A Stability Index for Traveling Waves in Activator-Inhibitor Systems

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Motivation

- The *Maslov index* is a topological invariant assigned to curves of Lagrangian subspaces. It has been used to analyze the spectra of self-adjoint operators.

- Our goal is to develop the theory of the Maslov index for traveling waves in activator-inhibitor systems.

- Our main result relates the parity of the Maslov index to the sign of the derivative of the Evans function at $\lambda = 0$.

The Problem

- Consider the reaction-diffusion system

$$\begin{aligned} u_t &= u_{xx} + f(u) - \sigma v \\ v_t &= v_{xx} + \alpha u + g(v), \end{aligned} \tag{1}$$

  where $\sigma, \alpha > 0$ and $u, v, x, t \in \mathbb{R}$.

- The signs of σ and α are chosen so that this system is of activator-inhibitor type. We assume that $(0, 0)$ is a stable steady state of the reaction equation.

- We assume that (1) possesses a transversely constructed traveling pulse and study its stability.

Traveling Wave Equation

- A traveling pulse of (1) is a solution $\varphi = (\hat{u}, \hat{v})$ of one variable $z = x - ct$ to the ODE

$$\begin{aligned} 0 &= u_{zz} + cu_z + f(u) - \sigma v \\ 0 &= v_{zz} + cv_z + \alpha u + g(v) \end{aligned} \tag{2}$$

that decays exponentially to $(0,0)$ as $z \to \pm\infty$.

- Setting $u_z = \sigma w$ and $v_z = \alpha y$, we can write (2) as a first order system

$$\begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix}' = \begin{pmatrix} \sigma w \\ \alpha y \\ -cw + v - f(u)/\sigma \\ -cy - u - g(v)/\alpha \end{pmatrix}. \tag{3}$$

Stability Analysis

- The (nonlinear) stability of φ is determined by analyzing the spectrum of the linearization L of (2) about φ.

- Written as a first-order system, the eigenvalue problem $Lp = \lambda p$ becomes

$$\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}' = \begin{pmatrix} 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & \alpha \\ \frac{\lambda - f'(\hat{u})}{\sigma} & 1 & -c & 0 \\ -1 & \frac{\lambda - g'(\hat{v})}{\alpha} & 0 & -c \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} \tag{4}$$

- For $\lambda \in \mathbb{C}$ to be an eigenvalue, there must exist a bounded solution to (4), which we write $Y'(z) = A(\lambda, z)Y(z)$.

The Evans Function

- Fact: $\sigma_{\text{ess}}(L) \subset \{z \in \mathbb{C} : \text{Re } z < 0\}$. For Re $\lambda \geq 0$, the asymptotic matrix $A(\lambda)$ has two-dimensional stable and unstable subspaces $W^s(\lambda)$ and $W^u(\lambda)$.

- From standard theory, we therefore have two-dimensional spaces of solutions of (4), $E^s(\lambda, z)$ and $E^u(\lambda, z)$, decaying to 0 as $z \to \infty$ and as $z \to -\infty$ respectively.

- Furthermore, $E^s(\lambda, z)$ (resp. $E^u(\lambda, z)$) is asymptotically tangent to $W^s(\lambda)$ (resp. $W^u(\lambda)$) as $z \to \infty$ (resp. $z \to -\infty$).

- The Evans function $D(\lambda) = e^{2cz} E^s(\lambda, z) \wedge E^u(\lambda, z)$ determines whether these subspaces intersect, and hence whether λ is an eigenvalue.

Symplectic Structure

- The evolution of two-planes can be tracked by using Plücker coordinates. A basis $\{e_i\}$ yields a basis $\{e_i \wedge e_j\}$ of $\bigwedge^2(\mathbb{R}^4)$, on which (4) induces an equation.
- Denoting p_{ij} the $e_i \wedge e_j$ component of a plane, one computes that $\frac{d}{dz}(p_{13} - p_{24}) = -c(p_{13} - p_{24})$.
- The two-form ω dual to $p_{13} - p_{24}$ is symplectic, hence the set of ω-Lagrangian planes is invariant under the flow. Moreover, the form $\Omega(\cdot, \cdot) := e^{cz}\omega(\cdot, \cdot)$ is invariant on any two solutions of (4).
- Key fact: $E^{s/u}(\lambda, z)$ are ω-Lagrangian for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{R}$.

The Symplectic Evans Function

- Assume that we have spanning solutions $u_i(\lambda, z)$ such that $E^s(\lambda, z) = \text{sp}\{u_1, u_2\}$ and $E^u(\lambda, z) = \text{sp}\{u_3, u_4\}$.
- Using the standard volume form, we can rewrite the Evans function as

$$\begin{aligned} D(\lambda) &= e^{2cz} E^s(\lambda, z) \wedge E^u(\lambda, z) \\ &= e^{2cz} \det\left[u_1, u_2, u_3, u_4\right] \text{vol}. \end{aligned} \tag{5}$$

- The following formula due to Chardard and Bridges ('14) allows us to exploit the symplectic structure:

$$D(\lambda) = -e^{2cz} \begin{bmatrix} \omega(u_1, u_3) & \omega(u_1, u_4) \\ \omega(u_2, u_3) & \omega(u_2, u_4) \end{bmatrix} \text{vol}. \tag{6}$$

$D'(0)$ Calculation

- $\lambda = 0$ is an eigenvalue due to translation invariance. Generically, we have $\varphi'(z) = u_2(0, z) = u_3(0, z)$.

