# The Term Structure of Liquidity: A Liquidation Game Approach

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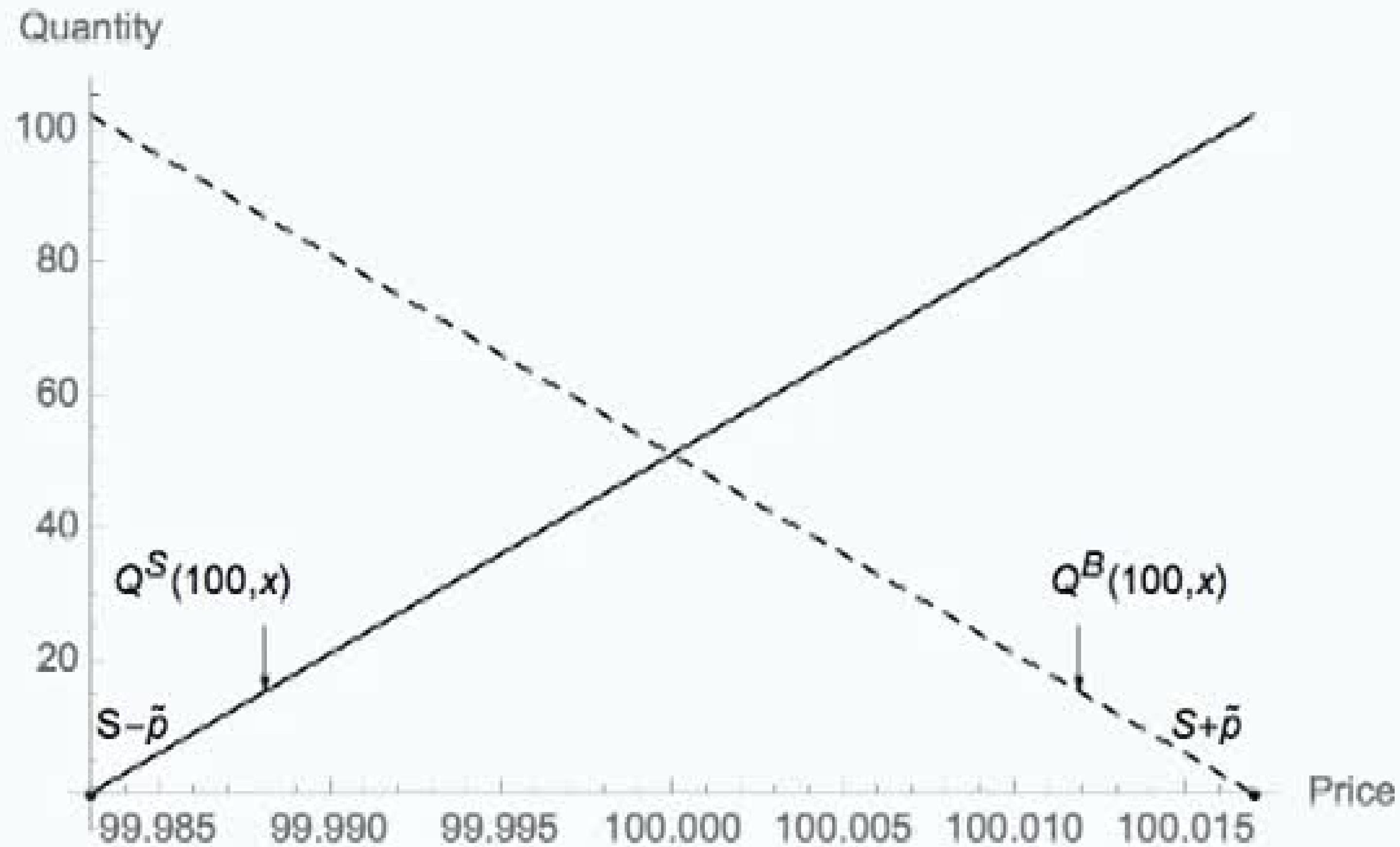
# Empirical Patterns of Execution

