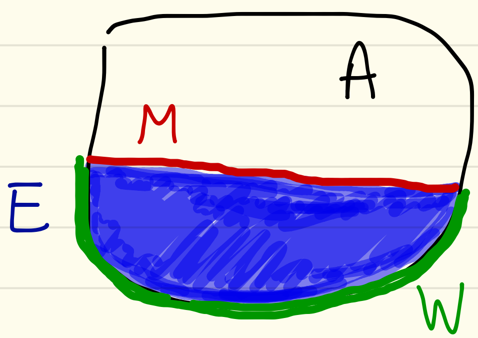
HYPERSURFACES WITH ALMOST CONSTANT MEAN CURVATURE & CAPILLARITY THEORY

FRANCESCO MAGGI
UT AUSTIN

CAPILLARITY FUNCTIONAL

A CONTAINER $\quad M$ LIQUID/AIR INTERFACE $\quad W$ WETTED SURFACE

E DROPLET



$\sigma : \partial A \to (-1,1)$ RELATIVE ADHESION COEFFICIENT

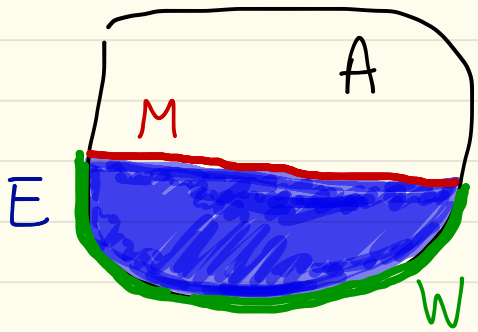
$$\boxed{\mathcal{F}(E) = \underbrace{\mathrm{Area}(M) + \int_W \sigma}_{\text{SURFACE TENSION}} + \underbrace{\int_E g(x)\,dx}_{\text{POTENTIAL ENERGY}}}$$

VOLUME CONSTRAINT $|E| = m$

CAPILLARITY FUNCTIONAL

A CONTAINER   M LIQUID/AIR INTERFACE   W WETTED SURFACE

E DROPLET $\sigma : \partial A \to (-1,1)$ RELATIVE ADHESION COEFFICIENT



$$\boxed{\mathcal{F}(E) = \underbrace{\text{Area}(M) + \int_W \sigma}_{\text{SURFACE TENSION}} + \underbrace{\int_E g(x)\,dx}_{\text{POTENTIAL ENERGY}}}$$

VOLUME CONSTRAINT $|E| = m$

CAPILLARITY: $\underbrace{\text{SURFACE TENSION}}_{O(m^{n-1/n})} \gg \underbrace{\text{POTENTIAL ENERGY}}_{O(m)}$

GOAL: UNDERSTANDING GLOBAL/LOCAL MINIMIZERS

& CRITICAL POINTS IN THE SMALL VOLUME REGIME

GLOBAL MINIMIZERS IN $\mathbb{R}^n$ (NO CONTAINER)

$$\boxed{\begin{array}{l} P(E)+\int_E g \leq P(F)+\int_F g \\ \text{FOR EVERY } |F|=|E|=m \end{array}} \implies \boxed{\begin{array}{l} P(E)+\int_E g \leq P(B)+\int_B g \\ \text{FOR EVERY BALL } |B|=m \end{array}}$$

$$P(E)-P(B) \leq \int_B g - \int_E g \leq C\,|E \Delta B|$$

$$= O(m)$$

GLOBAL MINIMIZERS IN $\mathbb{R}^n$ (NO CONTAINER)

$$\boxed{\begin{array}{l} P(E)+\int_E g \le P(F)+\int_F g \\ \text{FOR EVERY } |F|=|E|=m \end{array}} \Rightarrow \boxed{\begin{array}{l} P(E)+\int_E g \le P(B)+\int_B g \\ \text{FOR EVERY BALL } |B|=m \end{array}}$$

$$\underbrace{c\,P(B)\left(\frac{|E\Delta(x+B)|}{|E|}\right)^2}_{\text{SHARP QUANTITATIVE ISOPERIMETRIC INQ}} \le P(E)-P(B) \le \int_B g-\int_E g \le C\,|E\Delta B| = O(m)$$

FUSCO M. PRATELLI (08)

GLOBAL MINIMIZERS IN $\mathbb{R}^n$ (NO CONTAINER)

$$\boxed{\begin{array}{l} P(E)+\int_E g \le P(F)+\int_F g \\ \text{FOR EVERY } |F|=|E|=m \end{array}} \implies \boxed{\begin{array}{l} P(E)+\int_E g \le P(B)+\int_B g \\ \text{FOR EVERY BALL } |B|=m \end{array}}$$

$$\underbrace{c\,P(B)\left(\frac{|E\Delta(x+B)|}{|E|}\right)^2 \le P(E)-P(B)}\le \int_B g-\int_E g \le C\,|E\Delta B|$$

