Compatible Discretizations of Partial Differential Equations

Franco Brezzi

 IMATI-C.N.R., Pavia, Italy

IUSS-Istituto Universitario di Studi Superiori, Pavia, Italy

From joint works with:
A. Buffa, K. Lipnikov, M. Shashkov, and V. Simoncini

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Outline

How to identify a function

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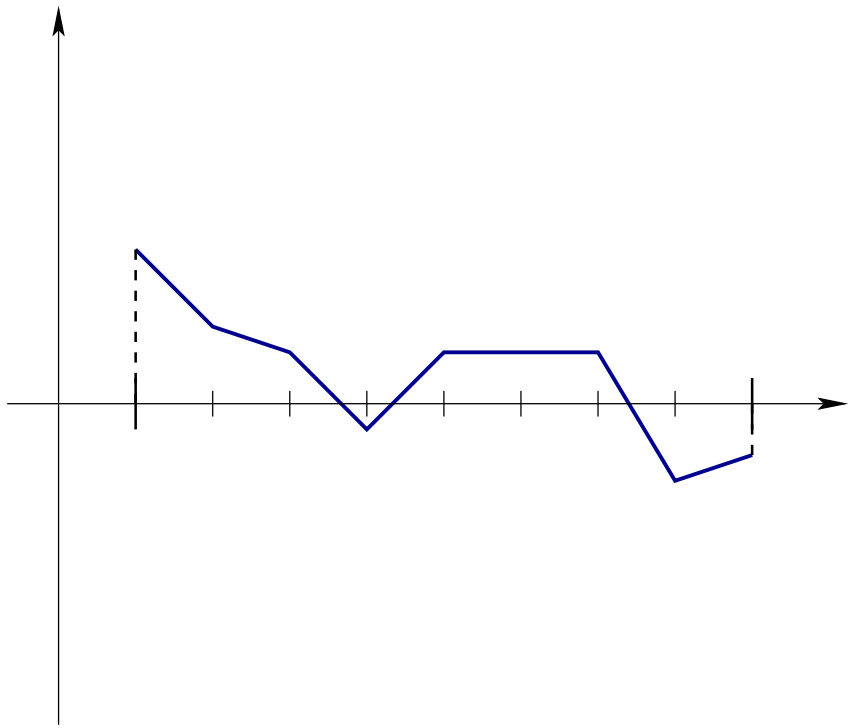
This corresponds to use either a basis or a dual basis

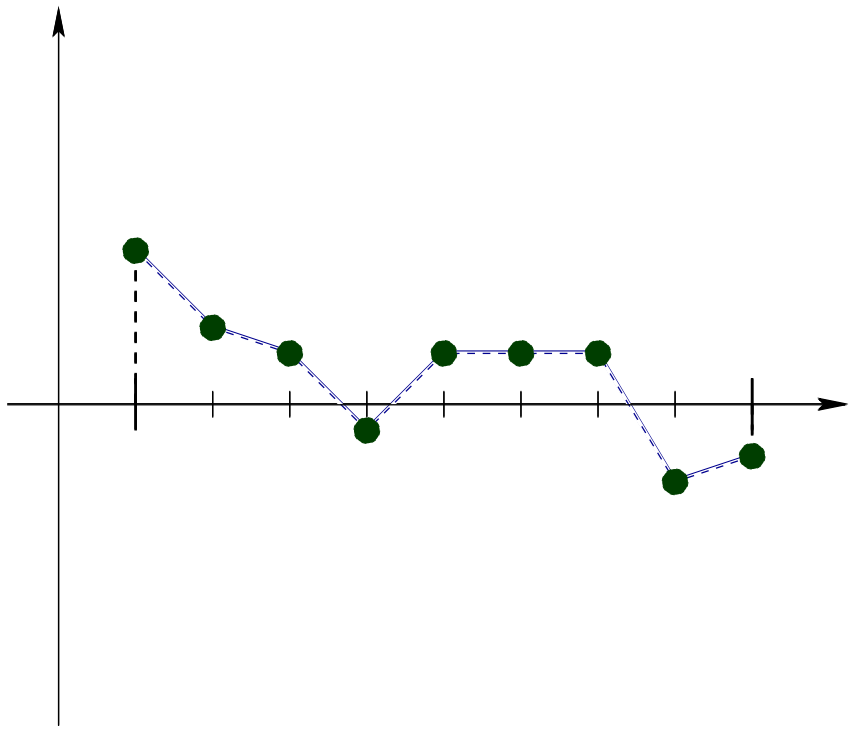
The Difference

Most of the times, the difference is just psychological.
You can think of a piecewise linear function



The Difference

Most of the times, the difference is just psychological. You can think of a piecewise linear function or of its nodal values



Fourier and Wavelets

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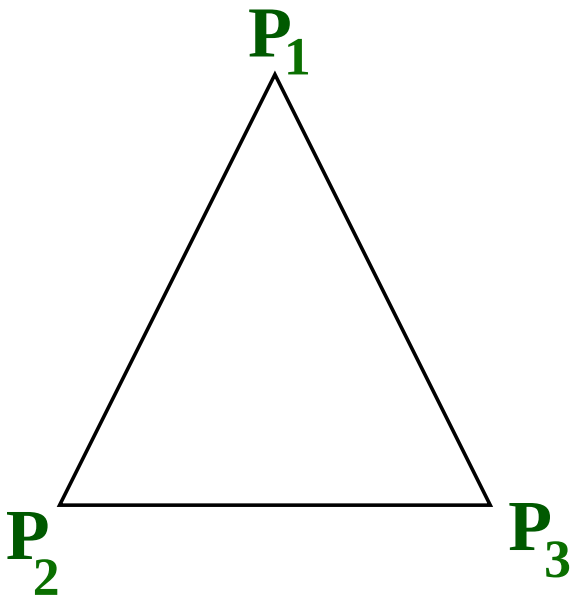
Valuable instruments are often available in order to pass from one way of identification (e.g. the point values) to another (e.g. the Fourier Coefficients). For instance the Fast Fourier Transform (FFT), the Fast Wavelet Transform (FWT), etc.