- Using Jacobi's formula, we calculate:

$$D'(0) = \Omega(u_1, u_4)\partial_\lambda \Omega(u_2, u_3)|_{\lambda=0}$$
$$= \Omega(u_1, u_4) \int_{-\infty}^{\infty} e^{cz} \left(\frac{(\hat{u}')^2}{\sigma} - \frac{(\hat{v}')^2}{\alpha} \right) dz. \tag{7}$$

- $\Omega(u_1, u_4)$ is called the *Lazutkin-Treschev* invariant. Its sign is not obvious.

The Maslov Index

- A plane $V \in \mathrm{Gr}_2(\mathbb{R}^4)$ is called Lagrangian if $\omega|_V = 0$. The set of Lagrangian planes is a compact $3-$manifold $\Lambda(2)$ with $\pi_1(\Lambda(2)) = \mathbb{Z}$.

- For fixed $V \in \Lambda(2)$ and curve $\gamma : [a,b] \to \Lambda(2)$, the *Maslov index* $\mu(\gamma, V)$ counts how many times $\gamma(t) \cap V \neq \{0\}$.

- We consider the curve $z \mapsto E^u(0,z)$ and count intersections with the plane $E^s(0,\tau)$, $\tau \gg 1$. The domain of the curve is $(-\infty, \tau]$, which forces a crossing at $z = \tau$.

- Chen and Hu ('07) showed that this definition is independent of τ, provided that $E^s(\tau', 0) \cap W^u(0) = \{0\}$ for all $\tau' \geq \tau$.

The Maslov Index

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- A *conjugate point* is a value $z = z^*$ such that $E^u(0, z^*) \cap E^s(0, \tau) \neq \{0\}$. At such a point, the crossing form defined on the intersection is given by

$$\Gamma(z^*)(\zeta) = \omega(\zeta, A(0, z^*)\zeta). \tag{8}$$

- The Maslov index of the homoclinic orbit is then given by

$$\text{Maslov}(\varphi) := \frac{1}{2} + \sum_{z^* \in (-\infty, \tau)} \text{sign}\Gamma(z^*) + \frac{1}{2}\text{sign}\Gamma(\tau), \tag{9}$$

where the sum is taken over all interior conjugate points.

Bridging the Gap

- We relate the Evans function and Maslov index through the Lazutkin-Treschev invariant. The two-form

$$\pi(\cdot,\cdot) = \det\left[e^{-\mu_1(0)}u_1, e^{-\mu_2(0)}u_2, \cdot, \cdot\right] \tag{10}$$

detects crossings with the train of $E^s(0,\tau)$. In particular,

$$\beta(z) := \pi\left(E^u(0,z)\right) \tag{11}$$

vanishes precisely at conjugate points.

- Furthermore, the multiplicity of z^* as a root of β has the same parity as the signature of $\Gamma(z^*)$.

Stability Index

- A simple calculation shows that

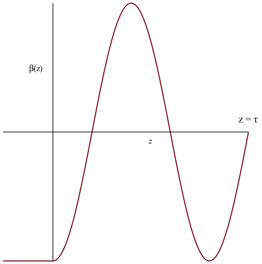
$$\beta'(\tau) = \Omega(u_1, u_4)\Omega(\varphi'(\tau), \varphi''(\tau)). \tag{12}$$

Also, $\beta < 0$ for large $z < 0$.

- Intuitively, the sign of $\beta'(\tau)$ is determined by the number of zeros of β prior to τ.
- This allows us to prove

$$(-1)^{\mathrm{Maslov}(\varphi)+1} = \operatorname{sign}\Omega(u_1, u_4). \tag{13}$$

Stability Index



$$\beta'(\tau) = \Omega(u_1, u_4)\Omega(\varphi'(\tau), \varphi''(\tau))$$

Maslov(φ)	$\beta'(\tau)$	$\Gamma(\tau)$	$\Omega(u_1, u_4)$
e	$+$	$-$	$-$
e	$-$	$+$	$-$
o	$+$	$+$	$+$
o	$-$	$-$	$+$

Application: a FitzHugh-Nagumo System

- It can be shown that the FitzHugh-Nagumo system

$$\begin{aligned} u_t &= u_{xx} + f(u) - v \\ v_t &= v_{xx} + \epsilon\,(u - \gamma v) \end{aligned} \tag{14}$$

  has fast traveling wave solutions for $0 < \epsilon \ll 1$.

- The same waves with no diffusion on v were shown to be stable by Jones ('84) using a $D'(0)$ calculation.

- The stability question reduces to finding $D'(0)$ in this case as well, so our result applies. Techniques from geometric singular perturbation theory should allow us to calculate the Maslov index.