SHARP QUANTITATIVE

ISOPERIMETRIC INQ

FUSCO M. PRATELLI ('08)

AS $|E\Delta B| \le 2m$ $\quad P(B)=m^{(n-1)/n}$

WE HAVE

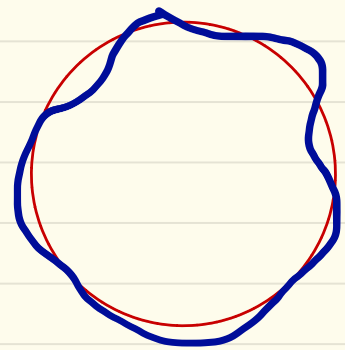
$$\boxed{\frac{|E\Delta(x+B)|}{|E|} \le C\, m^{1/2n}}$$

THM (FIGALLI M. 11)

IF E GLOBAL MINIMIZER OF $P(E)+\int_E g$ WITH $g \in C^1_{loc}(\mathbb{R}^n)$ & g COERCIVE THEN $|E| = m < m_0(n, g)$ IMPLIES

$$\frac{R^{OUT}(E)}{R^{IN}(E)} \leq 1 + C\, m^{1/n^2}$$

$$\left\| m^{1/n} k_i^{\partial E} - c(n) \right\|_{C^0(\partial E)} \leq C\, m^{\frac{2}{n+2}}$$



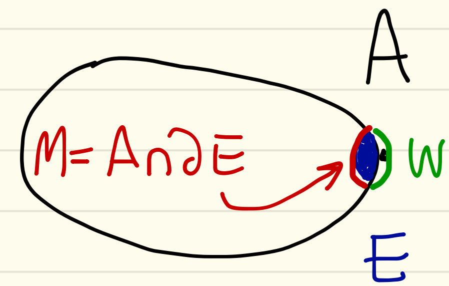
∂E

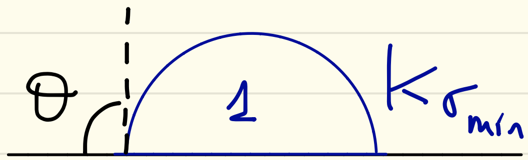
IN PARTICULAR E IS CONVEX

THM (MIHAILA, M. 15)

IF E GLOBAL MINIMIZER OF $\mathcal{H}^{n-1}(M)+\int_W \sigma+\int_E g$ WITH

$\partial A\in C^{1,1}$ $\sigma\in Lip(A)$ $g\in Lip(A)$





$\cos\theta=\sigma_{min}$

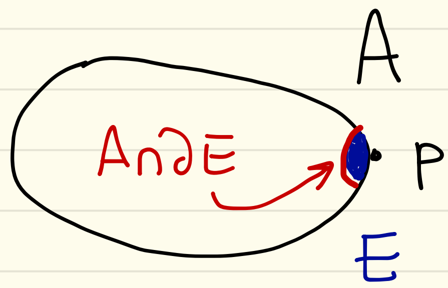
THM (MIHAILA, M. 15)

IF E GLOBAL MINIMIZER OF $\mathcal{H}^{n-1}(M) + \int_W \sigma + \int_E g$ WITH

$\partial A \in C^{1,1}$ $\sigma \in Lip(A)$ $g \in Lip(A)$ THEN $|E| = m < m_0(n, g)$ IMPLIES

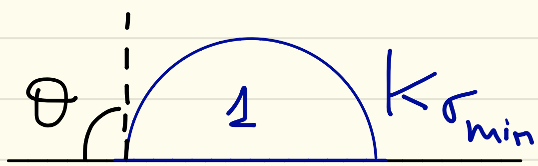
$\exists\, p \in \partial A$ & L ISOMETRY S.T.

$E \subseteq B_{cm^{1/n}}(p)$



$\sigma(p) - \sigma_{min} \leq C\, m^{1/n}$

$hd\left(L\left(\frac{A \cap \partial E - p}{m^{1/n}}\right), K_\sigma\right) \leq C\, m^{1/2n^2}$



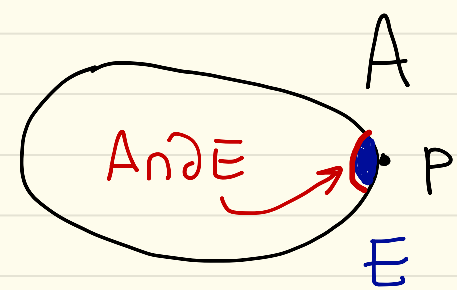
$\cos\theta = \sigma_{min}$

THM (MIHAILA, M. 15)

IF E GLOBAL MINIMIZER OF $\mathcal{H}^{n-1}(M)+\int_W \sigma+\int_E g$ WITH

$\partial A\in C^{1,1}$ $\sigma\in Lip(A)$ $g\in Lip(A)$ THEN $|E|=m<m_0(n,g)$ IMPLIES

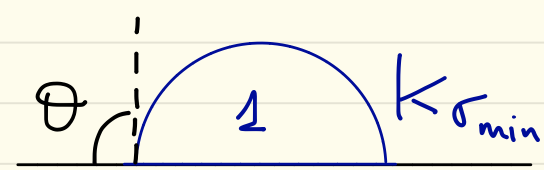
$\exists\, p\in\partial A$ & L ISOMETRY S.T.