- Empirical study of the market impact of metaorders: Zarinelli, Treccani, Farmer, and Lillo (2015)
- Most comprehensive study using execution data from Ancerlo Ltd, 7 million metaorders in Russell 3000 index stocks from 2007-2009
- Key findings:
  - Participation rate, defined as the ratio of order size to the market volume over the same trading period, negatively correlates with the duration of liquidation.
  - Price impact subsides before the end of the liquidation: it decays as the metaorder is being executed
  - After liquidation, price impact decays in (square root of) time, irrespective of duration.
  - Price impact at the end of liquidation is concave in the order size

# Objective

- Endogeneize both the demand and supply of liquidity:
  - HFTs intermediate between randomly arriving buyers and sellers and a large liquidating institution
  - HFTs strategically compete over the traded quantities
  - The liquidity-demanding investor optimally chooses the liquidation strategy to minimize its expected costs of execution
  - HFTs set prices taking into account the execution strategy of the liquidating investor
  - Each HFT maximizes the expected discounted trading revenue minus the flow costs of inventory holdings

# Demand and Supply



# Market Environment: the Institutional Investor

- Duration of liquidation $D$ is sampled from an independent exponential distribution with mean $1/\nu$.
- The institutional investor can conduct:
  - Stealth trading: the sampled duration is not revealed to the market makers, and the same liquidation rate is used for all durations
  - Sunshine trading: the sampled duration is revealed to the market makers, and the liquidation rate may depend on the value of the sampled duration
- Let $b_t$ be the bid price offered by the market makers at time $t$, then the institutional investor's objective is

$$\sup_{\bar{f}\geq 0} \mathbb{E}\left[\int_0^D e^{-\beta t}\bar{f} \times (b_t - S_t + \tilde{p})dt \,\middle|\, D \text{ iff sunshine}\right]$$

# Market Environment: Market Makers

- $N$ market makers split the liquidation stream from the institutional investor
- Market maker $n$ chooses the amount it plans to buy from/sell to the randomly arriving sellers and buyers at time $t$: $x_t^{b,n}$, $x_t^{a,n}$, $n = 1, \ldots, N$
- The aggregated strategies of the $N$ market makers collectively determine the ask and bid prices via market clearing:

$$\begin{cases} \sum_{n=1}^{N} x_t^{a,n} = c(S_t + \tilde{p} - a_t) \\ \sum_{n=1}^{N} x_t^{b,n} = c(b_t - S_t + \tilde{p}) \end{cases} \Rightarrow \begin{cases} a_t = S_t + \tilde{p} - \frac{1}{c} \sum_{n=1}^{N} x_t^{a,n} \\ b_t = S_t - \tilde{p} + \frac{1}{c} \sum_{n=1}^{N} x_t^{b,n} \end{cases}$$

- $(x_t^{a,n})$ and $(x_t^{b,n})$ are Markov predictable strategies (dependent on $t$, $\bar{f}$ and the inventory level)

# The Objective of Market Makers

- Market maker $n$ solves

$$\max_{(x_\cdot^{a,n}, x_\cdot^{b,n}) \in \mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\left(dW_t^{(x,n)} - \Theta\left(I_t^{(x^n,n)}\right)^2 dt\right)\right]$$

where $\mathcal{A}$ is the collection of all admissible strategies subject to:

$$dW_t^{(x,n)} = -b_t \cdot \frac{\bar{f}}{N} 1_{t \le D} dt + a_t \cdot x_t^{a,n} dN_t^B - b_t \cdot x_t^{b,n} dN_t^S + S_t dI_t^{(x^n,n)}$$

$$dI_t^{(x^n,n)} = \underbrace{\frac{\bar{f}}{N} 1_{t \le D} dt}_{\text{Shares liquidated by institution}} + \underbrace{x_t^{b,n} dN_t^S}_{\text{Shares bought from sell investors}} - \underbrace{x_t^{a,n} dN_t^B}_{\text{Shares sold to buy investors}}$$