Dual bases

On a given triangle T with vertices P_1, P_2, P_3, in the space of linear functions, one could identify three **basis functions** λ^1, λ^2, λ^3 such that



$$\lambda^i(P_j) = \delta^i_j$$

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or by

- the values of the integral of its normal components on two given edges (non parallel)

$$\int_{e_1} \mathbf{v} \cdot \mathbf{n}^1 de \qquad \int_{e_2} \mathbf{v} \cdot \mathbf{n}^2 de$$

The RT Spaces

A vector $\mathbf{v}$ of the form

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- the values of the integral of its normal components on the edges of a given triangle.

$$\int_{e_1} \mathbf{v} \cdot \mathbf{n}^1 de, \quad \int_{e_2} \mathbf{v} \cdot \mathbf{n}^2 de, \quad \int_{e_3} \mathbf{v} \cdot \mathbf{n}^3 de$$

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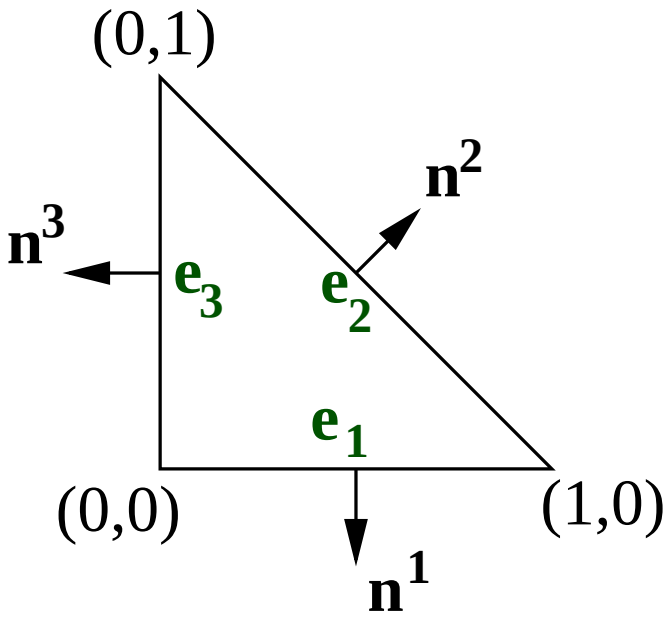
On a given triangle T with edges e_1, e_2, e_3, in the space of RT vectors, one could identify three **basis functions** $\boldsymbol{\tau}^1$, $\boldsymbol{\tau}^2$, $\boldsymbol{\tau}^3$such that



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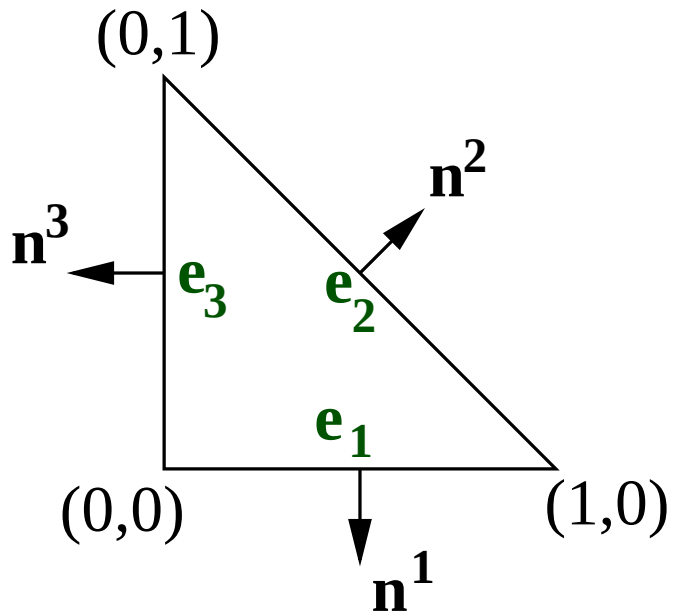

$$\int_{e_j} \boldsymbol{\tau}^i \cdot \mathbf{n}^j \, de = \delta^i_j$$

$$\boldsymbol{\tau}^1 = (x, y-1), \quad \boldsymbol{\tau}^2 = (x, y), \quad \boldsymbol{\tau}^3 = (x-1, y)$$

The Mixed Zoo

(Arnold-Falk-Winther, Acta Numerica 2006)
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- scalars given by **element integrals** $\int_{element_j} f$

Two Philosophies - Unknowns

We can identify two different philosophies:

The unknown is a function, to be chosen within a given finite dimensional space (Ex: Finite Elements, Spectral Methods, Wavelets,...)

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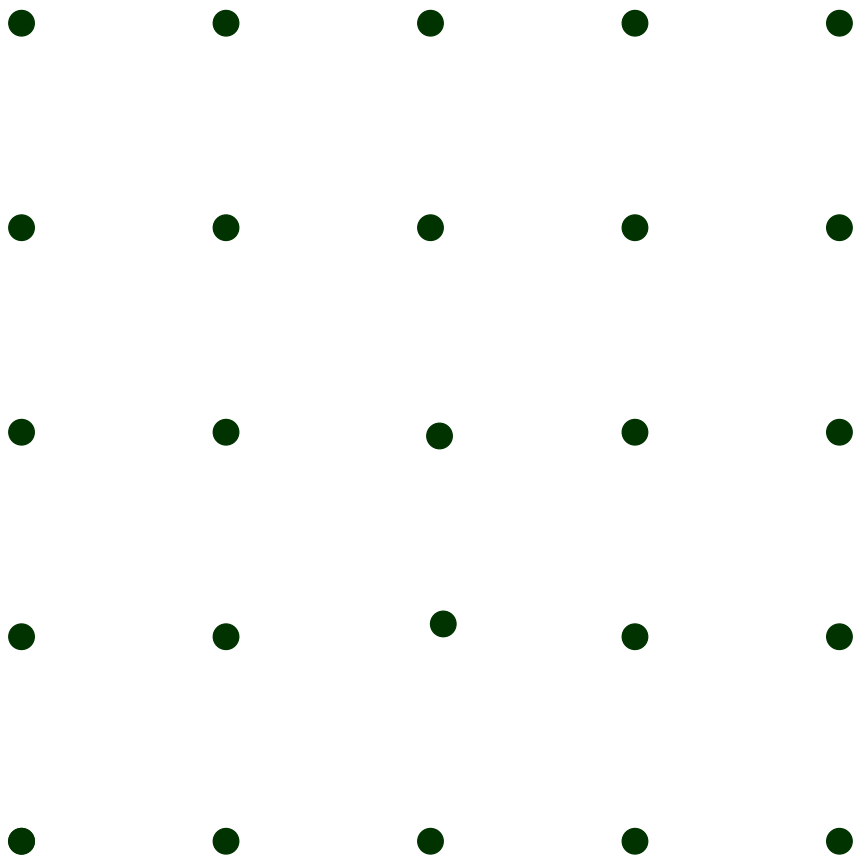
Once a finite dimensional space has been chosen, the difference in the two approaches is just *psychological*. But this is not always the case.

Nodal Functions

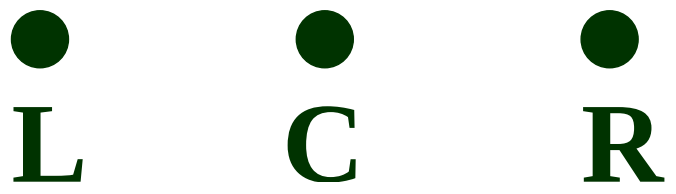
In finite difference methods, the *unknowns* are the *nodal values*. We do not assume to have an underlying finite dimensional space



Differential Operators on Nodal Functions

How can we write *differential operators* on *nodal functions*?