$E\subseteq B_{cm^{1/n}}(p)$

$\sigma(p)-\sigma_{min}\le C\, m^{1/n}$

$$hd\left(L\left(\frac{A\cap\partial E-p}{m^{1/n}}\right),K_\sigma\right)\le C\, m^{1/2n^2}$$



$\cos\theta=\sigma_{min}$

MOREOVER $A\cap\partial E$ IS $C^{1,\alpha}$ DIFFEO TO K_σ

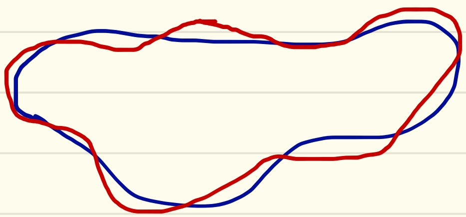
WITH $f\approx p+m^{1/n}L$ & $\text{ENERGY}(E)=m^{\frac{n-1}{n}}c(\sigma_{min})\left(1+O(m^{1/n})\right)$

PHYSICAL MOTIVATION $\Longrightarrow$ LOCAL MINIMIZERS / STATIONARY SETS

$\Longrightarrow$ NO COMPARISON WITH BALLS !!!

LOCAL MINIMIZERS

$$P(E) + \int_E g \le P(F) + \int_F g$$



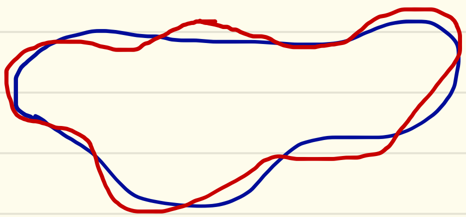
whenever $F \Delta E \subset\subset \underbrace{I_\varepsilon(\partial E)}_{\varepsilon\text{-NEIGHBORHOOD}}$, $|F| = |E|$

PHYSICAL MOTIVATION $\Longrightarrow$ LOCAL MINIMIZERS / STATIONARY SETS

$\Longrightarrow$ NO COMPARISON WITH BALLS !!!

LOCAL MINIMIZERS

$$P(E) + \int_E g \le P(F) + \int_F g$$



whenever $F \Delta E \subset\subset \underbrace{I_\varepsilon(\partial E)}_{\varepsilon\text{-NEIGHBORHOOD}}$, $|F| = |E|$

STATIONARY SETS

$\exists\, \lambda \in \mathbb{R}$ s.t.

$$H_{\partial E}(x) + g(x) = \lambda \quad \text{for every } x \in \partial E$$

$$= \text{MEAN CURVATURE OF } \partial E = \sum_{i=1}^{n} \kappa_i^{\partial E}$$

STATIONARY SETS $H_{\partial E}(x) + g(x) = \lambda$ FOR EVERY $x \in \partial E$

$g \equiv 0 \implies H_{\partial E}$ CONSTANT ON ∂E

ALEXANDROV'S THM Σ EMBEDDED BOUNDED CMC HYPERSURFACE

$\implies \Sigma$ IS A SPHERE

SMALL MASS REGIME $\Longrightarrow$ ALMOST CMC HYPERSURFACES

$$H_{\partial E} + g = \lambda \text{ ON } \partial E \Longrightarrow \lambda = H^0_{\partial E} + \frac{1}{|E|}\int_E \operatorname{div}\big(g(x)x\big)\,dx$$

$$H^0_{\partial E} = \frac{n\,P(E)}{(n+1)|E|} \approx m^{-1/n}$$

SMALL MASS REGIME $\Longrightarrow$ ALMOST CMC HYPERSURFACES

$$H_{\partial E} + g = \lambda \text{ ON } \partial E \implies \lambda = H^0_{\partial E} + \frac{1}{|E|}\int_E \operatorname{div}(g(x)x)\,dx$$

$$H^0_{\partial E} = \frac{n P(E)}{(n+1)|E|} \approx m^{-1/n}$$

THUS E CRITICAL FOR $P(E) + \int_E g(x)\,dx$ IMPLIES

$$\underbrace{\frac{\|H_{\partial E} - H^0_{\partial E}\|_{C^0(\partial E)}}{H^0_{\partial E}}} \le C(n,g)\, m^{1/n}$$