- Look for symmetric equilibria

# Dynamic Programming Formulation

- Fix a liquidation strategy $f \equiv \bar{f} 1_{t \le D}$.
- Given $I_t^{(x^n,n)} = i$, consider the value function

$$V_n(t,i;f)$$
$$:= \sup_{(x_\cdot^{a,n}, x_\cdot^{b,n}) \in \mathcal{A}} \mathbb{E}\left[\int_0^\infty e^{-\beta(u-t)} (dW_u^{(x,n)} - \Theta \left(I_u^{(x^n,n)}\right)^2 du) | I_t^{(x^n,n)} = i\right]$$

- Value independent of fundamental since revenue is calculated relative to the fundamental
- Transition of $I_t^{(x^n,n)}$ given $I_{t-}^{(x^n,n)} = i$ and $(x_t^{a,n}, x_t^{b,n})$

$$I_t^{(x^n,n)} = \begin{cases} i - x_t^{a,n}, & \text{w.p. } \lambda dt, \\ i + x_t^{b,n}, & \text{w.p. } \lambda dt, \\ i, & \text{else} \end{cases}$$

# The Value Function: Stealth Trading

## Theorem 3.1

*Let $A$ be the unique positive root to the following equation*

$$\Theta - \beta A = \frac{8c\lambda A^2(1+cA)}{(N+1+2cA)^2}.$$

*Then the optimal value of market maker $n$ is given by*

$$V_n(t,i;f) = -Ai^2 + B(t,\bar{f})i + C(t,\bar{f}),$$

*where $B(t,\bar{f}) = -\bar{f}\frac{\delta-\beta}{2c\lambda}\frac{N+2cA}{N}\frac{1}{\nu+\delta}1_{t\le D}$ and $\delta = \Theta/A$. Optimal prices are given by*

$$\begin{cases} a_t(i,\bar{f}) = S_t + \dfrac{p(1+2cA) - 2NAi + NB(t,\bar{f})}{N+1+2cA} \\ b_t(i,\bar{f}) = S_t + \dfrac{-p(1+2cA) - 2NAi + NB(t,\bar{f}) - \frac{\bar{f}}{c\lambda}1_{t\le D}}{N+1+2cA} \end{cases}$$

# Price Policy Implications: Stealth Trading

- Before the liquidation is terminated, the price policy functions are stationary, i.e. independent of $t$
- Constant bid-ask spread during and after the investor's liquidation
- Liquidation widens the bid-ask spread
- Liquidation drives down both ask and bid prices when the inventory level stays put - price pressure from liquidation. Sudden price corrections at $t = D$.

## Corollary 3.2

*If $I_0^{(x^n,n)} = 0$, the expected inventory at $t \leq D$ is given by ($M = \frac{4c\lambda A}{N+1+2cA}$)*

$$g(t) \equiv \mathbb{E}[I_t^{(x^n,n)}] = \frac{\bar{f}}{N} \frac{N+2cA}{N+1+2cA} \frac{\beta+\nu}{\delta+\nu} \frac{1-e^{-Mt}}{M}$$

- Hence, expected price trajectories are monotonically decreasing in $t$

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$$V_n(t, i; f) = -Ai^2 + B(t, \bar{f})i + C(t, \bar{f}),$$

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# Price Impact and Liquidation Size

- Zarinelli, Treccani, Farmer, and Lillo (2015) find empirically that price impact is concave in the size of the liquidated order
- We define the price impact as the absolute value of the expected midquote deviation from the fundamental at $D$, i.e.

$$PI = \frac{2NAg(D) + N|B(D,\bar{f})| + \frac{\bar{f}}{2c\lambda}}{N + 1 + 2cA}$$

- $|B(D,\bar{f})|$ is independent of $D$, and $g(D)$ is concave in $D$.
- Hence, **price impact is concave in the total liquidation size** $\bar{f}D$.

