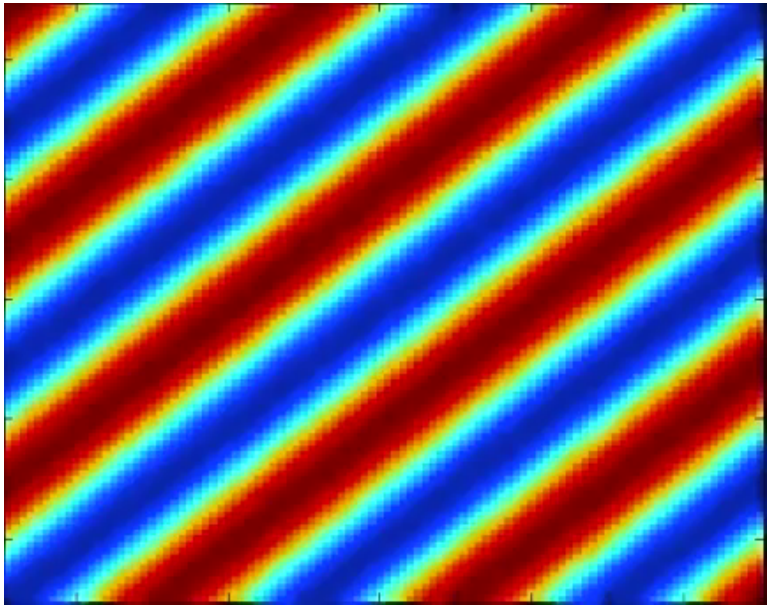
Basic wave solutions:

TW:	Traveling waves	(A,0,0,0)
SW:	Standing waves	(A,0,A,0)
TR_x:	Traveling rectangles in x	(A,0,0,A)
TR_y:	Traveling rectangles in y	(A,A,0,0)
SR:	Standing rectangles	(A,A,A,A)
AW:	Alternating waves	(A,iA,A,iA)

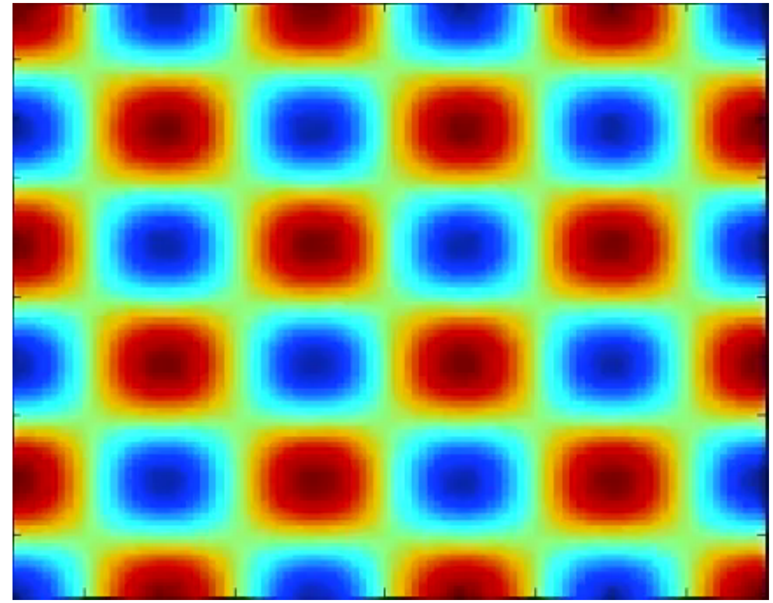
Other solutions: Quasiperiodic waves in 4d