ALEXANDROV'S DEFICIT: SCALE INVARIANT, ≥ 0 QUANTITY

$= 0 \iff E = B + x$ FOR $x \in \mathbb{R}^{n+1}$

$$\delta(E) = \frac{\| H_{\partial E} - H^{\circ}_{\partial E} \|_{C^{\circ}(\partial E)}}{H^{\circ}_{\partial E}} \qquad H^{\circ}_{\partial E} = \frac{n P(E)}{(n+1)|E|}$$

QUESTION DOES $\delta(E)$ SMALL IMPLY ∂E CLOSE TO SPHERE?

$$\delta(E) = \frac{\| H_{\partial E} - H^{\circ}_{\partial E} \|_{C^{\circ}(\partial E)}}{H^{\circ}_{\partial E}} \qquad H^{\circ}_{\partial E} = \frac{n P(E)}{(n+1)|E|}$$

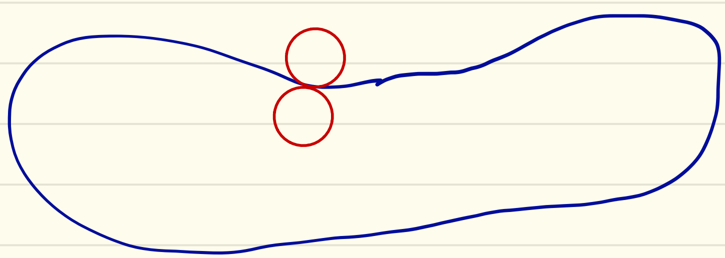
QUESTION DOES $\delta(E)$ SMALL IMPLY ∂E CLOSE TO SPHERE?

THM (CIRAOLO VEZZONI 2015)

E BOUNDED OPEN SET $\partial E \in C^2$ WITH EXT/INT BALL COND. $\rho > 0$

THEN $\delta(E) \leq \delta_0(n, P(E)/\rho^{n-1})$

GIVES $\frac{R^{out}(E)}{R^{in}(E)} - 1 \leq C\,\delta(E)$

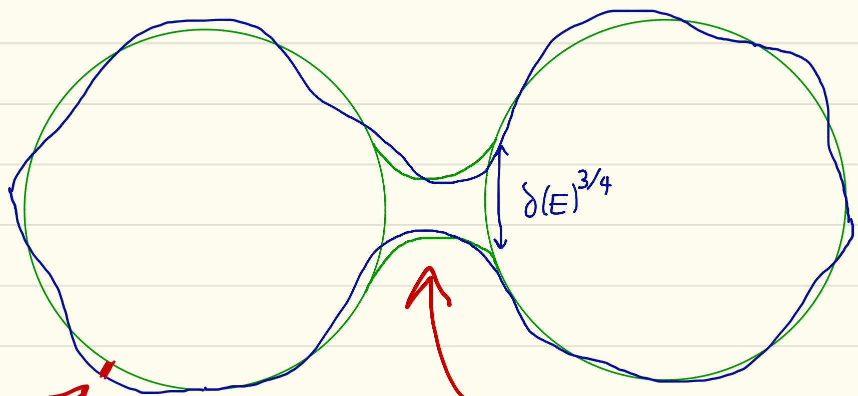


E

SHARP DECAY RATE

AN EXAMPLE BY BUTSCHER-MAZZEO

$\delta(E)$ SMALL ∂E ARRAY TANGENT BALLS



CATENOIDAL NECK OF LENGHT

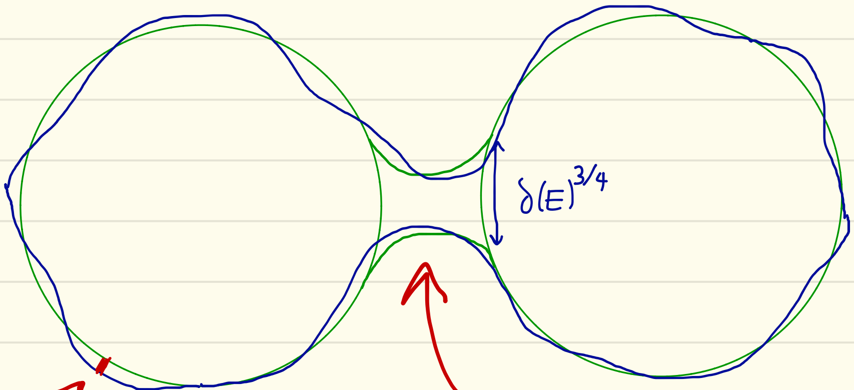
$\delta(E)\,|\log \delta(E)|$

NORMAL DEFORMATION

C^k-NORM OF ORDER $\delta(E)$

AN EXAMPLE BY BUTSCHER-MAZZEO

$\delta(E)$ SMALL ∂E ARRAY TANGENT BALLS



CATENOIDAL NECK OF LENGHT $\delta(E)\,|\log \delta(E)|$

NORMAL DEFORMATION
C^k-NORM OF ORDER $\delta(E)$

OTHER OPTION CHOPPING AN UNDULOID

THM (CIRAOLO-M. 2015)