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# The Value Function: Sunshine Trading

**Theorem 3.3**

*Let A be the unique positive root to the following equation*

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*Then the optimal value of market maker n is given by*

$$V_n(t,i;f) = -Ai^2 + \tilde{B}(t,\bar{f})i + \tilde{C}(t,\bar{f}),$$

*where* $\tilde{B}(t,\bar{f}) = -\bar{f}\frac{\delta-\beta}{2c\lambda}\frac{N+2cA}{N}\frac{1-e^{-\delta(D-t)}}{\delta}1_{t\leq D}$ *and* $\delta = \Theta/A$. *Optimal prices are given by*

$$\begin{cases} a_t(i,\bar{f}) = S_t + \dfrac{p(1+2cA) - 2NAi + N\tilde{B}(t,\bar{f})}{N+1+2cA} \\ b_t(i,\bar{f}) = S_t + \dfrac{-p(1+2cA) - 2NAi + N\tilde{B}(t,\bar{f}) - \frac{\bar{f}}{c\lambda}1_{t\leq D}}{N+1+2cA} \end{cases}$$

# Price Policy Implications: Sunshine Trading

- Before liquidation ends, the price policy functions are time-dependent, continuously converging to the stationary strategies at time $t = D$
- Constant bid-ask spread during and after the liquidation
- Liquidation widens the bid-ask spread
- Liquidation drives down both ask and bid prices when the inventory level stays put - price pressure from liquidation. No sudden price corrections to the ask price at $t = D$.

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# Price Trajectories under Sunshine Trading

**Corollary 3.4**

*If $I_0^{(x^n,n)} = 0$, the expected inventory at $t \leq D$ is given by*

$$g(t) \equiv \mathbb{E}[I_t^{(x^n,n)}] = \frac{\bar{f}}{N}\frac{N+2cA}{N+1+2cA}\left(\frac{\beta}{\delta}\frac{1-e^{-Mt}}{M} + \frac{\delta-\beta}{\delta}\frac{e^{\delta t}-e^{-Mt}}{M+\delta}e^{-\delta S}\right),$$

*where $\delta = \Theta/A$. For $t > D$, we have $g(t) = g(D)e^{-M(t-D)}$.*

- Recall that the expected ask and bid prices are

$$\begin{cases} \mathbb{E}[a_t(i,\bar{f})] = S_0 + \dfrac{p(1+2cA) - 2NAg(t) + NB(t,\bar{f})}{N+1+2cA} \\ \mathbb{E}[b_t(i,\bar{f})] = S_0 + \dfrac{-p(1+2cA) - 2NAg(t) + NB(t,\bar{f}) - \frac{\bar{f}}{c\lambda}1_{t\leq D}}{N+1+2cA} \end{cases}$$

# Participation Rate

- Participation rate measures the percentage of the liquidated order over the total trading volume in the same period.
- We formally define participation rate for liquidation duration $D$ as $R(D)$:

$$R(D) = \frac{D \cdot \bar{f}^*(D)}{\mathbb{E}[\text{total volume}]}$$

- $1/R(D)$ is strictly increasing in $D$:

$$\frac{1}{R(D)} = \frac{N + 2cA}{N + 1 + 2cA} + \frac{2N}{N + 1 + 2cA} \frac{c\lambda\tilde{p}}{\bar{f}^*(D)},$$

- Thus, **the participation rate strictly decreases with the duration** $D$ **of the liquidation**.

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# Optimal Liquidation Rate: Sunshine Trading

## Corollary 3.5

*For sunshine trading, the institutional investor's expected proceeds are given by*

$$\tilde{P}(D)\bar{f} - \tilde{Q}(D)(\bar{f})^2$$

*for some positive functions of D,* $\tilde{P}(D)$ *and* $\tilde{Q}(D)$ *that depends on* $\beta, N, c, \lambda, \tilde{p}$. *The optimal liquidation rate for duration D is thus given by*

$$\bar{f}^*(D) = \frac{\tilde{P}(D)}{2\tilde{Q}(D)}.$$

*The optimal expected liquidation proceeds for duration D is* $\frac{(\tilde{P}(D))^2}{4\tilde{Q}(D)}$. *Moreover,* $\bar{f}^*(D)$ **is strictly decreasing in** $D$.