Various heteroclinic cycles

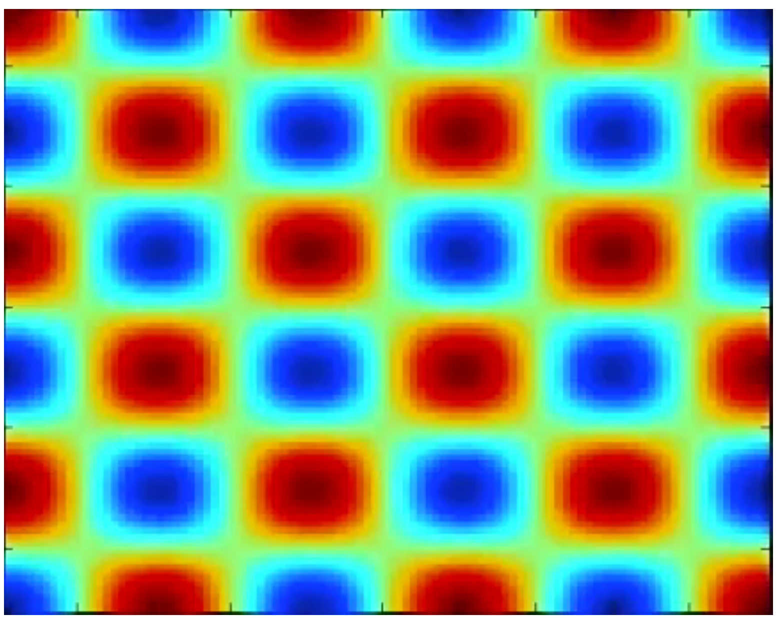
TW



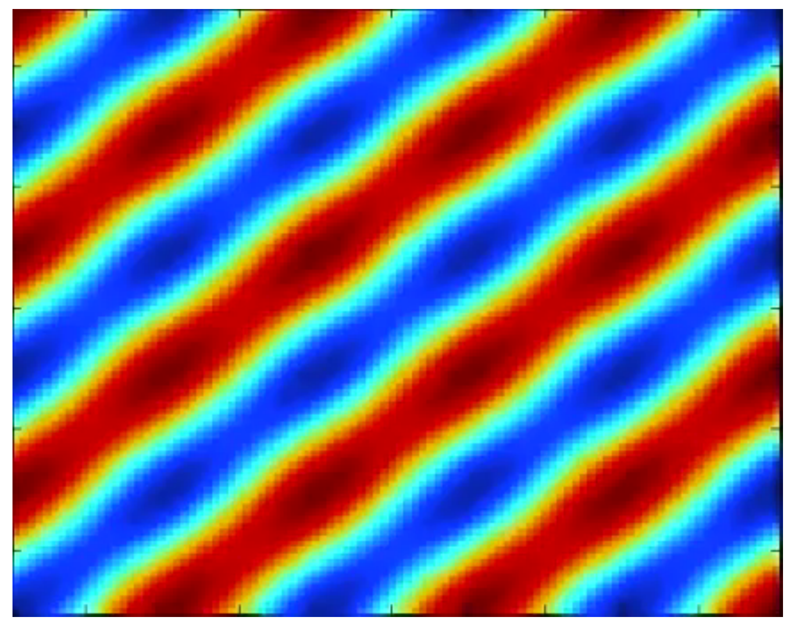
TR_y



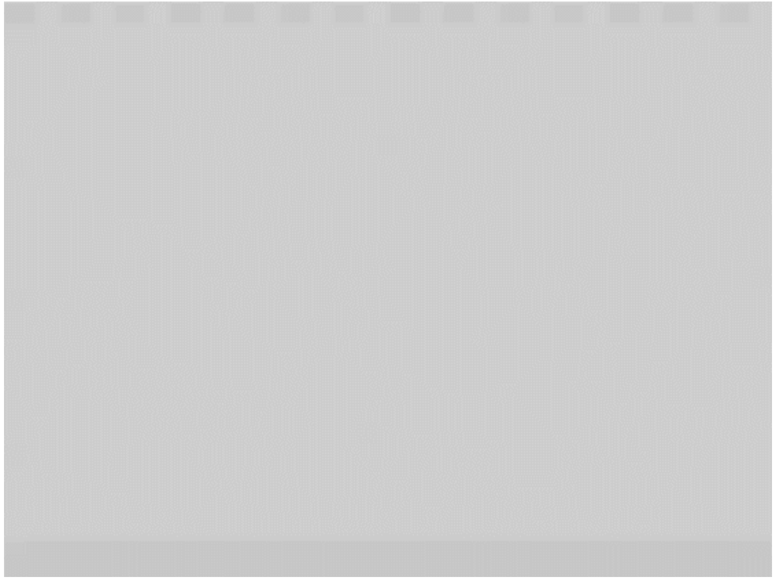
SR



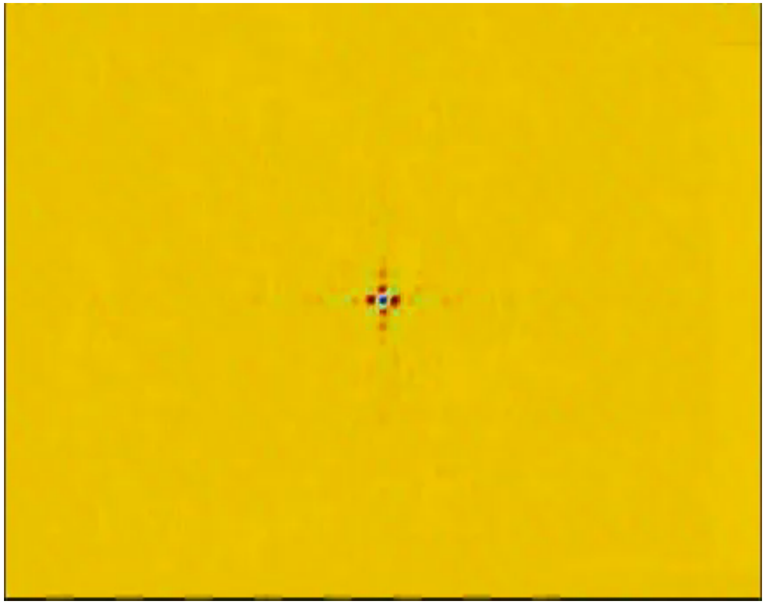
AW

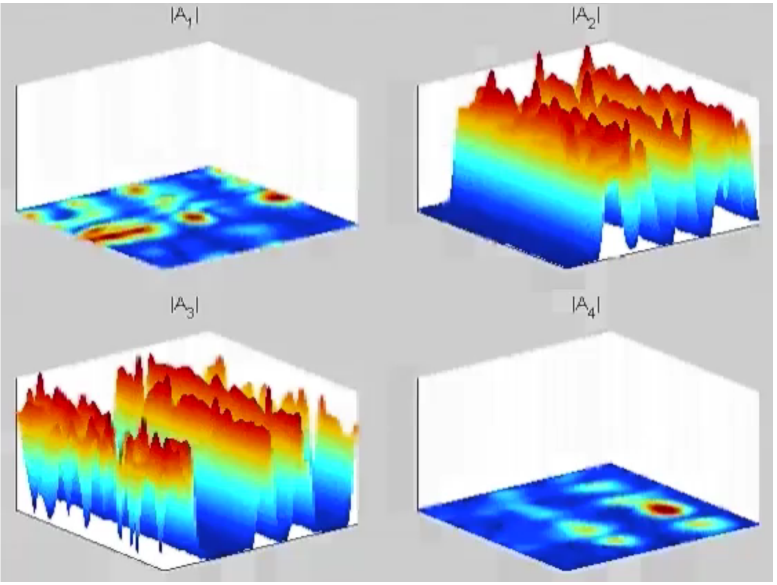