E OPEN BOUNDED C^2 IN $\mathbb{R}^{n+1}$

$$H^{\circ}_{\partial E} = \frac{n P(E)}{(n+1)|E|} = n \quad \text{(SCALING)}$$

$$P(E) \leq (L+1-a) P(B) \quad \left[\begin{array}{ll} \text{SOME} & L \in \mathbb{N} \\ \text{FIXED} & 0<a<1 \end{array} \right]$$

$$\delta(E) \leq \delta_0(n, L, a)$$

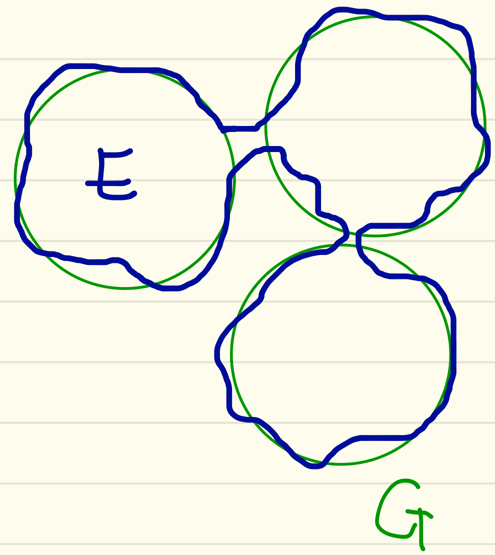
THM (CIRAOLO-M. 2015)

E OPEN BOUNDED C^2 IN $\mathbb{R}^{n+1}$

$$\boxed{\begin{array}{ll} H^0_{\partial E} = \dfrac{nP(E)}{(n+1)|E|} = n & \text{(SCALING)} \\ P(E) \le (L+1-a)P(B) & \left[\begin{array}{ll}\text{SOME} & L \in \mathbb{N} \\ \text{FIXED} & 0<a<1\end{array}\right] \\ \delta(E) \le \delta_0(n,L,a) & \end{array}}$$

THEN $\exists$ $\{B(z_j)\}_{j\in J}$ DISJOINT BALLS RADIUS 1 WITH $\# J \le L$ SUCH THAT

FOR $G = \bigcup_{j\in J} B(z_j)$ ONE HAS



THM (CIRAOLO–M. 2015)

E OPEN BOUNDED C^2 IN $\mathbb{R}^{n+1}$

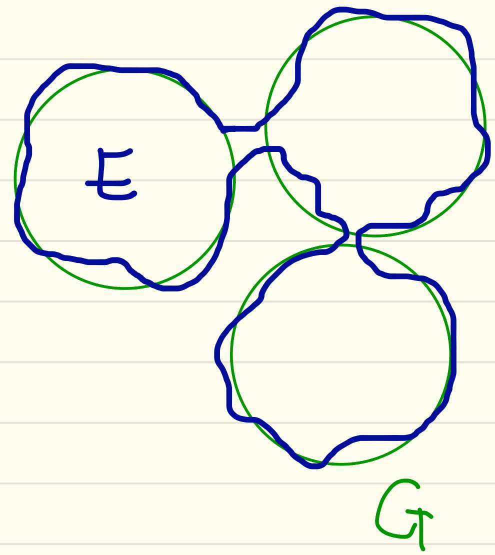
$$\boxed{\begin{array}{ll} H^{\circ}_{\partial E} = \dfrac{nP(E)}{(n+1)|E|} = n & \text{(SCALING)} \\ P(E) \le (L+1-a)P(B) & \left[\begin{array}{ll}\text{SOME} & L \in \mathbb{N} \\ \text{FIXED} & 0<a<1\end{array}\right] \\ \delta(E) \le \delta_0(n,L,a) & \end{array}}$$

THEN $\exists$ $\{B(z_j)\}_{j\in J}$ DISJOINT BALLS RADIUS 1 WITH $\# J \le L$ SUCH THAT

FOR $G = \bigcup_{j\in J} B(z_j)$ ONE HAS

$$\frac{|E \Delta G|}{|E|} + \frac{|P(E) - \# J\, P(B)|}{P(E)} \le C\, \delta(E)^{\frac{1}{2(n+2)}}$$

$$\frac{hd(\partial E, \partial G)}{\operatorname{diam}(E)} \le C\, \delta(E)^{\alpha} \qquad \alpha = O(n^{-4})$$



CONTINUED...