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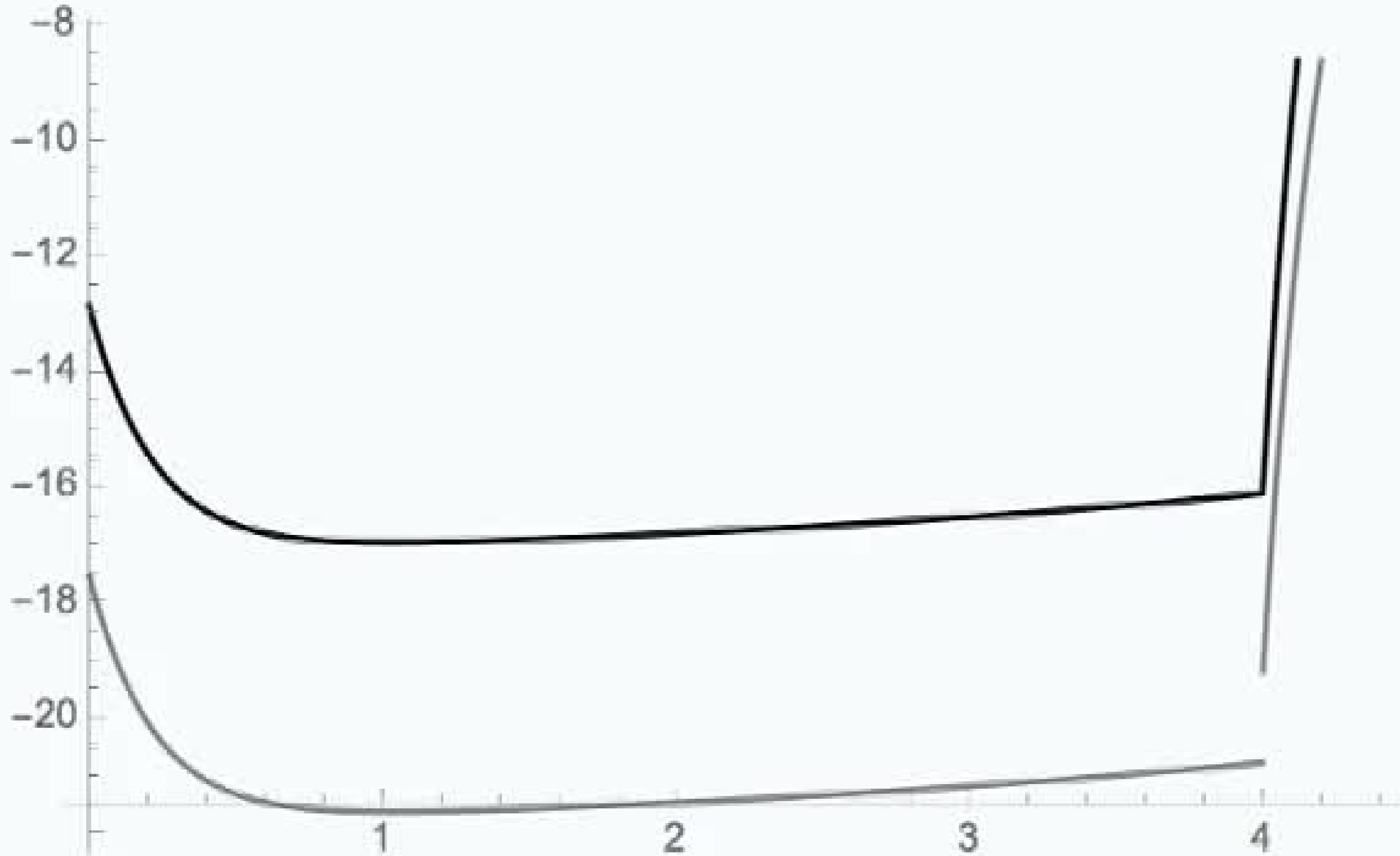
# Expected Price Pressures ($D = 4$)