Examples of Patterns shown by the GL System

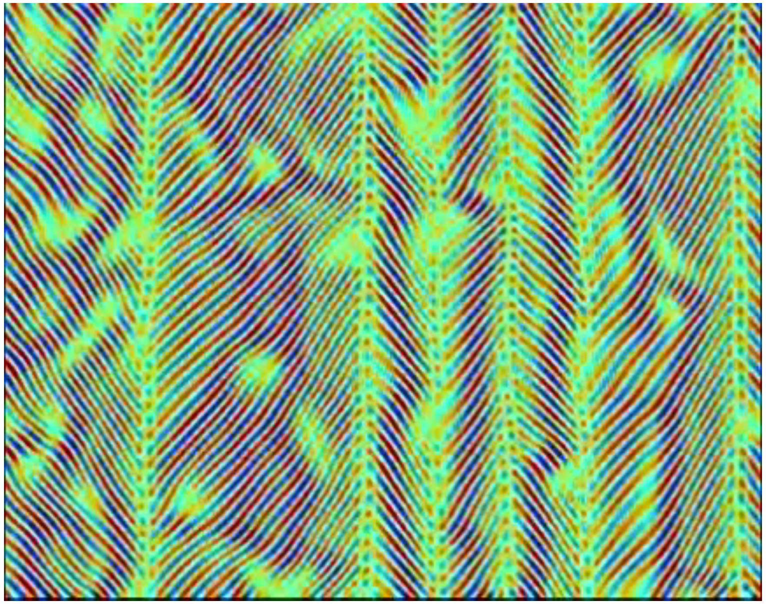


Stable TW

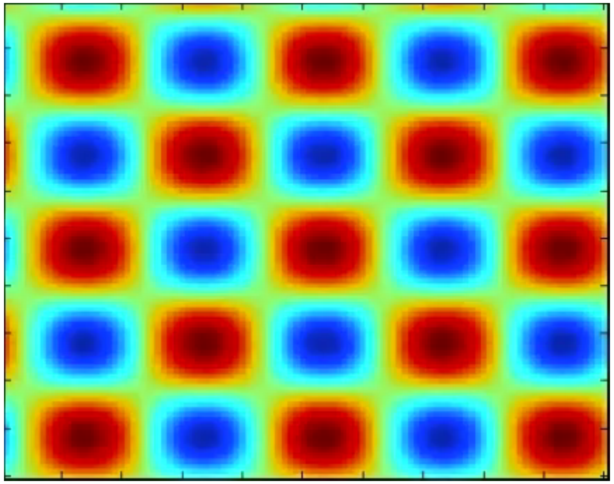


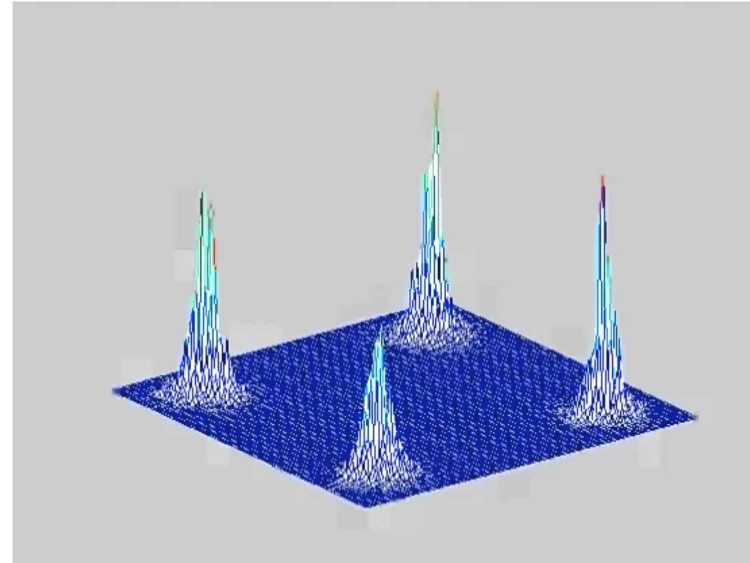


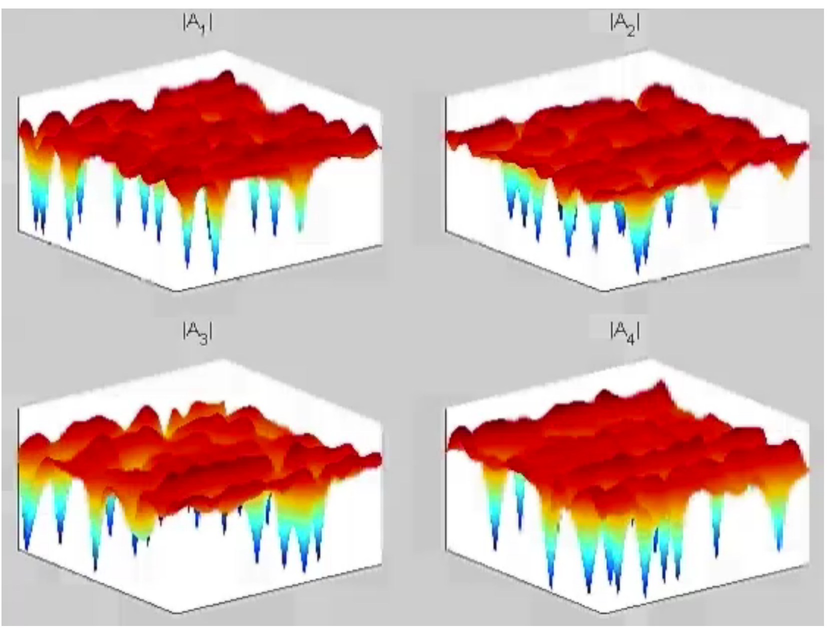
Unstable TW

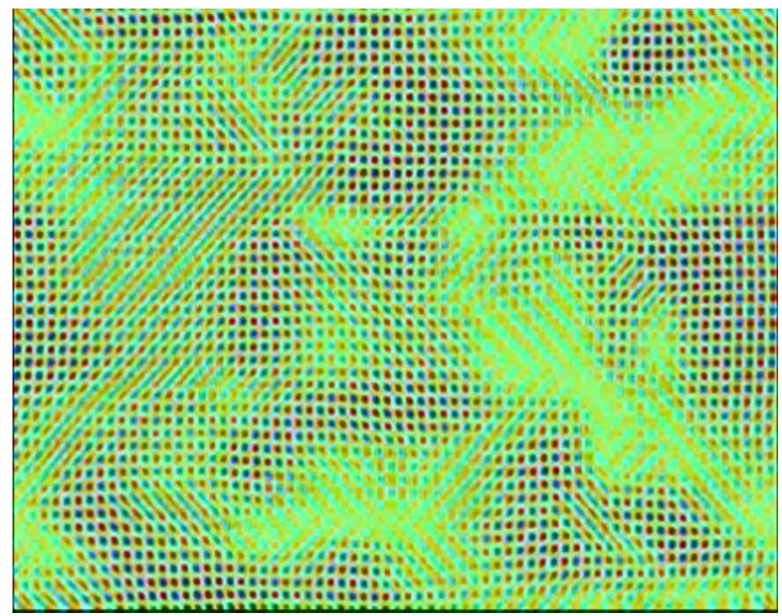


