Large Eddy Simulation Reduced Order Models

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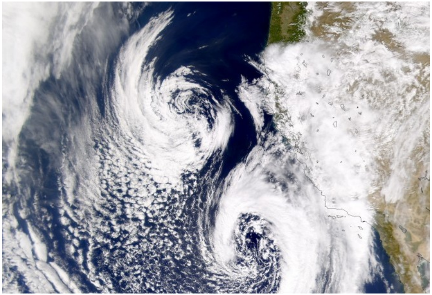
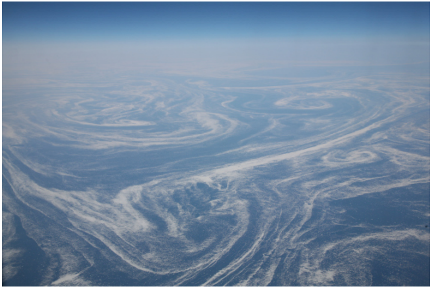
Goal – Structure Dominated Turbulence





Conjecture

LES-ROM is the answer !

Standard ROM (G-ROM)

- $\{\varphi_1, \ldots, \varphi_r, \varphi_{r+1}, \ldots\ldots, \varphi_d\}$
- $\boldsymbol{u} \approx \boldsymbol{u}_r(\mathbf{x}, t) \equiv \sum_{j=1}^{r} a_j(t)\varphi_j(\mathbf{x}),$
- $\frac{\partial \boldsymbol{u}_r}{\partial t} - Re^{-1}\Delta \boldsymbol{u}_r + \boldsymbol{u}_r \cdot \nabla \boldsymbol{u}_r + \nabla p = 0$
  - $r = d$ ☺
  - $r << d$ ☹
- G-ROM
  - $\left(\frac{\partial \boldsymbol{u}_r}{\partial t}, \varphi_k\right) + \frac{2}{Re}\left(\mathbb{D}(\boldsymbol{u}_r), \nabla \varphi_k\right) + \left((\boldsymbol{u}_r \cdot \nabla)\boldsymbol{u}_r, \varphi_k\right) = 0$

What Is LES-ROM ?

- Filter the NSE to obtain the *spatially filtered NSE (SF-NSE)*.
  - $G \Big| \quad \frac{\partial \boldsymbol{u}}{\partial t} - Re^{-1} \Delta \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = 0$
  - $\frac{\partial \overline{\boldsymbol{u}}}{\partial t} - Re^{-1} \Delta \overline{\boldsymbol{u}} + \overline{\boldsymbol{u} \cdot \nabla \boldsymbol{u}} + \nabla \overline{p} = 0$
  - $\overline{\boldsymbol{u} \cdot \nabla \boldsymbol{u}} = function(\overline{u})$
  - $\left(\frac{\partial \overline{\boldsymbol{u}}}{\partial t}, \phi \right) + \frac{2}{Re} \left(\mathbb{D}(\overline{\boldsymbol{u}}), \nabla \phi \right) + \left(function(\overline{\boldsymbol{u}}), \phi \right) = 0$
- Use the SF-NSE and the ROM approximation to obtain the LES-ROM.
  - $\left(\frac{\partial \boldsymbol{w}_r}{\partial t}, \varphi_k \right) + \frac{2}{Re} \left(\mathbb{D}(\boldsymbol{w}_r), \nabla \varphi_k \right) + \left(function(\boldsymbol{w}_r), \varphi_k \right) = 0$

Spatially Filtered ROM

- $\frac{\partial \overline{\boldsymbol{u}}}{\partial t} - Re^{-1} \Delta \overline{\boldsymbol{u}} + \nabla \cdot \left(\overline{\boldsymbol{u}\, \boldsymbol{u}} \right) + \nabla \overline{p} = 0$
- $\overline{\boldsymbol{u}\, \boldsymbol{u}} = \overline{\boldsymbol{u}}\, \overline{\boldsymbol{u}} + \left(\overline{\boldsymbol{u}\, \boldsymbol{u}} - \overline{\boldsymbol{u}}\, \overline{\boldsymbol{u}} \right)$
- $\frac{\partial \overline{\boldsymbol{u}}}{\partial t} - Re^{-1} \Delta \overline{\boldsymbol{u}} + \nabla \cdot (\overline{\boldsymbol{u}}\, \overline{\boldsymbol{u}}) + \nabla \cdot \tau + \nabla \overline{p} = 0$
  - subfilter-scale stress tensor $\boxed{\tau = \overline{\boldsymbol{u}\, \boldsymbol{u}} - \overline{\boldsymbol{u}}\, \overline{\boldsymbol{u}}}$