- Price pressure: the deviation of prices from the fundamental, i.e. $a_t - S_t$, $b_t - S_t$

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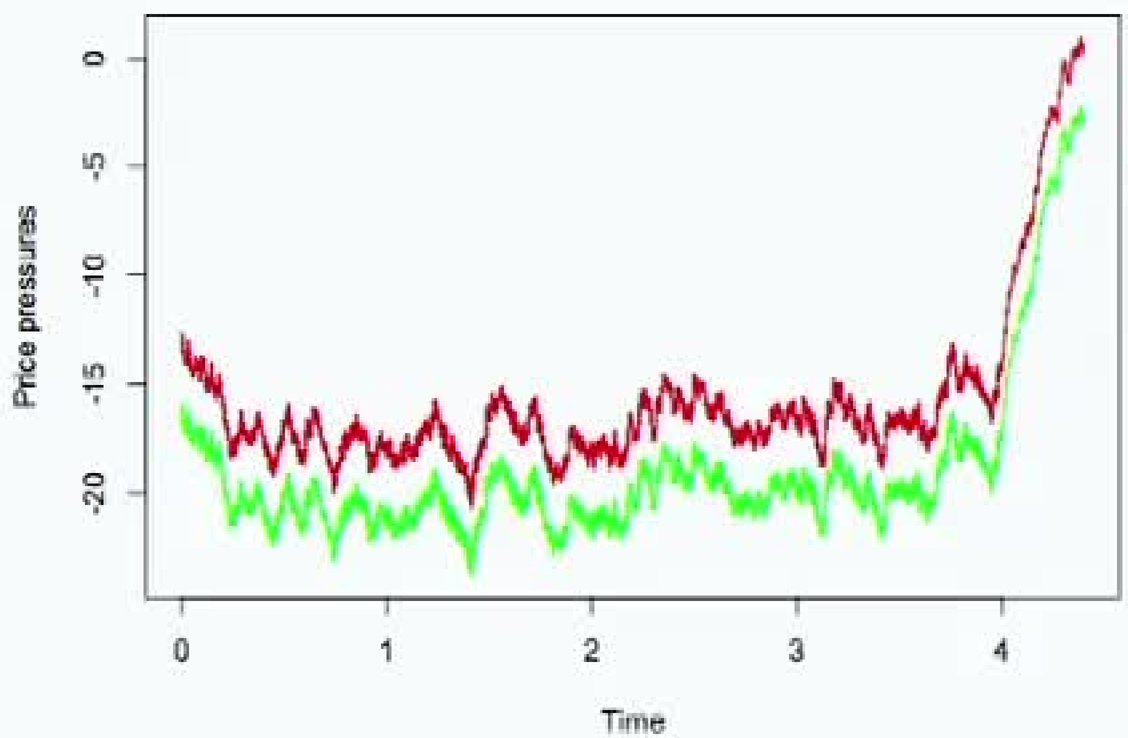
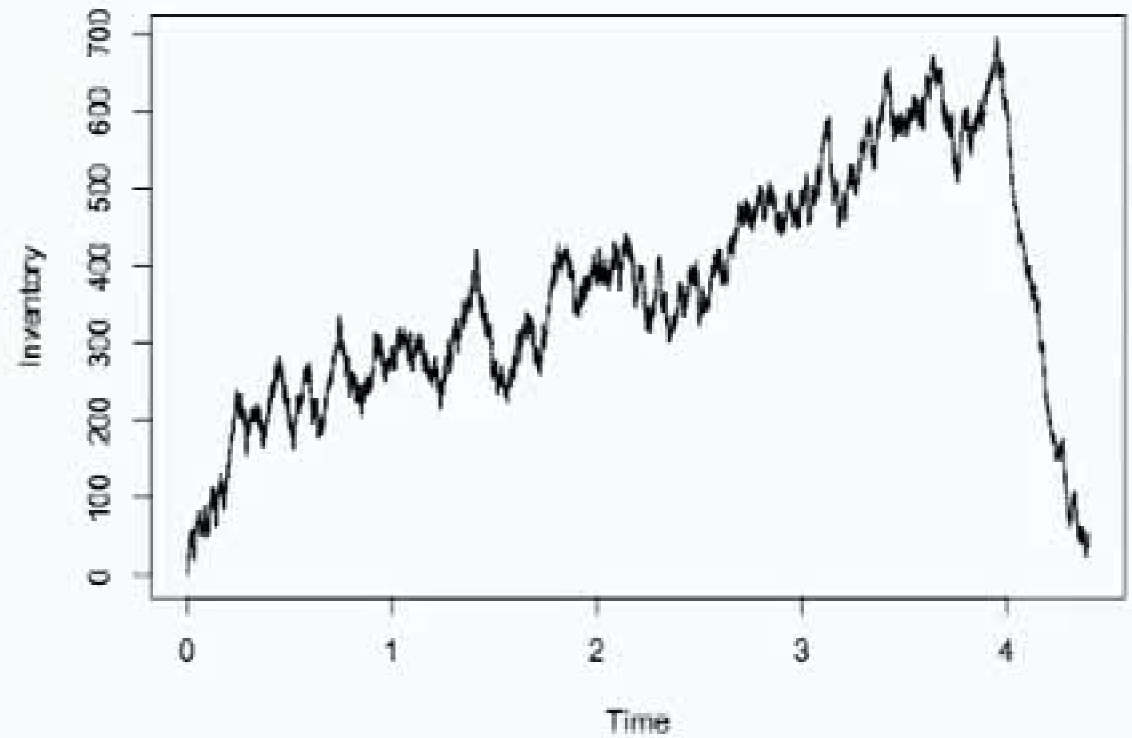
$$\bar{f}^*(D) = \frac{\tilde{P}(D)}{2\tilde{Q}(D)}.$$