NF has attracting heteroclinic cycle TR_x-SR-AW



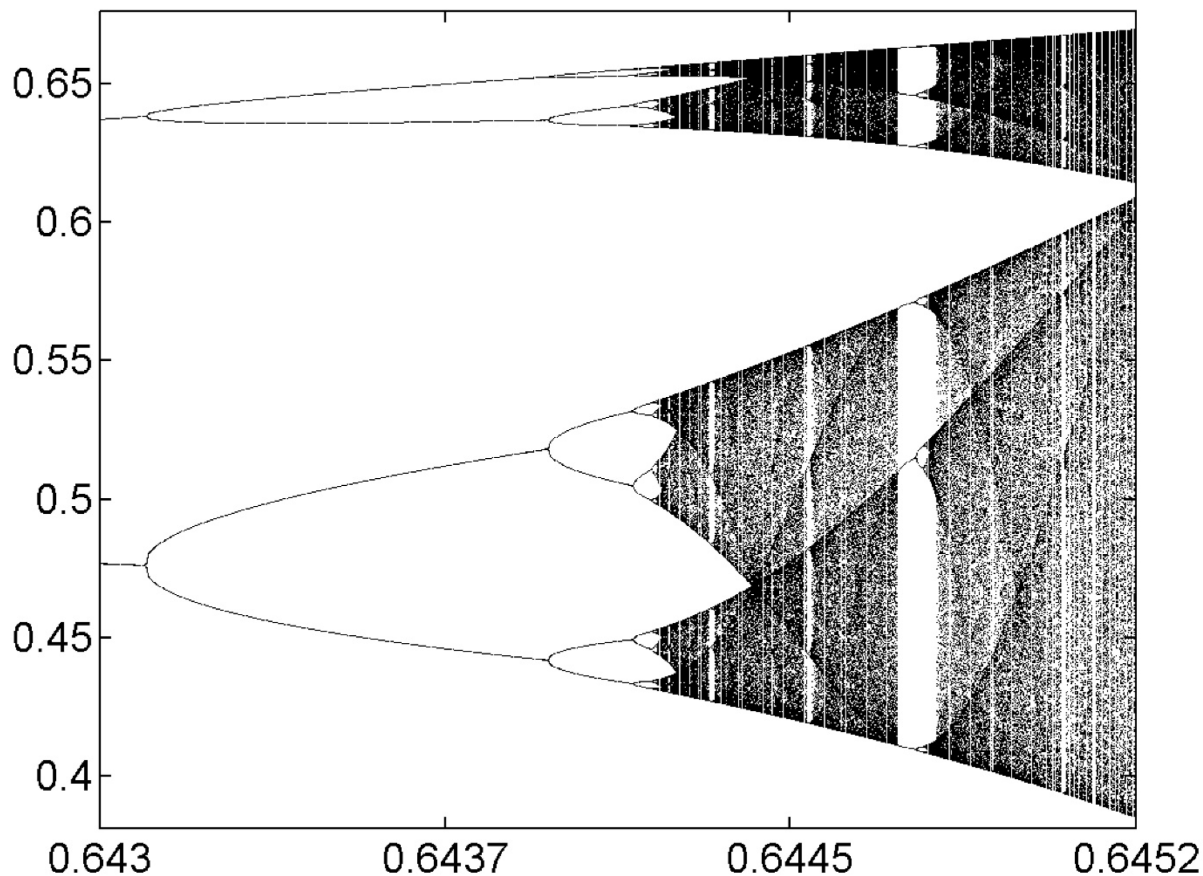






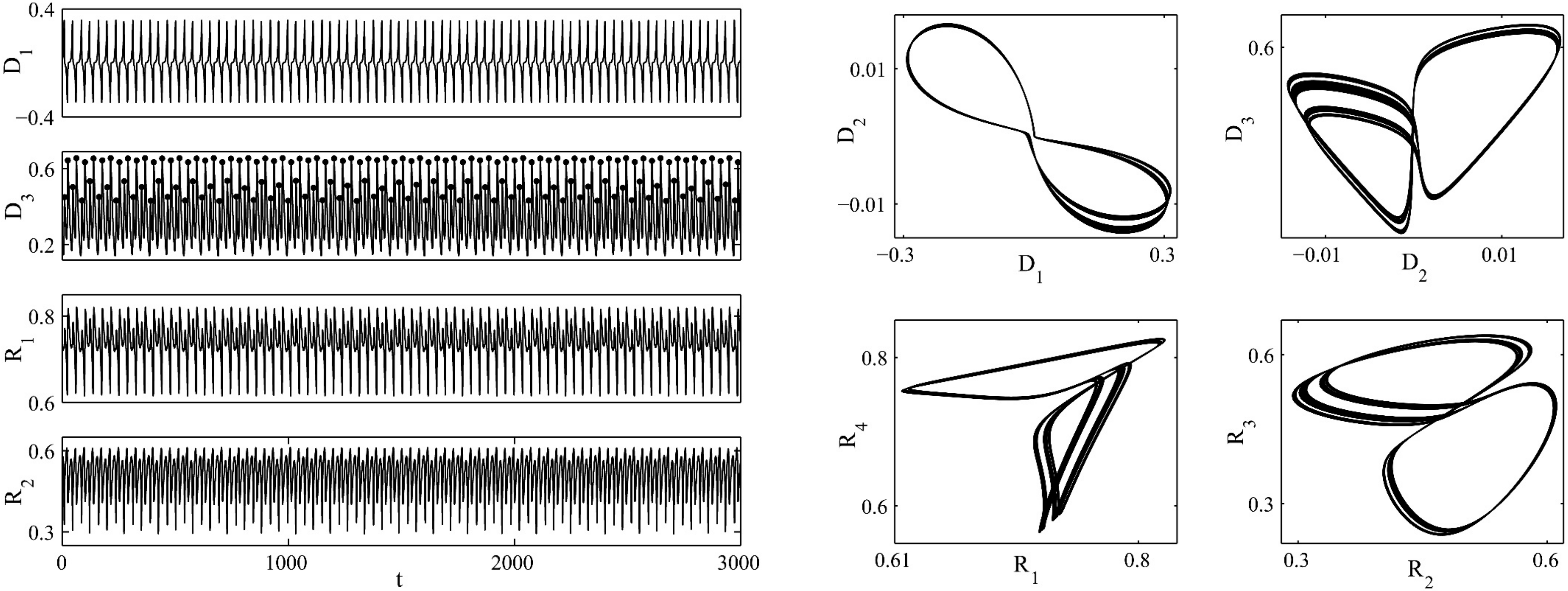
Normal Form Chaos

For certain parameters, NF shows a chaotic attractor created through a period doubling cascade; no stable basic waves.