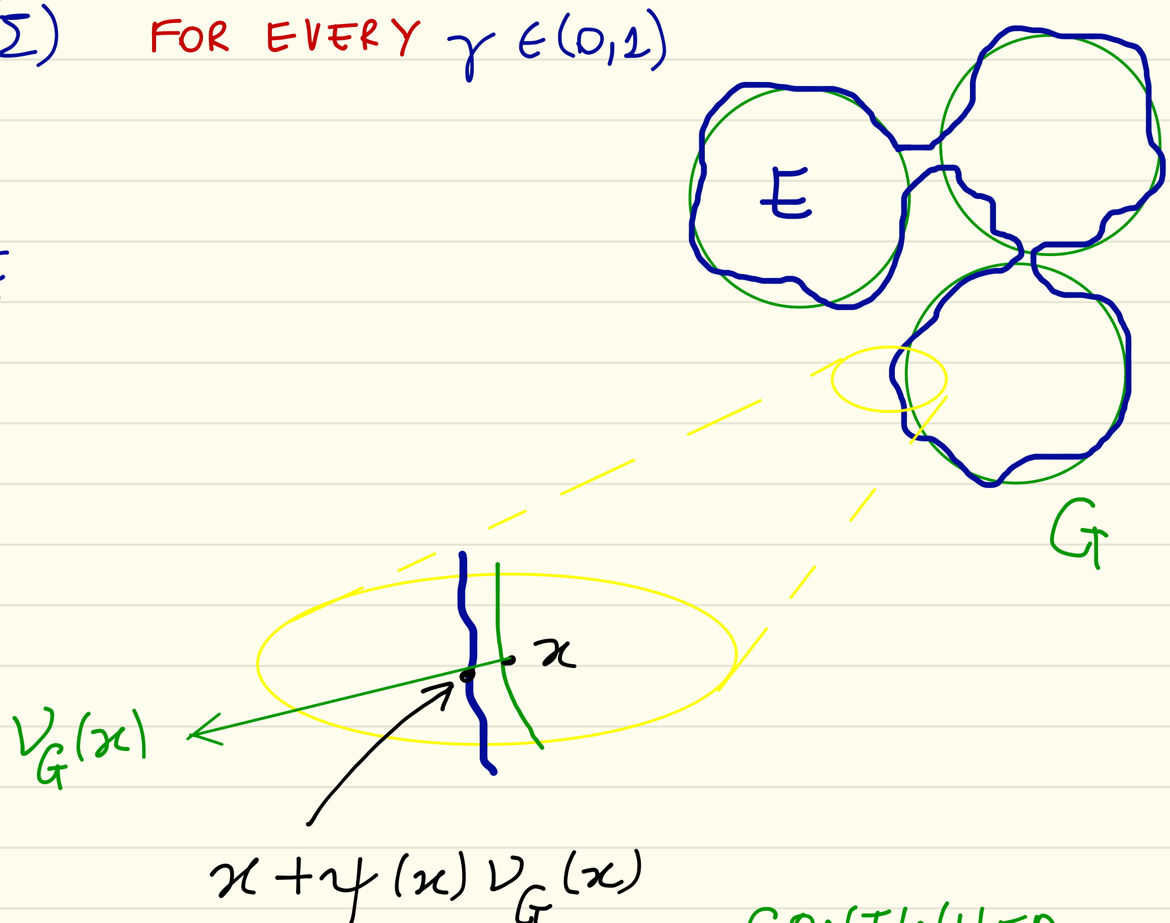
MOREOVER:

$$\exists\ \Sigma = \partial G \setminus \{\text{AT MOST } C(n,L) \text{ MANY SPHERICAL CAPS OF DIAMETER} \le C\,\delta(E)^{\alpha} \quad \alpha = O(n^{-2})\}$$

$$\exists\ \psi \in C^{1,\gamma}(\Sigma) \quad \text{FOR EVERY } \gamma \in (0,1)$$

SUCH THAT

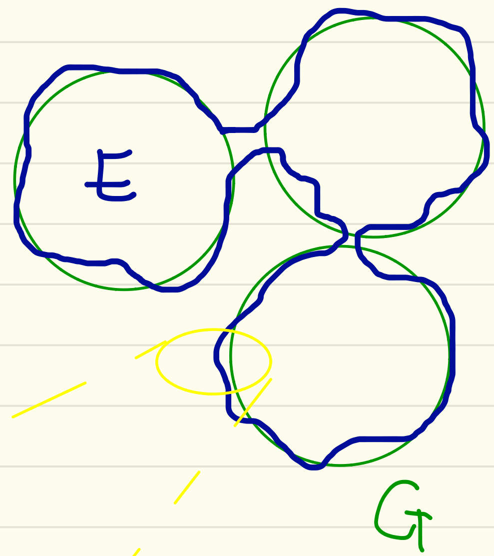
$$(Id + \psi\,\nu_G)(\Sigma) \subseteq \partial E$$



CONTINUED...