For instance, the second derivative at C can be approximated as

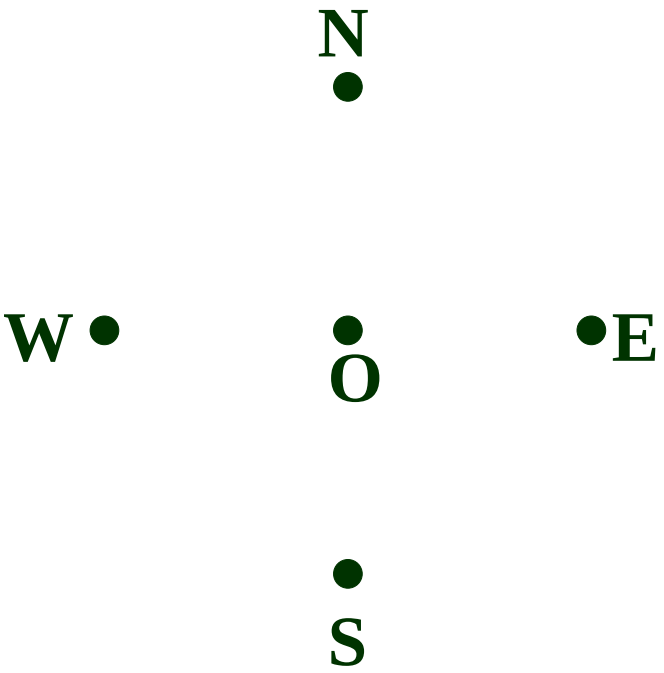


$$\frac{d^2 f}{dx^2}(C) \simeq \frac{f(R) - 2f(C) + f(L)}{|R - C|^2}$$

and the approximation is *exact* whenever f is a polynomial of degree ≤ 3.

The Five-point Scheme

Similarly, the famous *five point method* reads

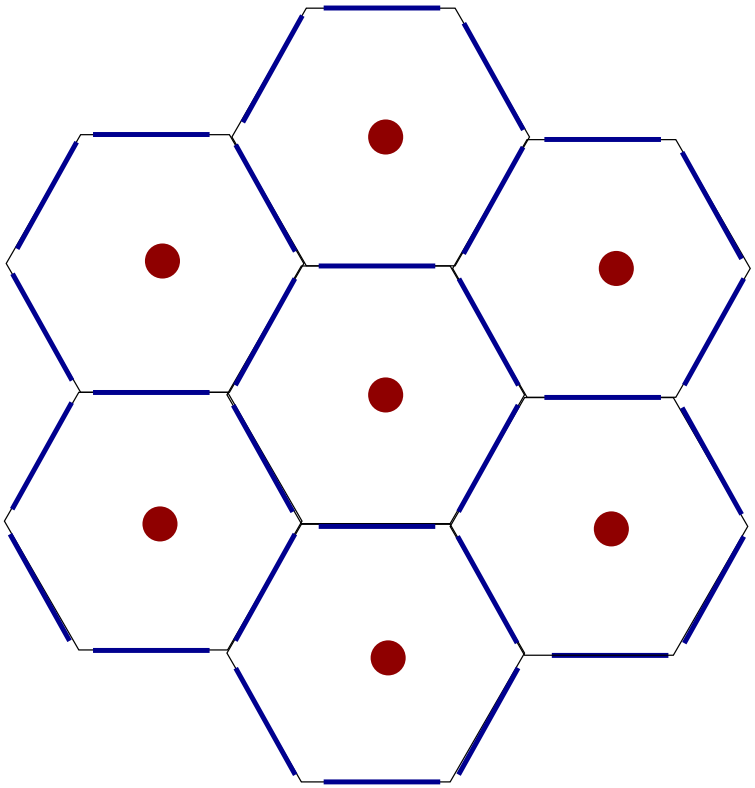


$$-h^2\,\Delta_h f(O) =$$

$$4f(O) - f(N) - f(S) - f(W) - f(E)$$

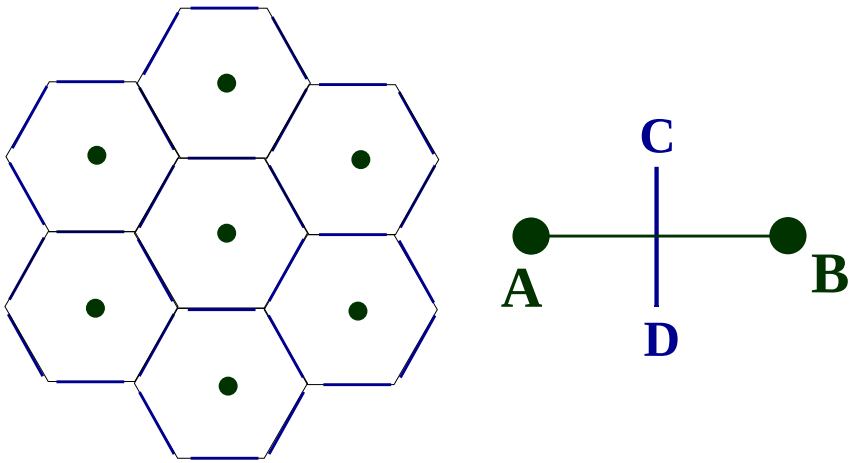
Finite Volumes-1

In other situations one can mix, e.g., fluxes and volume integrals. For instance, in Finite Volumes, one applies the divergence theorem to get (*exactly*) the integral of the divergence on each cell, given the fluxes on every edge.



Finite Volumes-2

However, the fluxes can be expressed as a function of the nodal values at the centers of the cells only in an approximate way



$$p(B) - p(A) \simeq \int_A^B \frac{\partial p}{\partial x} \simeq$$

$$\simeq \frac{|B - A|}{|C - D|} \int_{[C,D]} \frac{\partial p}{\partial \mathbf{n}}$$

Approximating the space

Revisiting the *Two Philosophies* we can say that typically function-based methods use integral (weak) formulations. For instance the problem $-\Delta p = f$ in Ω with $p = 0$ on $\partial\Omega$ becomes: *find* $p_h \in L_h$ *such that*

$$\int_\Omega \nabla p_h \cdot \nabla q_h = \int f q_h \qquad \forall q_h \in L_h$$

where L_h is the space of piecewise linear functions vanishing on $\partial\Omega$.