LES-ROM

- LES-ROM *(spatially filtered ROM)*

$$\left(\frac{\partial \overline{\boldsymbol{u}}_r}{\partial t}, \varphi_k\right) + \frac{2}{Re}\left(\mathbb{D}(\overline{\boldsymbol{u}}_r), \nabla \varphi_k\right) + \left((\overline{\boldsymbol{u}}_r \cdot \nabla)\overline{\boldsymbol{u}}_r, \varphi_k\right) + (\tau, \nabla \varphi_k) = 0$$

- better than G-ROM
  - larger spatial scales
  - $r \ll d$ ☺
- challenges
  - ROM closure model

ROM Spatial Filtering: Projection

- POD basis $\mathbf{X}^r = \{\varphi_1, \ldots, \varphi_{r_1}, \varphi_{r_1+1}, \ldots\ldots\ldots\ldots, \varphi_r\}$
- POD projection basis $\mathbf{X}^{r_1} = \{\varphi_1, \ldots, \varphi_{r_1}\}$
- given $\boldsymbol{u}^r \in \mathbf{X}^r$
- find $\overline{\boldsymbol{u}}^r \in \mathbf{X}^{r_1}$
- $\boxed{\left(\overline{\boldsymbol{u}}^r, \varphi_j^r\right) = \left(\boldsymbol{u}^r, \varphi_j^r\right)}$ $\quad \forall j = 1, \ldots r_1$
- Kunisch, Volkwein, *Numer. Math.*, 2001
- Wang, Akhtar, Borggaard, Iliescu, *Comput. Meth. Appl. Mech. Eng.*, 2012

ROM Spatial Filtering: Differential Filter

- Germano, *Phys. Fluids*, 1986
- Sabetghadam, Jafarpour, *Appl. Math. Comput.*, 2012
- Wells et al, *arXiv*, 2015
- Xie et al, *Comput.Meth.Appl.Mech.Eng.* in revision, 2016
- smoothing
- given $\boldsymbol{u}^r \in \mathbf{X}^r$
- find $\overline{\boldsymbol{u}}^r \in \mathbf{X}^r$
- $\boxed{\left(\left(I - \delta^2 \Delta \right) \overline{\boldsymbol{u}}^r , \varphi_j \right) = \left(\boldsymbol{u}^r , \varphi_j \right)} \qquad \forall j = 1, \ldots r$

ROM Closure

- stabilization
- non-LES
  - Noack et al, Iollo et al, Karniadakis et al, Farhat et al, Amsallem et al, Carlberg et al, Kalashnikova et al, Balajewicz et al, . . .
  - calibration models, power balance ROM, LSPG, etc
- LES-ROM
  - EV-ROMs: model the function of τ
    - mixing length: Lumley et al
    - Smagorinsky: Noack et al, Ullman & Lang, Wang et al
    - variational multiscale: Iollo et al, Wang et al
    - *dynamic subgrid-scale*: Wang et al
  - AD-ROM: fundamentally different
    - structural

Approximate Deconvolution ROM–Structural

- image processing, inverse problems
- deconvolution
  - given $\overline{\boldsymbol{u}}_r := G\,\boldsymbol{u}_r$
  - find $\boldsymbol{u}_r$
  - *structural* ROM closure model
- exact deconvolution $\boldsymbol{u}_r{}^{ED} = G^{-1}\,\overline{\boldsymbol{u}}_r$
  - very bad idea
  - notoriously ill-posed: noise amplification
- approximate deconvolution $\boldsymbol{u}_r{}^{AD} \approx \boldsymbol{u}_r{}^{ED} = G^{-1}\,\overline{\boldsymbol{u}}_r$
  - Lavrentiev regularization $\boxed{\boldsymbol{u}_r{}^{AD} = (G + \mu\, I)^{-1}\,\overline{\boldsymbol{u}}_r}$

Approximate Deconvolution ROM (AD-ROM)

- approximate deconvolution ROM closure model

$$\left(\frac{\partial \overline{\boldsymbol{u}}_r}{\partial t}, \varphi_k\right) + \frac{2}{Re}\left(\mathbb{D}(\overline{\boldsymbol{u}}_r), \nabla\varphi_k\right) + \left(\overline{\left(\boldsymbol{u}_r^{AD} \cdot \nabla\right) \boldsymbol{u}_r^{AD}}, \varphi_k\right) = 0$$