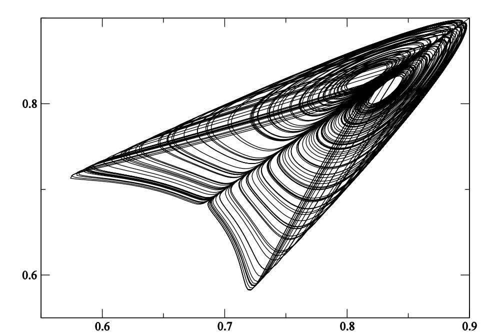


Iterates (variable of a suitable Poincare map) vs $-a_{3r}$

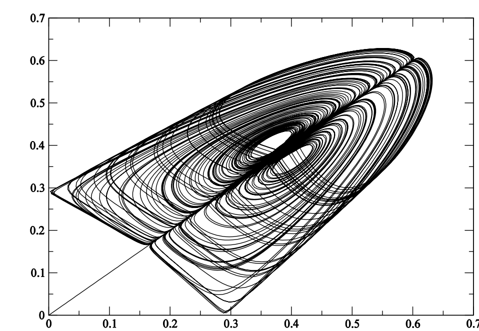
Chaotic NF-Attractors at $-a_{3r}$=0.6442: 4 symmetry-conjugated copies (R_j=$|A_j|$)



Nearby Symmetrized Attractors (2 copies)