Approximating the operator

Revisiting the *Two Philosophies* we can say that, on the other hand, value-based methods use a discretization of the differential operator. For instance the problem $-\Delta p = f$ in Ω with $p = 0$ on $\partial\Omega$ will be approximated by: *Find a nodal function* p_h *such that*

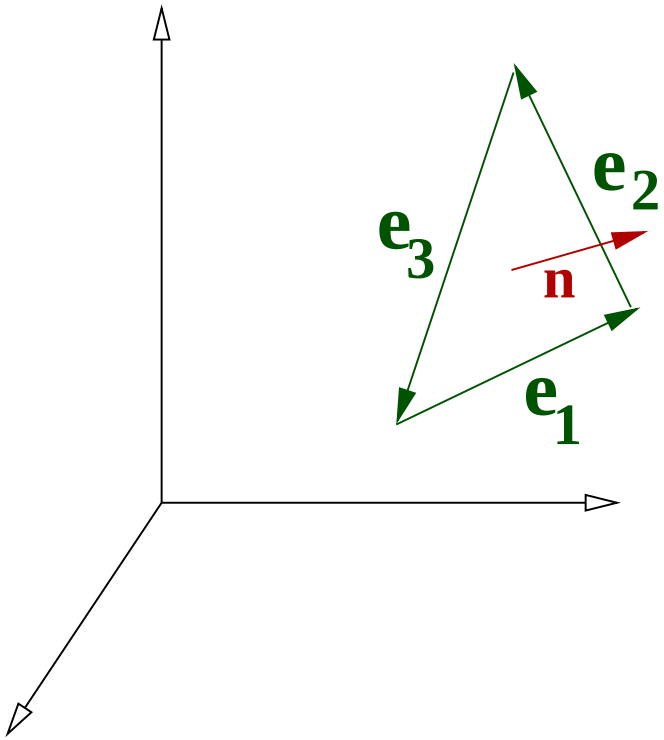
$$-\Delta_h p_h = f_h \qquad \text{and } p_h = 0 \text{ on } \partial\Omega$$

where f_h is made by the nodal values of f.

Fundamental Laws

$$\mathbf{curl}\ \mathbf{H}=\mathbf{j}$$

H=Magnetic field **j**=current density

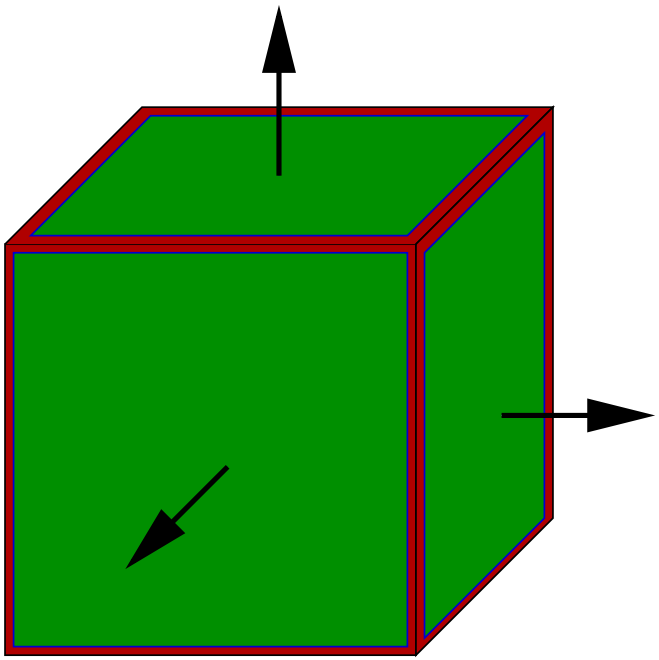


$$\sum_{k=1}^{3} \int_{e_k} \mathbf{H} \cdot \mathbf{t}_k = \int_{face} \mathbf{j} \cdot \mathbf{n}$$

Fundamental Laws

$$\text{div}\ \mathbf{D} = \rho$$

$\mathbf{D}$=Electric Displacement ρ=Charge density

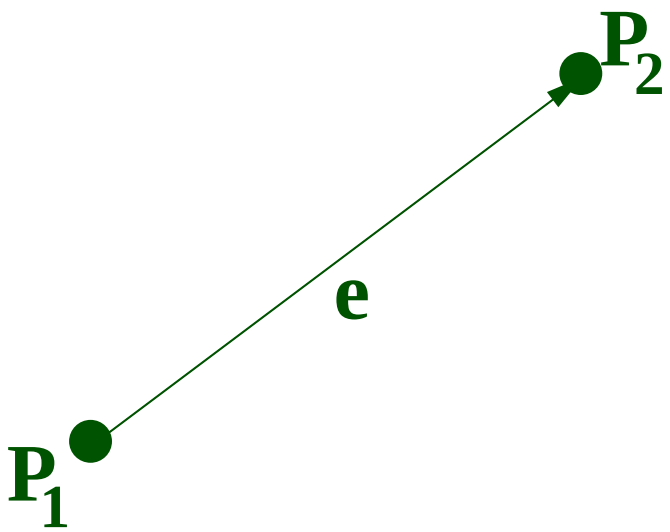


$$\sum_{k=1}^{6} \int_{face_k} \mathbf{D} \cdot \mathbf{n}_k = \int_{element} \rho$$

Fundamental Laws

$$\mathbf{curl}\,\mathbf{E} = \mathbf{0} \qquad \mathbf{E} = -\nabla V$$

$\mathbf{E}$=Electric Field V=Electric Potential



$$V(P_1) - V(P_2) = \int_{edge} \mathbf{E} \cdot \mathbf{t}$$

Less Fundamental Laws

$$\mathbf{D} = \varepsilon \mathbf{E}$$

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Compatible Methods (as Mixed FEM) try, as much as possible, to reproduce *exactly* the **Fundamental Laws**

The Mixed Zoo - Compatible Animals

In the literature on Mixed FEM, the finite dimensional spaces are made (in 3 dimensions) by **four** types of **basic ingredients**:

- scalars given by **nodal values** $f(point_j)$
- vectors given by **face fluxes** $\int_{face_j} \boldsymbol{\tau} \cdot \mathbf{n}^j$
- vectors given by **edge works** $\int_{edge_j} \boldsymbol{\tau} \cdot \mathbf{t}^j$
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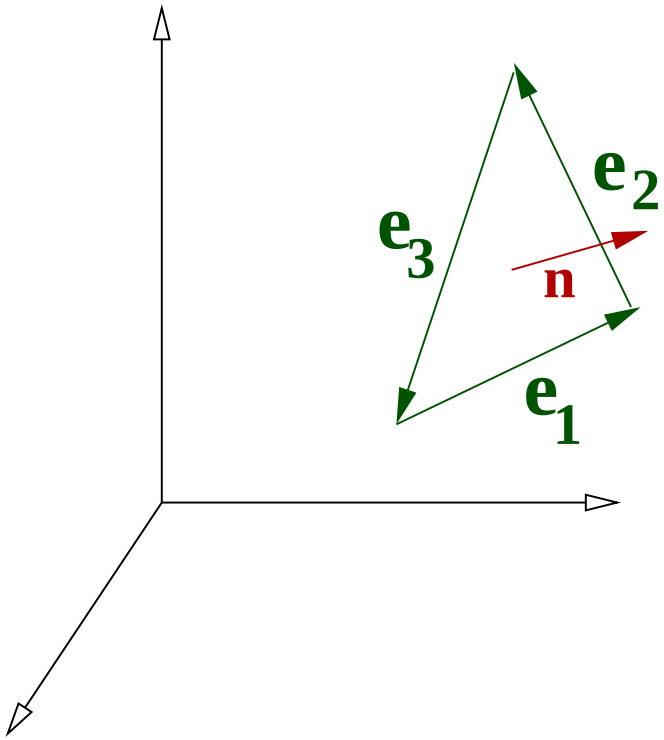
From edges to faces