- $\boldsymbol{w}_r := \overline{\boldsymbol{u}}_r$

- approximate deconvolution ROM (AD-ROM)

$$\boxed{\left(\frac{\partial \boldsymbol{w}_r}{\partial t}, \varphi_k\right) + \frac{2}{Re}\left(\mathbb{D}(\boldsymbol{w}_r), \nabla\varphi_k\right) + \left(\overline{\left(\boldsymbol{w}_r^{AD} \cdot \nabla\right) \boldsymbol{w}_r^{AD}}, \varphi_k\right) = 0}$$

  - $\boldsymbol{w}_r^{AD} = (G + \mu\, I)^{-1}\, \boldsymbol{w}_r$

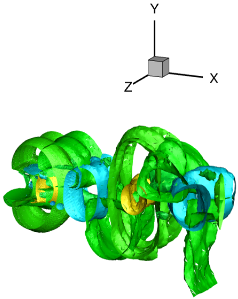
- $\boldsymbol{w}_r^{AD} \approx \boldsymbol{u}_r$

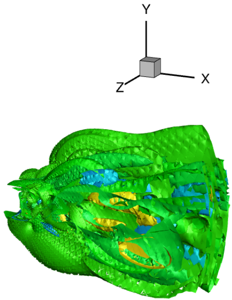
AD-ROM

- $\dot{\mathbf{a}} = \mathbf{b} + \mathbf{A}\mathbf{a} + \mathbf{A}^{AD}\mathbf{a} + \mathbf{a}^T\mathbf{B}^{AD}\mathbf{a}$
- $\mathbf{b} = -(\boldsymbol{U}\cdot\nabla\boldsymbol{U}, \overline{\varphi}) - \frac{2}{Re}(\frac{\nabla\boldsymbol{U}+\nabla\boldsymbol{U}^T}{2}, \nabla\overline{\varphi})$
- $\mathbf{A} = -\frac{2}{Re}(\frac{\nabla\varphi+\nabla\varphi^T}{2}, \nabla\varphi)$
- $\mathbf{A}^{AD} = -(\boldsymbol{U}\cdot\nabla\varphi^{AD}, \overline{\varphi}) - (\varphi^{AD}\cdot\nabla\boldsymbol{U}, \overline{\varphi})$
- $\mathbf{B}^{AD} = -(\varphi^{AD}\cdot\nabla\varphi^{AD}, \overline{\varphi})$

3D flow past a cylinder $Re = 1000$

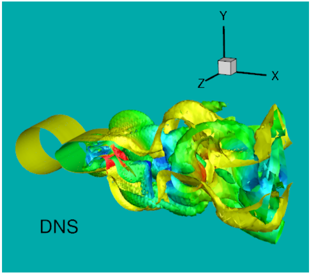
- POD modes

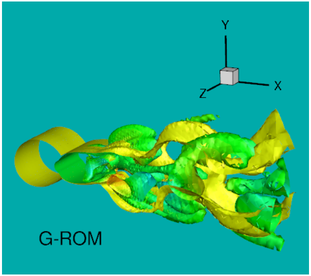


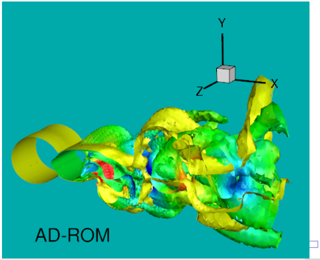


3D flow past a cylinder $Re = 1000$

- Snapshot at $t = 137.5s$,







3D cylinder flow $Re = 1000$

- Strouhal number

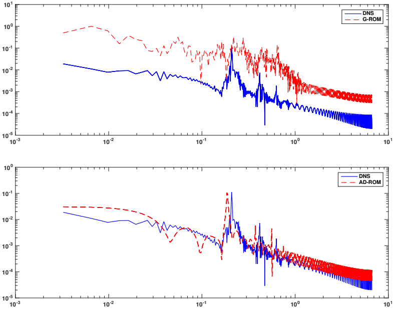
	DNS	G-ROM	AD-ROM
St	0.2083	-	0.1888

- speed up factor (online)