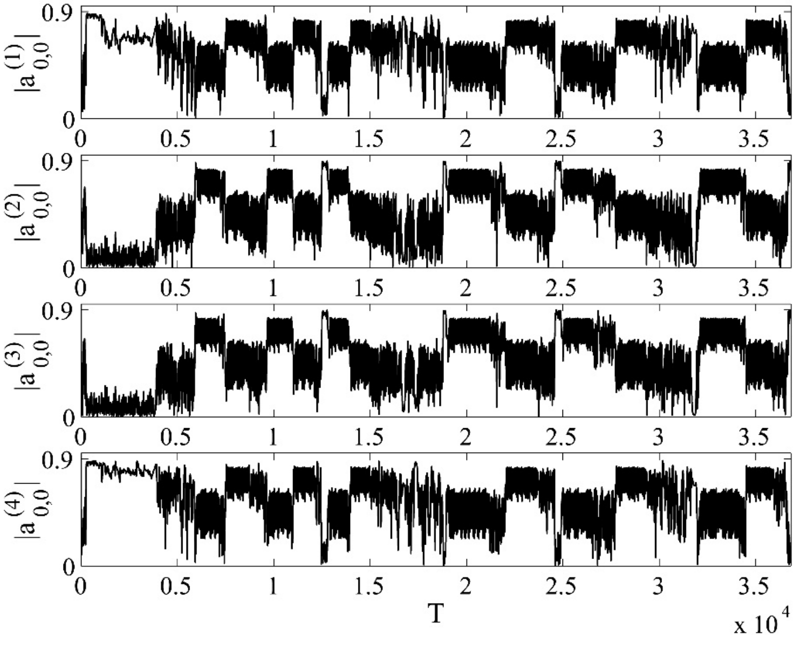


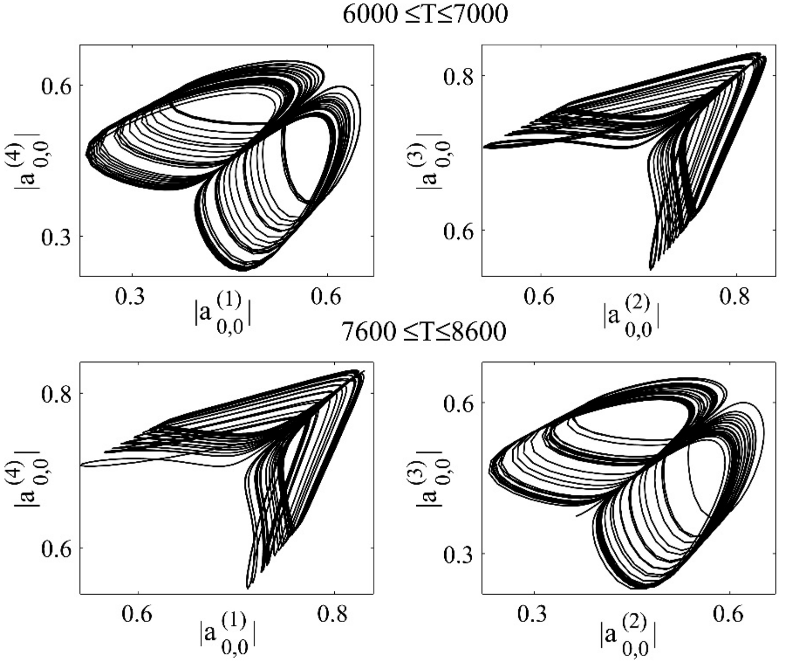
R_1 versus R_4



R_2 versus R_3

Corresponding GL-Dynamics





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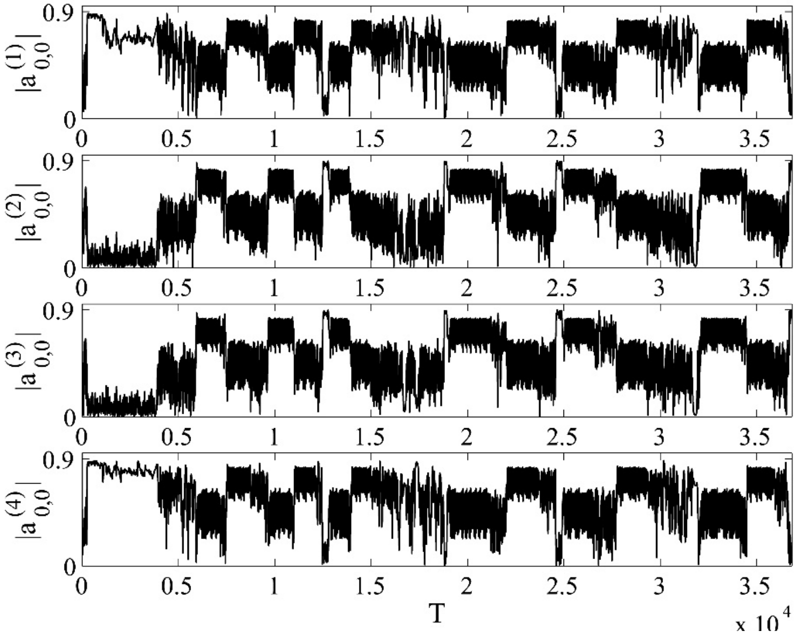
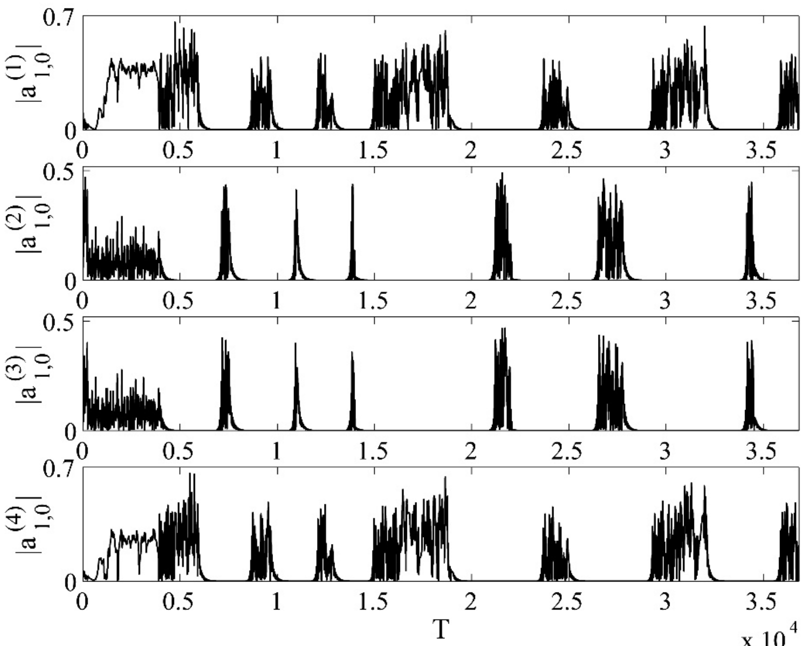
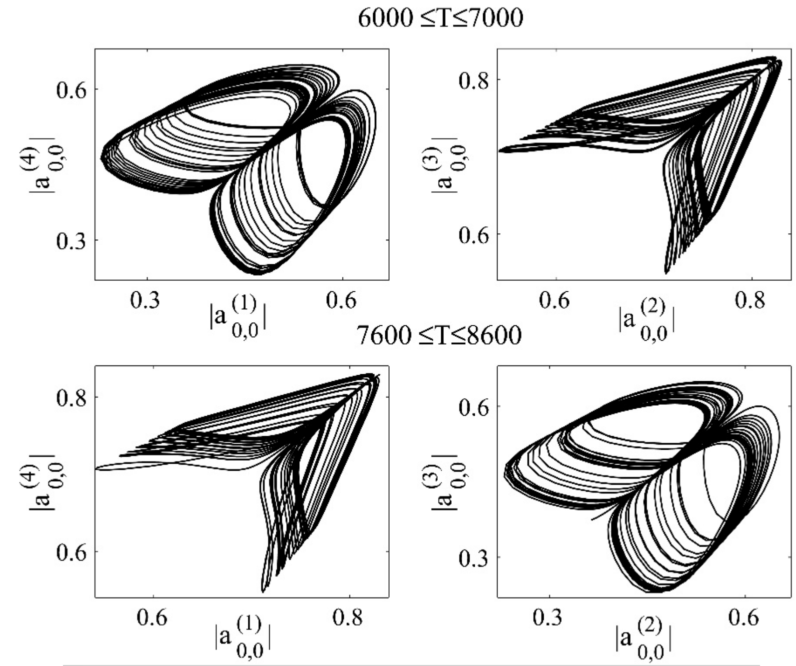
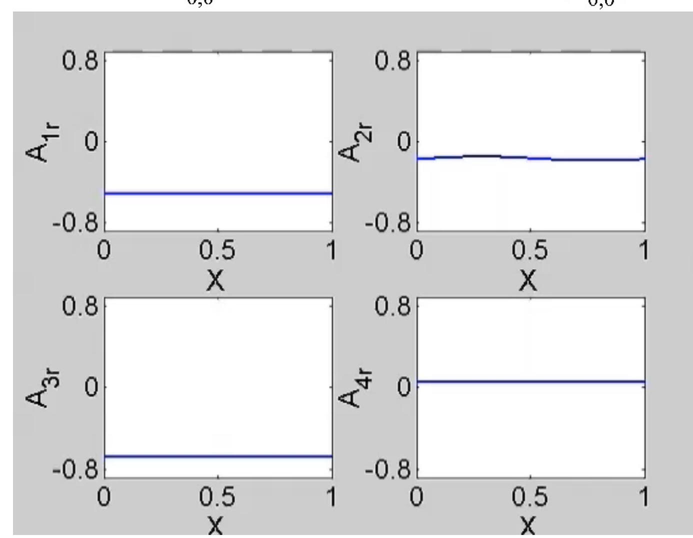
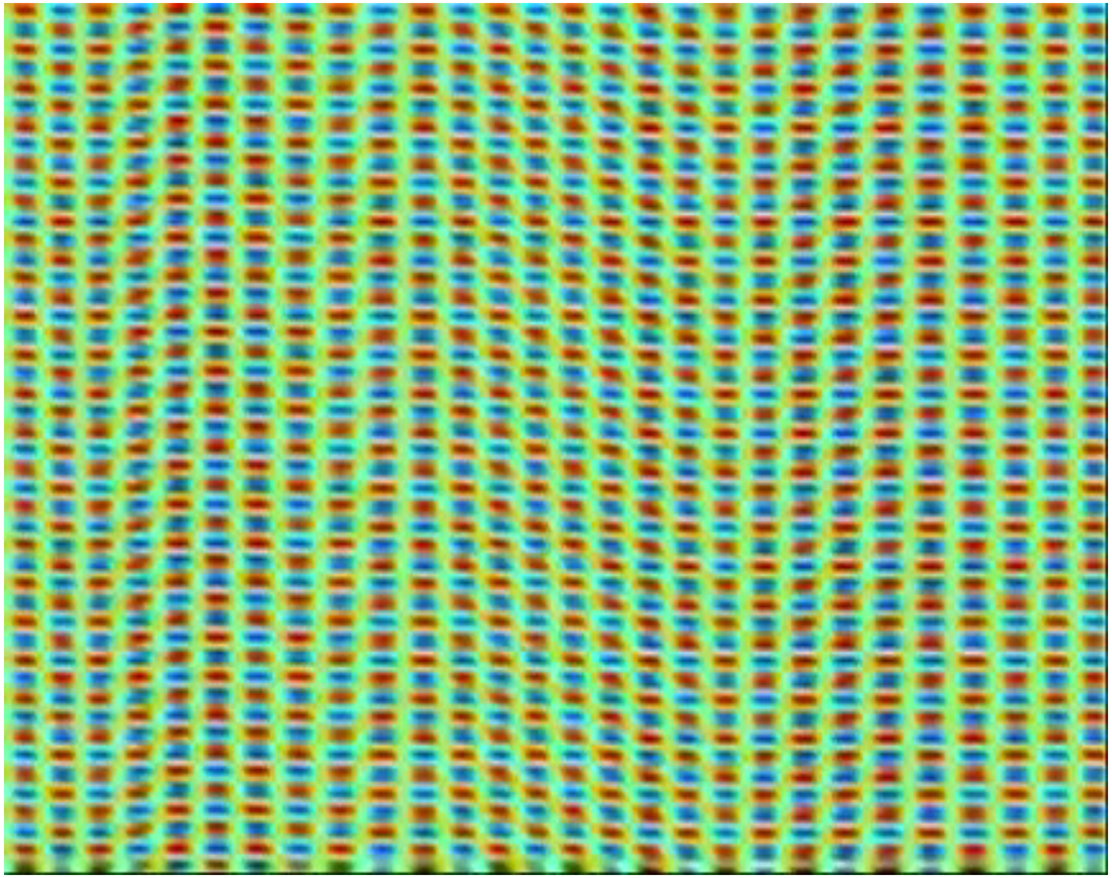