$$\mathbf{curl}\ \mathbf{H}=\mathbf{j}$$

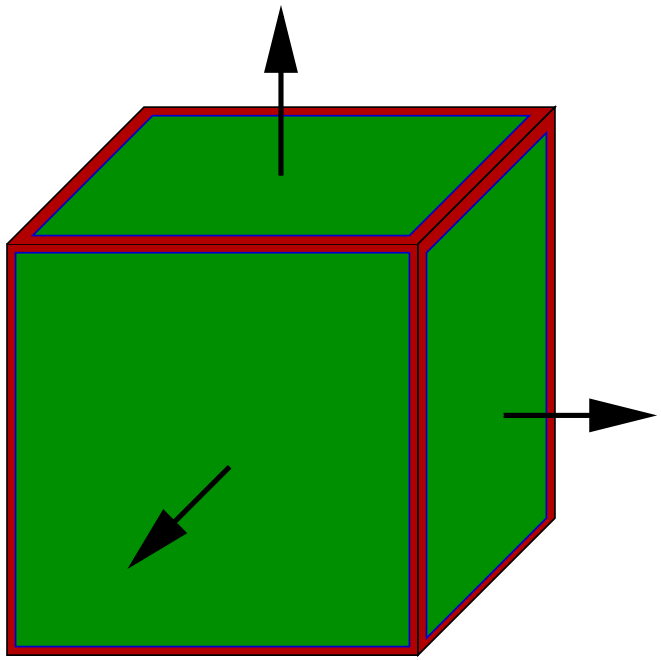


$$\sum_{k=1}^{3} \int_{e_k} \mathbf{H} \cdot \mathbf{t}_k = \int_{face} \mathbf{j} \cdot \mathbf{n}$$

$\int_{edge} \mathbf{H} \cdot \mathbf{t}_k$ on each edge $\Rightarrow \int_{face} \mathbf{j}$ on each face

From faces to elements

$$\text{div}\ \mathbf{D} = \rho$$

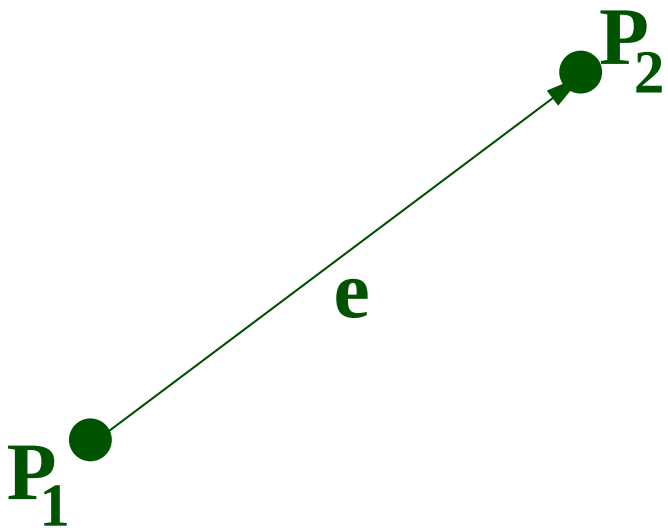


$$\sum_{k=1}^{6} \int_{face_k} \mathbf{D} \cdot \mathbf{n}_k = \int_{element} \rho$$

$\int_{face} \mathbf{D} \cdot \mathbf{n}$ on each face $\Rightarrow \int_{ele} \rho$ on each element

From vertices to edges

$$\mathbf{curl}\,\mathbf{E} = \mathbf{0} \qquad \mathbf{E} = -\nabla V$$



$$V(P_1) - V(P_2) = \int_{edge} \mathbf{E} \cdot \mathbf{t}$$

$$V \text{ on each node } \Rightarrow \int_e \mathbf{E} \cdot \mathbf{t} \text{ on each edge}$$

An *Ideally Compatible* Scheme

Ideally, the *most compatible* scheme should be:

$$\text{Minimize } \|\mathbf{B} - \mathbf{H}\|$$

under the constraints

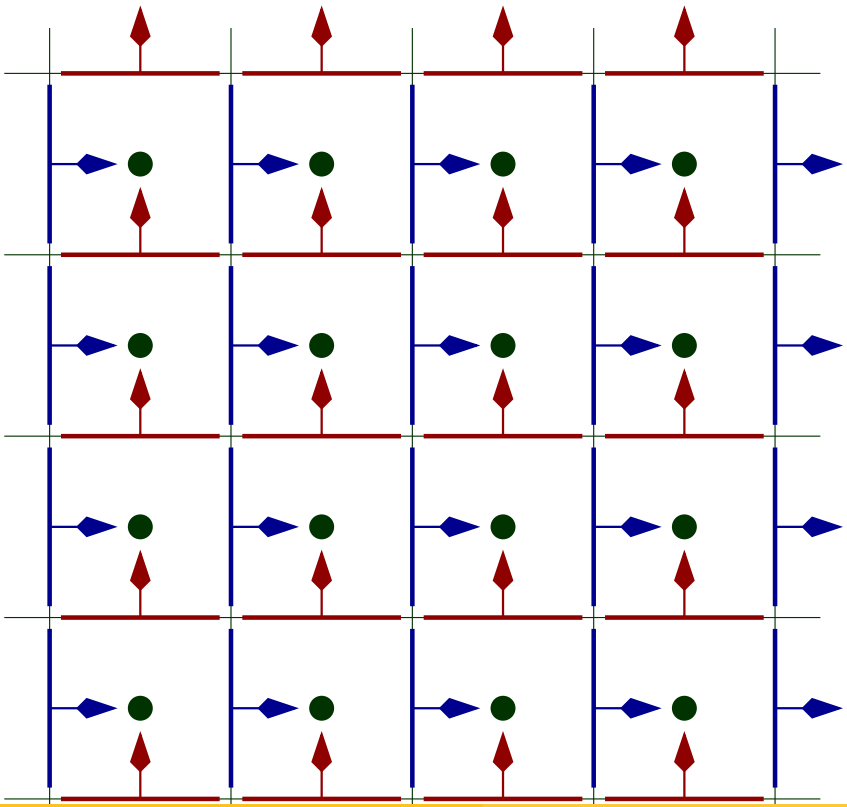
$$\mathrm{curl}\mathbf{H} = \mathbf{j} \quad \text{and } \mathrm{div}\mathbf{B} = 0$$

(J.Rikabi, A. Bossavit, I. Perugia)

But it is very expensive...

Other Compatible Schemes

The **Marker and Cell** (MaC) scheme (Harlow & Welch, 1965) for Darcy flows uses *horizontal velocities* on the vertical edges, *vertical velocities* on horizontal edges, and *pressure* at the center of each cell.