MOREOVER : $\exists\ \Sigma = \partial G \setminus \{$AT MOST $C(n,L)$ MANY SPHERICAL CAPS OF DIAMETER $\leq C\,\delta(E)^{\alpha}$ $\alpha = O(n^{-2})\}$

$\exists\ \psi \in C^{1,\gamma}(\Sigma)$ FOR EVERY $\gamma \in (0,1)$



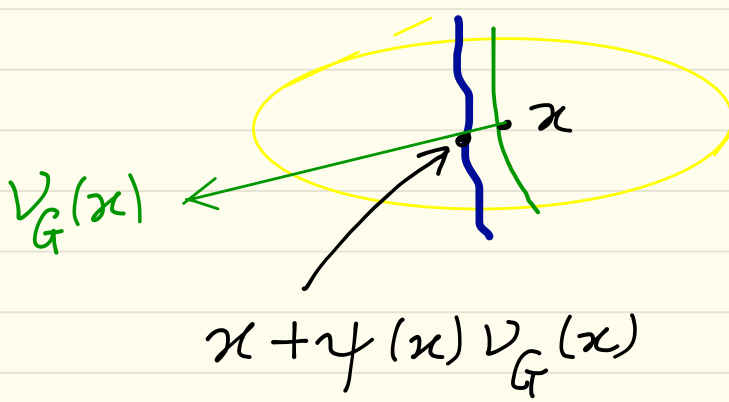
SUCH THAT

$$(Id + \psi\,\nu_G)(\Sigma) \subseteq \partial E$$

$$\mathcal{H}^n\left(\partial E \setminus (Id + \psi\,\nu_G)(\Sigma)\right) \leq C\,\delta(E)^{\alpha}$$

$$\|\psi\|_{C^1(\Sigma)} \leq C\,\delta(E)^{\alpha}$$

$$\alpha = O(n^{-3})$$



CONTINUED...

FINALLY :

① IF $\# J \geq 2$ THEN $\forall\, z_j\ \exists\, z_h$ SUCH THAT

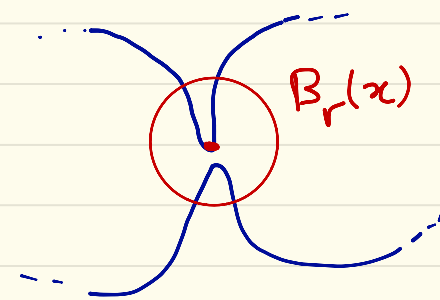
$$\big|\, |z_j - z_h| - 2 \,\big| \leq C\, \delta(E)^{\alpha} \qquad \alpha = O(n^{-2})$$

FINALLY: ① IF $\# J \geq 2$ THEN $\forall\, z_j\ \exists\, z_h$ SUCH THAT

$$\big|\, |z_j - z_h| - 2 \,\big| \leq C\, \delta(E)^{\alpha} \qquad \alpha = O(n^{-2})$$

② IF $\exists\, k > 0$ S.T. $|B_r(x) \setminus E| \geq k\, |B_r(x)|$ FOR EVERY $x \in \partial E$ $r < k$

THEN $\# J = 1$



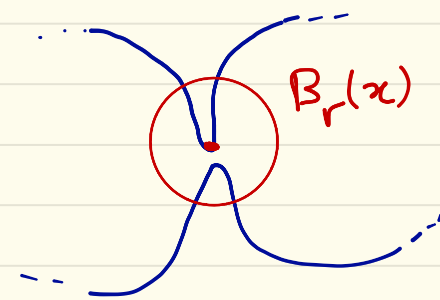
FINALLY: ① IF $\#J \geq 2$ THEN $\forall\, z_j \ \exists\, z_h$ SUCH THAT

$$\big|\,|z_j - z_h| - 2\,\big| \leq C\,\delta(E)^{\alpha} \qquad \alpha = O(n^{-2})$$

② IF $\exists\, k > 0$ S.T. $|B_r(x) \setminus E| \geq k\,|B_r(x)|$ FOR EVERY $x \in \partial E$ $\;r < k$

THEN $\#J = 1$

COROLLARY LOCAL MINIMIZERS WITH m SMALL ARE CLOSE TO SINGLE SPHERES!