*The optimal expected liquidation proceeds for duration* $D$ *is* $\frac{(\tilde{P}(D))^2}{4\tilde{Q}(D)}$. *Moreover,* $\bar{f}^*(D)$ **is strictly decreasing in** $D$.

# Simulated Price Pressures (D=4)




(a) Simulated Price Pressures




(b) Inventory

- Price reversal before the liquidation ends

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$$\bar{f}^*(D) = \frac{\tilde{P}(D)}{2\tilde{Q}(D)}.$$

*The optimal expected liquidation proceeds for duration $D$ is $\frac{(\tilde{P}(D))^2}{4\tilde{Q}(D)}$. Moreover, $\bar{f}^*(D)$* **is strictly decreasing in** *$D$.*

# Is Information about Duration Valuable?

- Should the institutional investor conduct stealth trading or sunshine trading?

# Private Information on Duration has Negative Value!

**Theorem 3.6**

*Suppose the duration $D$ is sampled from the exponential distribution with mean $1/\nu > 0$. Then, the optimal liquidation proceeds from sunshine trading, $\mathbb{E}[(\tilde{P}(D))^2/4\tilde{Q}(D)]$, are strictly higher than those under stealth trading, $P^2/4Q$.*

- Revealing information on duration helps market maker to continuously adjust price policy functions, and reduces the execution costs of the liquidating investor
- This is beneficial to the liquidating investor
- Even in the presence of a monopolistic HFT, the investor is better off if he reveals information about the duration of the liquidation

# Summary

- We study the time dimension of liquidity via a liquidation game of the Stackelberg type, with Cournot competition among market makers
- Liquidation reinforces price pressure and widens bid-ask spread
- Under stealth trading:
  - price impact is concave in the size of liquidation
  - price trajectories are monotone during liquidation
- Under sunshine trading:
  - participation rate negatively correlates with the liquidation duration
  - price reversal occurs prior to the end of liquidation
- Sharing information on duration is beneficial for the liquidating investor

Thank you for your attention!

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