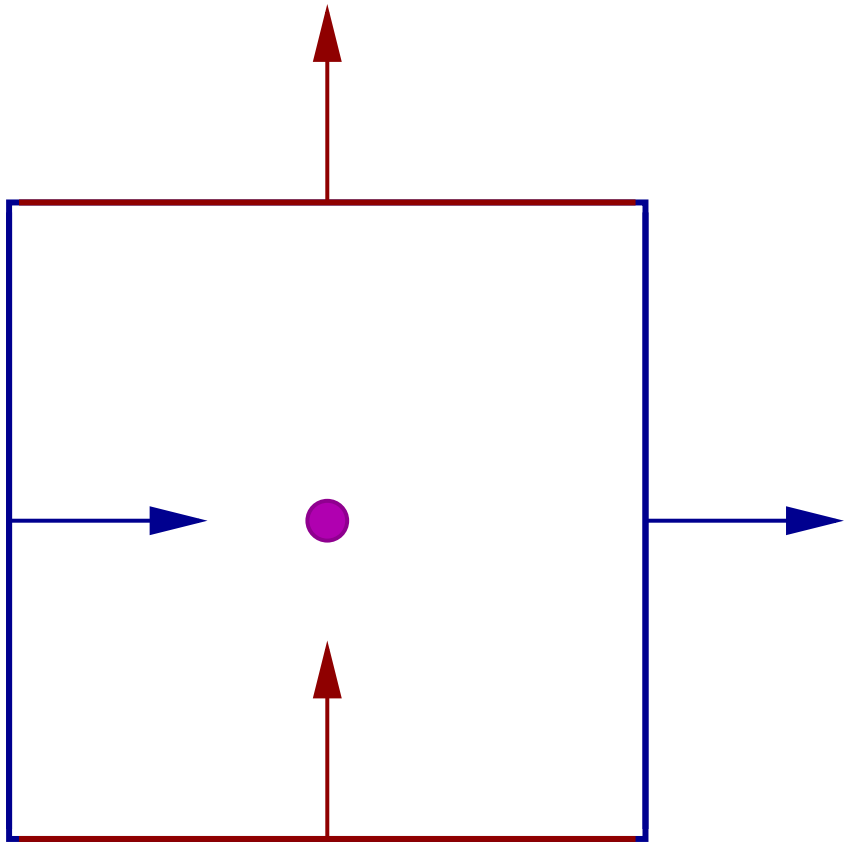


$$-\nabla p = \mathbf{u} \equiv (u, v)$$

$$\operatorname{div} \mathbf{u} = \textit{source}$$

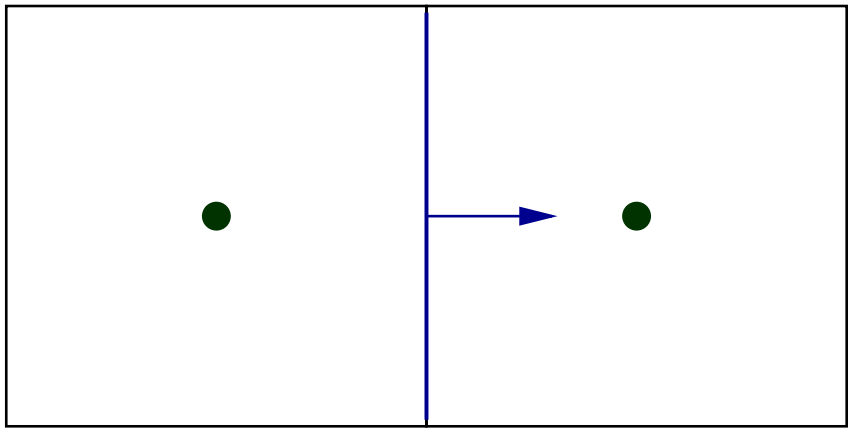
The Conservation equation in MaC

The *conservation equation*, e.g., $\mathrm{div}\,\mathbf{u} = 0$, is satisfied in a strong sense. This is *the compatible part* of the scheme.



The constitutive equation in MaC

On a regular grid (like ours) the fluxes are easily computed as functions of the pressure



The constitutive equation in MaC

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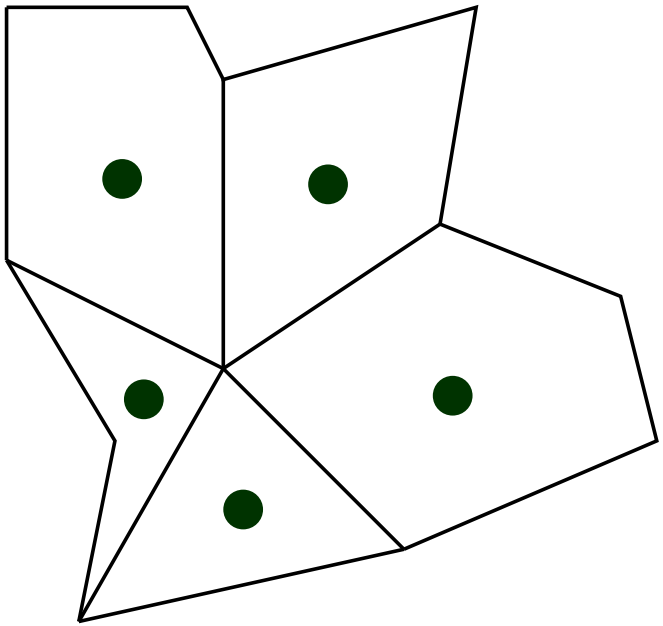

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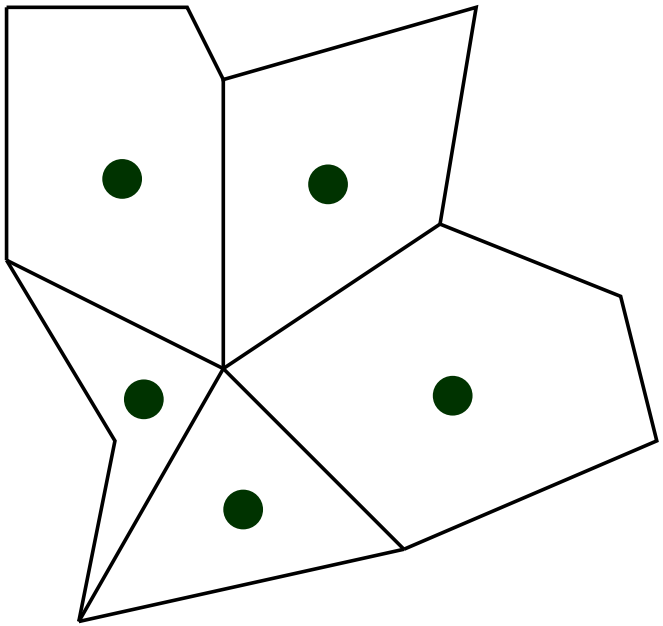
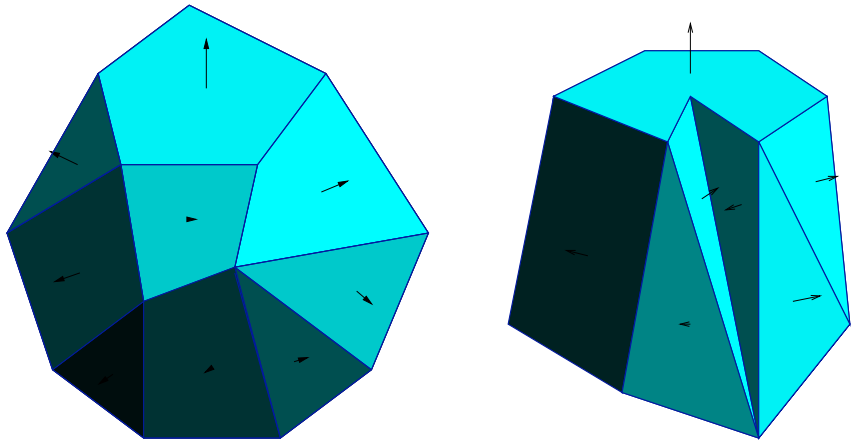