PROOF: PREVIOUS THM + DENSITY ESTIMATES

NONLOCAL CAPILLARITY

M. VALDINOCI (INCOMING!)

NO CONTAINER

FRACTIONAL PERIMETER

$$P_s(E) = \int_E \int_{E^c} \frac{dx\,dy}{|x-y|^{n+s}} \qquad 0<s<1$$

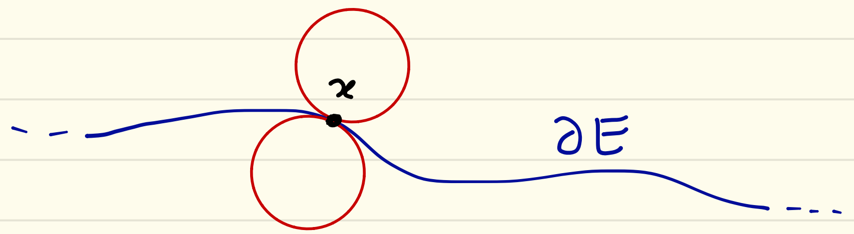
BOURGAIN BREZIS MIRONESCU
CAFFARELLI SOUGANIDIS
CAFFARELLI ROQUEJOFFRE SAVIN

$$w_E(x) = \int_{E^c} \frac{dy}{|x-y|^{n+s}} \approx \frac{1}{\mathrm{DIST}(x,\partial E)^s}$$

$\in L^1(E)$ BY COAREA

$$\min\left\{ P_s(E) + \int_E g(x)\,dx : |E| = m \right\}$$

$$\Longrightarrow \quad H^s_{\partial E}(x) + g(x) = \lambda \quad \text{ON } \partial E$$



$$H^s_{\partial E}(x) = \text{P.V.} \int_{\mathbb{R}^n} \frac{1_{E^c}(y) - 1_E(y)}{|x-y|^{n+s}}\,dy$$

WHEN m IS SMALL THEN $H^s_{\partial E} \approx$ CONSTANT

THM (CIRAOLO FIGALLI M. NOVAGA 15)

① $H^s_{\partial E}$ CONSTANT $\Rightarrow$ ∂E SPHERE (CABRÉ FALL SOLA-MORALES WETH)

THM (CIRAOLO FIGALLI M. NOVAGA 15)

① $H^s_{\partial E}$ CONSTANT $\Longrightarrow$ ∂E SPHERE (CABRÉ FALL SOLA-MORALES WETH)

② $$\frac{R^{OUT}(E) - R^{IN}(E)}{DIAM(E)} \leq C \, \frac{DIAM(E)^{2n+2s+1}}{|E|^2} \, Lip\left(H^s_{\partial E}\right) = C \eta_s(E)$$

THM (CIRAOLO FIGALLI M. NOVAGA 15)

① $H^s_{\partial E}$ CONSTANT $\Rightarrow$ ∂E SPHERE (CABRÉ FALL SOLA-MORALES WETH)

② $\dfrac{R^{out}(E) - R^{in}(E)}{\mathrm{DIAM}(E)} \leq C \dfrac{\mathrm{DIAM}(E)^{2n+2s+1}}{|E|^2} \mathrm{Lip}(H^s_{\partial E}) = C\eta_s(E)$

③ $\partial E = (Id + \varphi \nu_{B_1})(\partial B_1)$ $\varphi \in C^{2,\tau}$ $\|\varphi - 1\|_{C^{2,\tau}(\partial B_1)} \leq C \eta_s(E)$

$|E| = |B_1|$

E CONVEX

become a perfectly round ball, because in no other way can so small a surface be obtained. If, instead of taking so much water, we were to take a drop about as large as a pin's head, then the weight which tends to squeeze it out or make it fall would be far less, while the skin would be just as strong, and would in reality have a greater moulding power, though why I cannot now explain. We should therefore expect that by taking a sufficiently small quantity of water the moulding power of the skin would ultimately be able almost entirely to counteract the weight of the drop, so that very small drops should appear like perfect little balls. If you have found any difficulty in following this argument, a very simple illustration will make it clear. You many of you probably know how by folding paper to make this little thing which I hold in my hand (Fig.

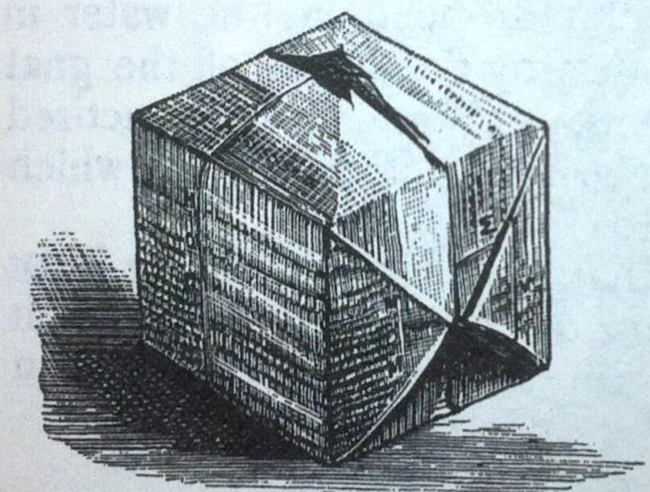


FIG. 16.