G-ROM	AD-ROM
≈ 389	≈ 256

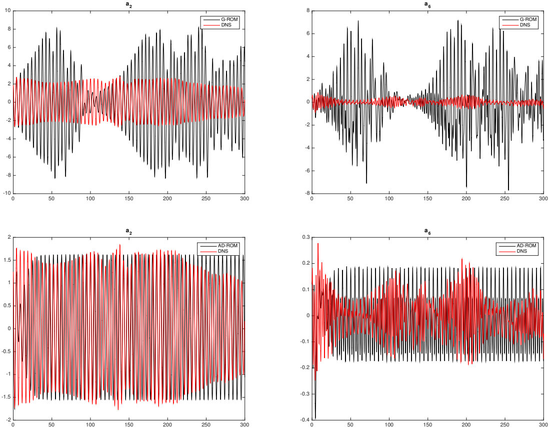
3D cylinder flow $Re = 1000$

- energy spectrum



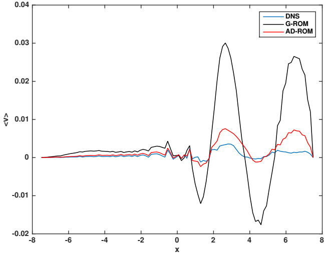
3D cylinder flow $Re = 1000$

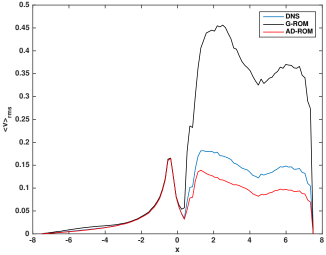
- time evolution of coefficients

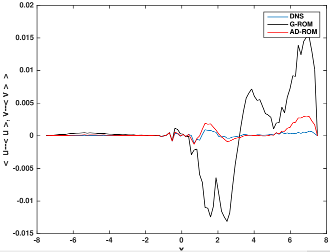


3D cylinder flow $Re = 1000$

- statistics: average, rms, Reynolds stress







Conclusions

- AD-ROM significantly better than G-ROM
- Structural ROM closure (AD-ROM) works well without any numerical dissipation mechanism
- The accuracy of AD-ROM is similar to that of EV-ROMs and Reg-ROMs without using any explicit numerical dissipation
- AD-ROM efficient (low cost) ☺
- AD-ROM predictive (use data on [0,75], run on [0,300])

Future Work

- Other regularization methods (AD)
- Turbulence ($Re \gg 10^4$)

- Parametrized ROM (Reduced Basis)

Future Work

- AD-ROM for Quasi-Geostrophic Equations (ocean model) ?

- Sparse Identification for POD modes ?

$$\Phi_\mu(u) := \|Gu - \overline{u}\|_2^2 + \mu\|u\|_2^2$$

$$\Phi_\mu(u) := \|Gu - \overline{u}\|_1^2 + \mu\|u\|_1^2$$

Thank you very much !

Supplementary slides

- $\boldsymbol{u} \approx \boldsymbol{U} + u_r$
- $W_r \approx \overline{\boldsymbol{U}} + \boldsymbol{w}_r$
- $\boldsymbol{u}^{AD} = \boldsymbol{U} + \boldsymbol{w}_r^{AD}$
- $\left(\frac{\partial \overline{\boldsymbol{U}} + \boldsymbol{w}_r}{\partial t}, \varphi_i\right) + \frac{2}{Re}\left(\mathbb{D}(\overline{\boldsymbol{U}} + \boldsymbol{w}_r), \nabla \varphi_i\right) + \left(\overline{\boldsymbol{U} \cdot \nabla \boldsymbol{U}}, \varphi_i\right) + \left(\overline{\boldsymbol{w}_r^{AD} \cdot \nabla \boldsymbol{U}}, \varphi_i\right) + \left(\overline{\boldsymbol{U} \cdot \nabla \boldsymbol{w}_r^{AD}}, \varphi_i\right) + \left(\overline{\boldsymbol{w}_r^{AD} \cdot \nabla \boldsymbol{w}_r^{AD}}, \varphi_i\right) = 0$
- C-DF is self adjoint.
- $\left(\frac{\partial \boldsymbol{w}_r}{\partial t}, \varphi_i\right) + \frac{2}{Re}\left(\mathbb{D}(\boldsymbol{U}), \nabla \overline{\varphi_i}\right) + \frac{2}{Re}\left(\mathbb{D}(\boldsymbol{w}_r), \nabla \varphi_i\right) + \left(\boldsymbol{U} \cdot \nabla \boldsymbol{U}, \overline{\varphi_i}\right) + \left(\boldsymbol{w}_r^{AD} \cdot \nabla \boldsymbol{U}, \overline{\varphi_i}\right) + \left(\boldsymbol{U} \cdot \nabla \boldsymbol{w}_r^{AD}, \overline{\varphi_i}\right) + \left(\boldsymbol{w}_r^{AD} \cdot \nabla \boldsymbol{w}_r^{AD}, \overline{\varphi_i}\right) = 0$

3D flow past a cylinder $Re = 1000$

- Computation domain, $144 \times 192 \times 32$ grid, $CFL \approx 0.2$