On a general geometry one needs all sorts of tricks in order to do it. All in a *quite incompatible* way!

General geometries

On the other hand also for Mixed Finite Element Methods, it is **very hard** to find the proper **finite dimensional spaces** in General geometries

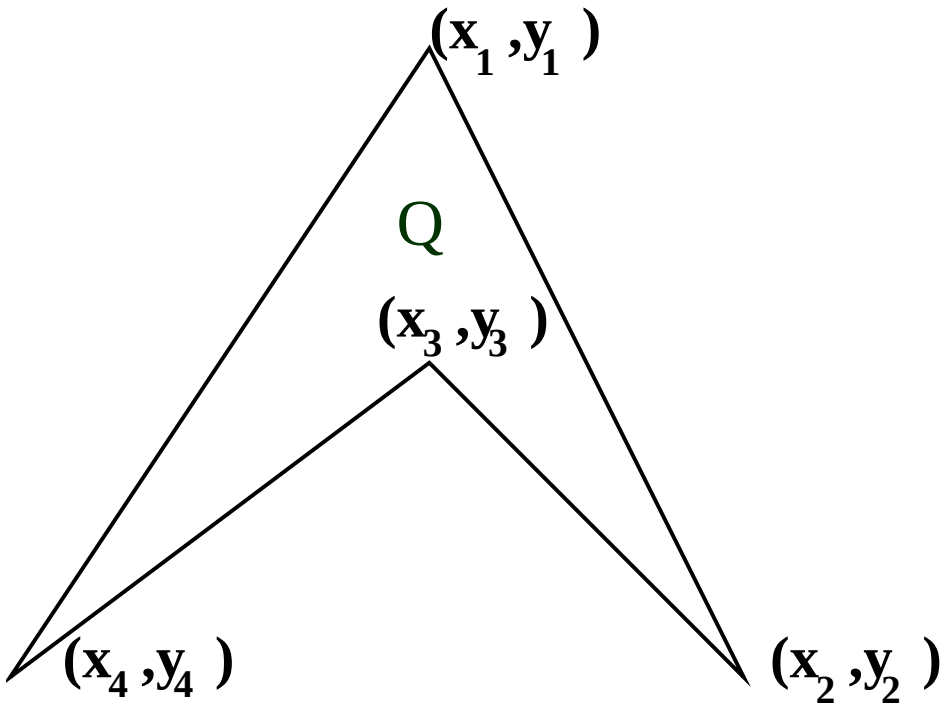


Two examples of pata-hedral elements

Mimetic Finite Differences

(Hyman-Scovel 1988)

Let us consider the **most simple** example of a nodal function on a quadrilateral.



Towards a local Scalar Product

We want to mimic the *scalar product*

$$\int_Q \nabla u \cdot \nabla v \, \mathrm{d}Q$$

but we lack the proper finite dimensional space (of dimension four, in a one-to-one correspondence with the nodal values at the vertices).

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Hence besides the functions "1", "x", "y" we would need a *fourth function* Ψ.

The nodal values of linear functions

We work on nodal functions. Hence, in our world (a copy of $\mathbb{R}^4$), the linear functions are represented by *the three-dimensional subspace generated by*:

- the nodal values of the function "1": $(1, 1, 1, 1)$
- the nodal values of the function "x": (x_1, x_2, x_3, x_4)
- the nodal values of the function "y": (y_1, y_2, y_3, y_4)

The local matrix-1

	"1"	"X"	"Y"	
"1"	0	0	0	0
"X"	0	|Q|	0	
"Y"	0	0	|Q|	
	0			

If both u **and** v are one of the functions 1, x, or y the scalar product $\int_Q \nabla u \cdot \nabla v \, \mathrm{d}Q$ is easy to compute.

The local matrix-2

We make the **further assumption that**: On each edge of ∂Q all the functions of our four-dimensional local space are integrated exactly by the *trapezoidal rule*

$$\int_a^b f \simeq \frac{b-a}{2}[f(a)+f(b)]$$

It **follows** that, for p_1 **linear**, the integral

$$\int_Q \nabla p_1 \cdot \nabla v \, \mathrm{d}Q = \int_{\partial Q} \frac{\partial p_1}{\partial \mathbf{n}} v$$

depends **only** on the nodal values of v.