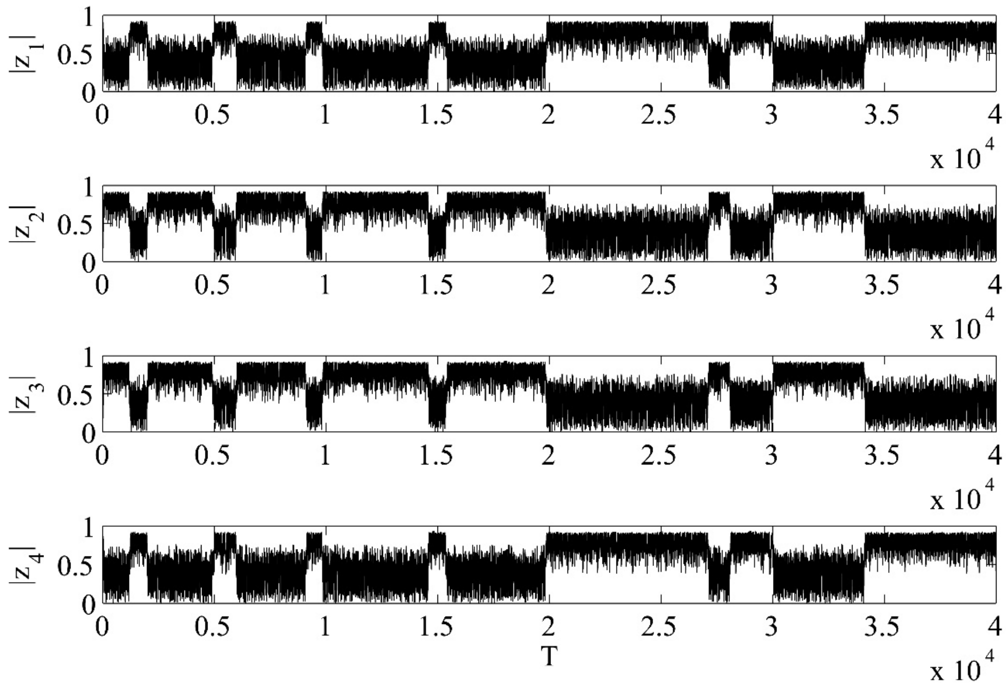
Associated Pattern Dynamics



Low-dimensional model for switches (PhD-work of Zou)

Perturbed NF: Breaking of y->y+ϕ

$$\frac{d}{dT}A_1 = (a_4+a_5)b^2A_4 + (a_0 + a_1|A_1|^2 + a_2|A_2|^2 + a_3|A_3|^2 + a_4|A_4|^2)A_1 + a_5A_2\bar{A}_3A_4$$