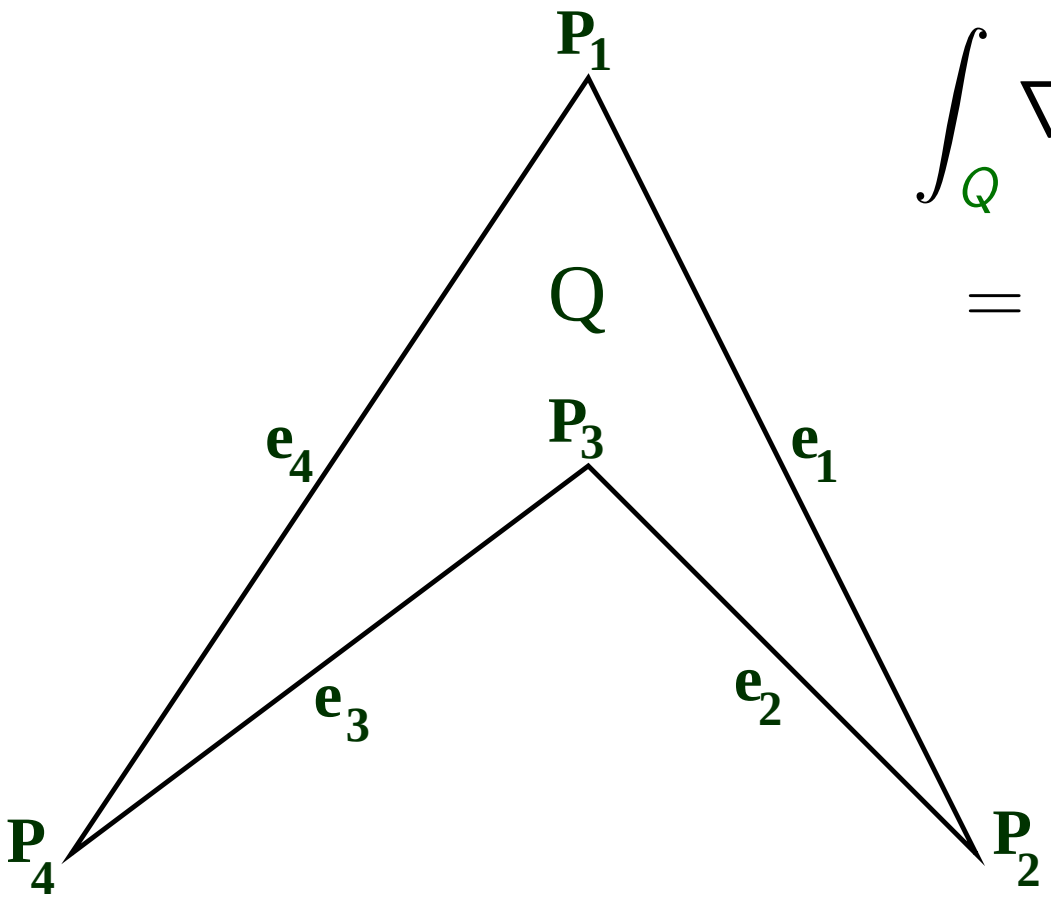
The Local Matrix-3

At this point, for every vector $(\Psi_1, \Psi_2, \Psi_3, \Psi_4)$ of nodal values (of an ideal function Ψ that we don't have), we can compute

$$\int_Q \nabla x \cdot \nabla \Psi \, \mathrm{d}Q \quad \text{and} \quad \int_Q \nabla y \cdot \nabla \Psi \, \mathrm{d}Q$$

using only the nodal values $(\Psi_1, \Psi_2, \Psi_3, \Psi_4)$.

The Local Matrix-4

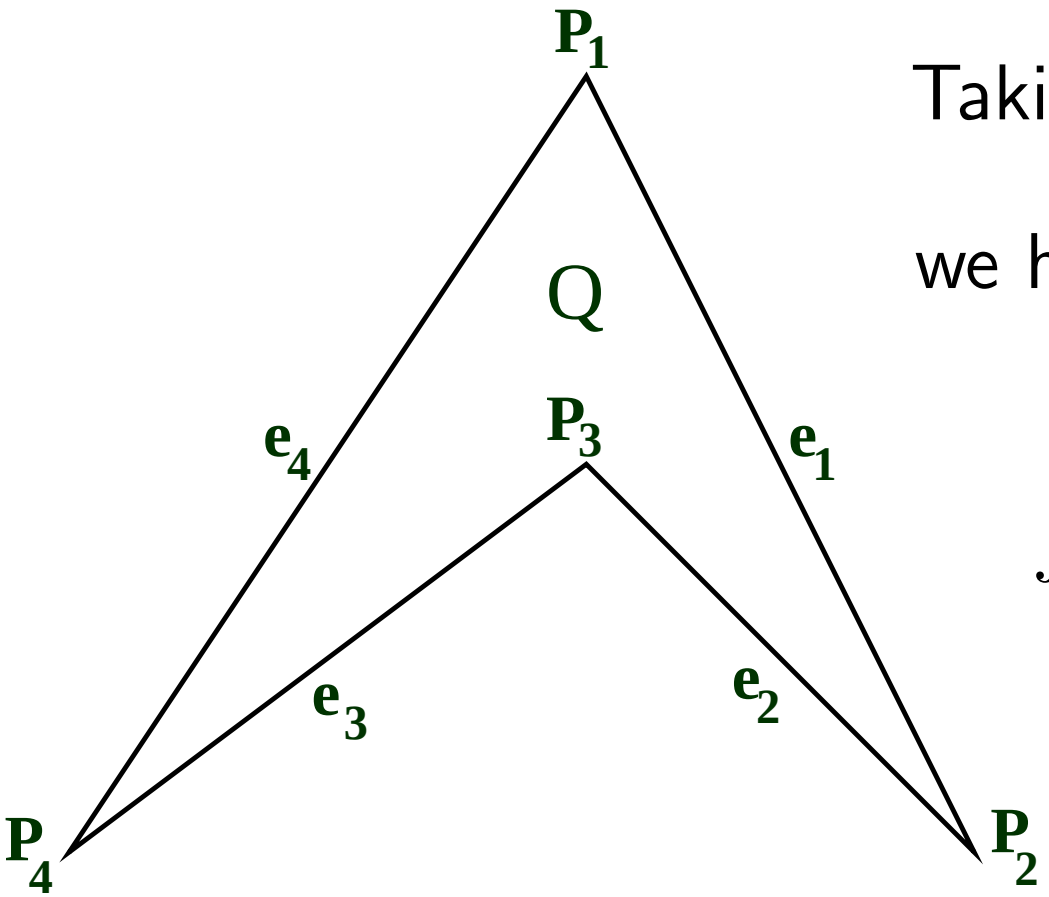


$$\int_Q \nabla x \cdot \nabla \psi \, \mathrm{d}Q = \int_{\partial Q} n_x \psi =$$

$$= |e_1| n_x^1 (\psi_1 + \psi_2)/2 + \ldots$$

$$\ldots + |e_4| n_x^4 (\psi_4 + \psi_1)/2$$

The Local Matrix-5



Taking $\Psi_A := (1, -1, 1, -1)$

we have $\int_Q \nabla x \cdot \nabla \Psi_A \, \mathrm{d}Q =$

$$\int_Q \nabla y \cdot \nabla \Psi_A \, \mathrm{d}Q = 0$$

The Local Matrix-6

	"1"	"X"	"Y"	"Ψ_A"
"1"	0	0	0	0
"X"	0	|Q|	0	0
"Y"	0	0	|Q|	0
"Ψ_A"	0	0	0	

We are left with the evaluation

of $\int_Q |\nabla \Psi_A|^2 \, \mathrm{d}Q =: \alpha$

which depends on the values

of Ψ_A inside

The Miracle

	"1"	"X"	"Y"	"Ψ_A"
"1"	0	0	0	0
"X"	0	|Q|	0	0
"Y"	0	0	|Q|	0
"Ψ_A"	0	0	0	α

Believe it or not, you can take **any positive value** for α, and there will be a function Ψ_A such that your scheme actually comes from a function-based approach (using linear functions plus Ψ_A)

In a more general geometry, the role of α is played by a symmetric and positive definite matrix.

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- You try to define, on every element, a scalar product among nodal functions.
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- You shamelessly cook up the remaining part of the local matrix.
- You find that you are using a function based approach.