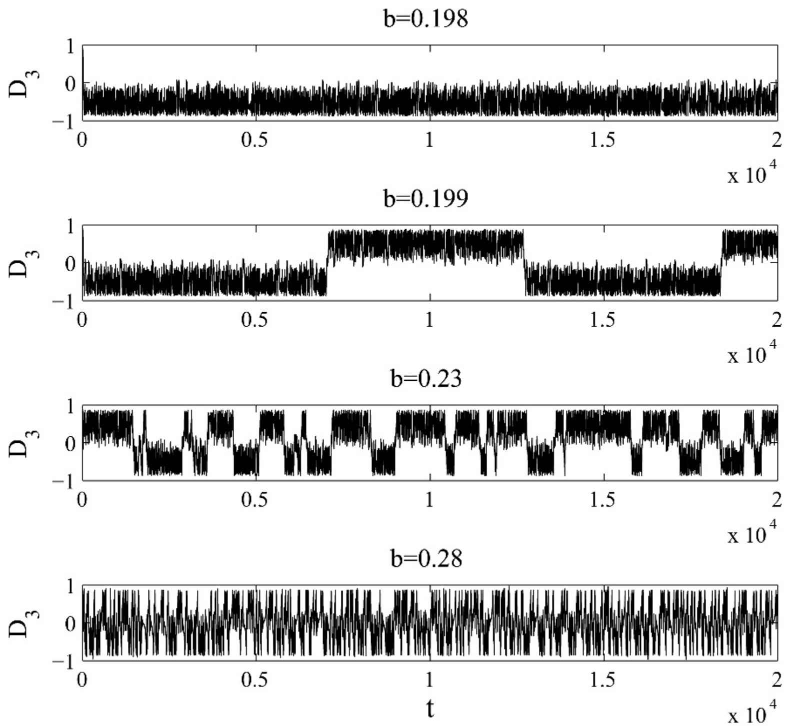


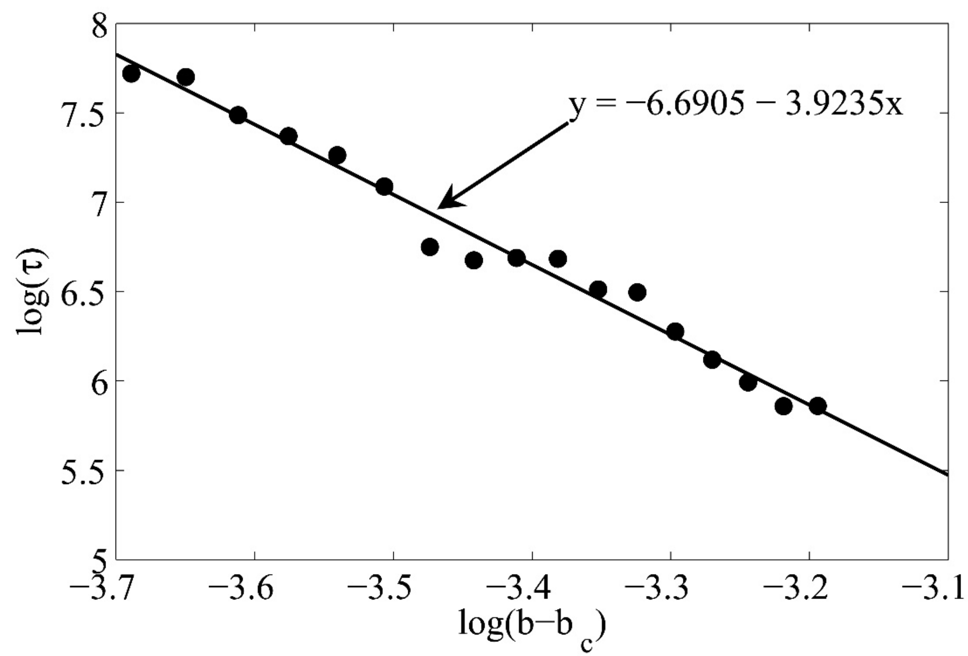
b=0.2

Transition occurs at $b_c \approx 0.199$.

Mean switch time:

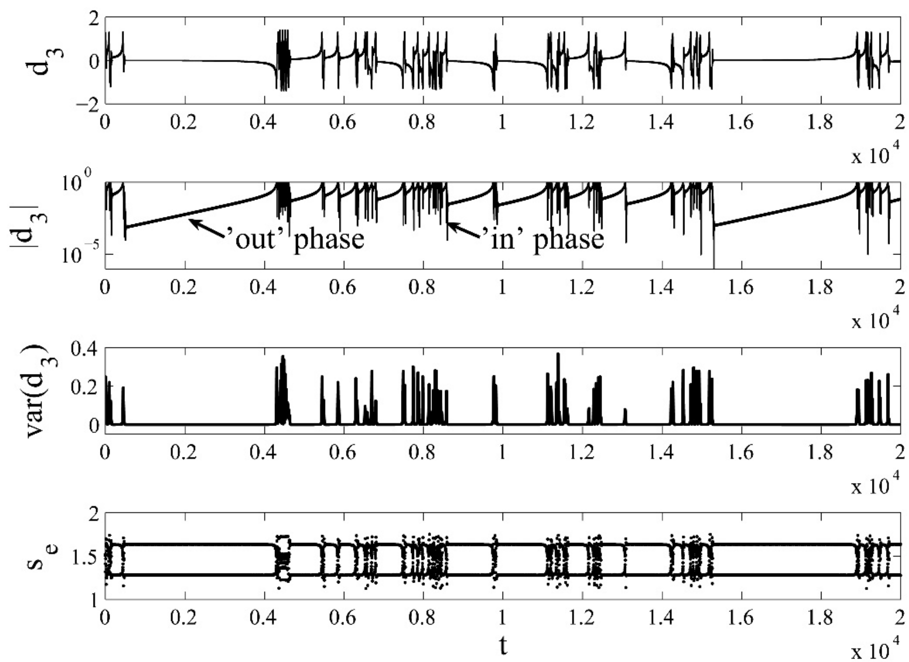
$$T_{switch} \sim |b - b_c|^{-\Upsilon}, \ \Upsilon \approx 3.9235$$

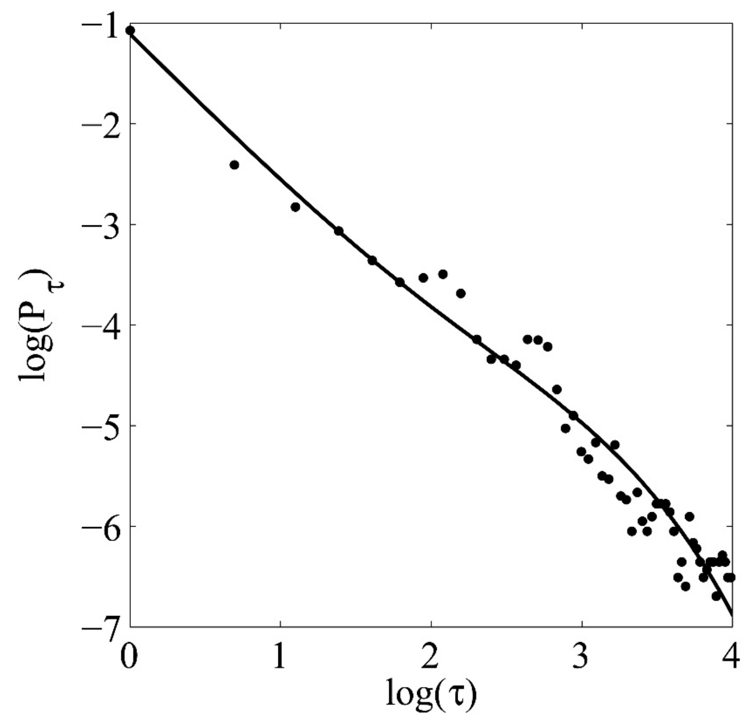




Further increase of b:
Transition to a periodic orbit via In-Out-Intermittency at $b_c \approx 0.308$

Probability of mean time between bursts:
$P(T_{mean}) \sim \alpha n^{-3/2} e^{-\beta n} + \Upsilon e^{-\delta n}$ (Ashwin's model), where
$n = 0.1 T_{mean}$, $\alpha \approx 0.3321$, $\beta \approx 0.0524$, $\Upsilon \approx 0.0156$, $\delta \approx 0.0506$





Thank You