Feedback stabilization of control systems

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W.T. and Idalia Reid Prize in Mathematics Lecture
2017 SIAM Annual Meeting, Pittsburgh, PA, USA, July 10-14 2017

siam Society for Industrial and Applied Mathematics

Outline

1. Examples of feedback laws
   - Inverted pendulum
   - Rivers
2. Some general results on stabilization in finite dimension
3. Stabilization of 1-D quasilinear hyperbolic systems

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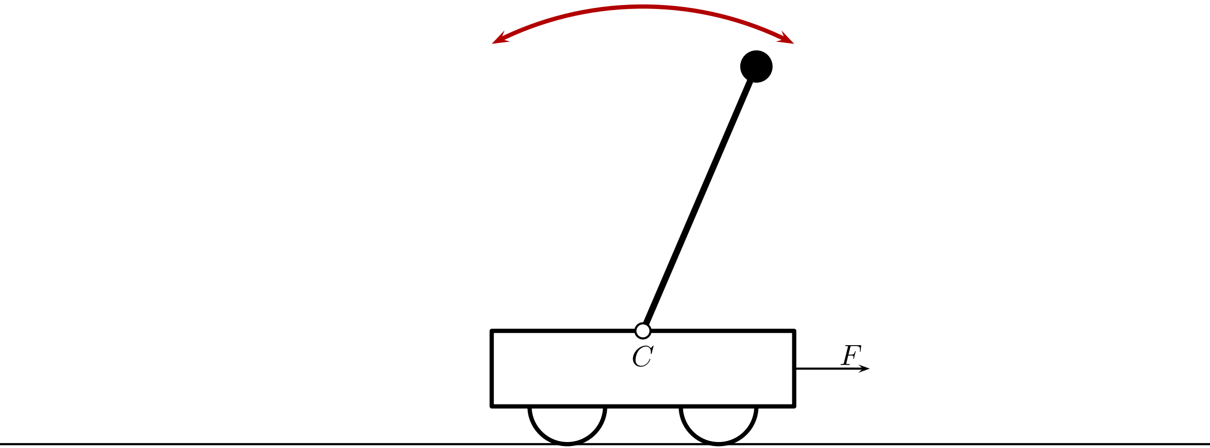
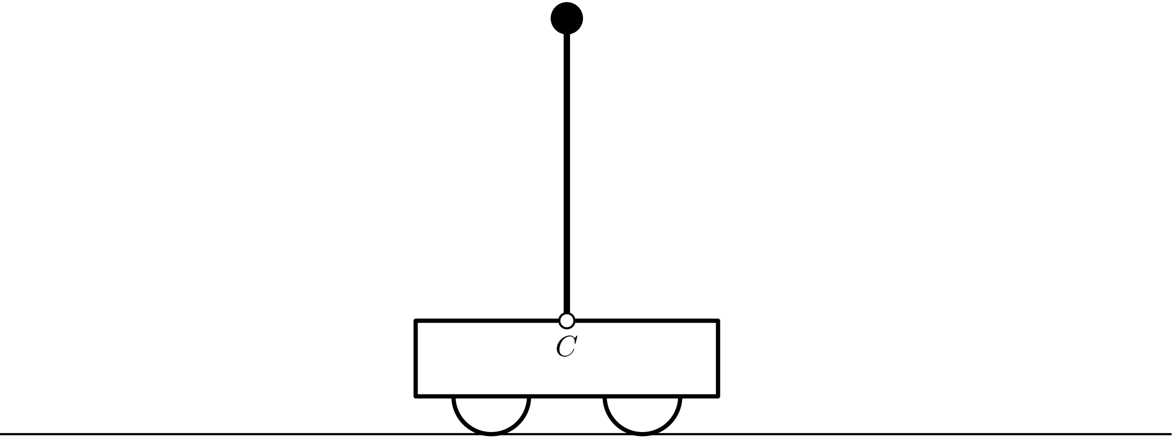
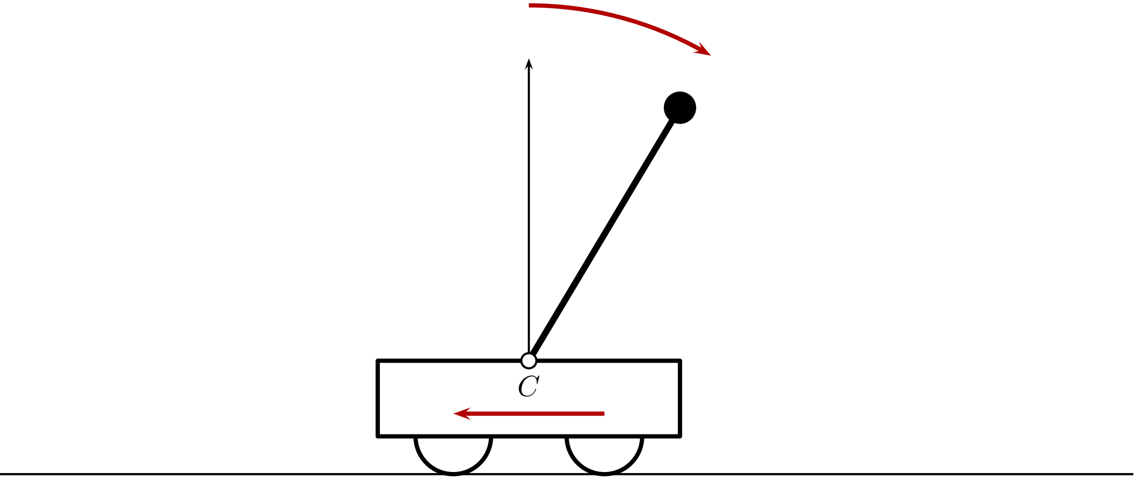
Cart inverted pendulum control system



Cart inverted pendulum: the equilibrium



Instability of the equilibrium



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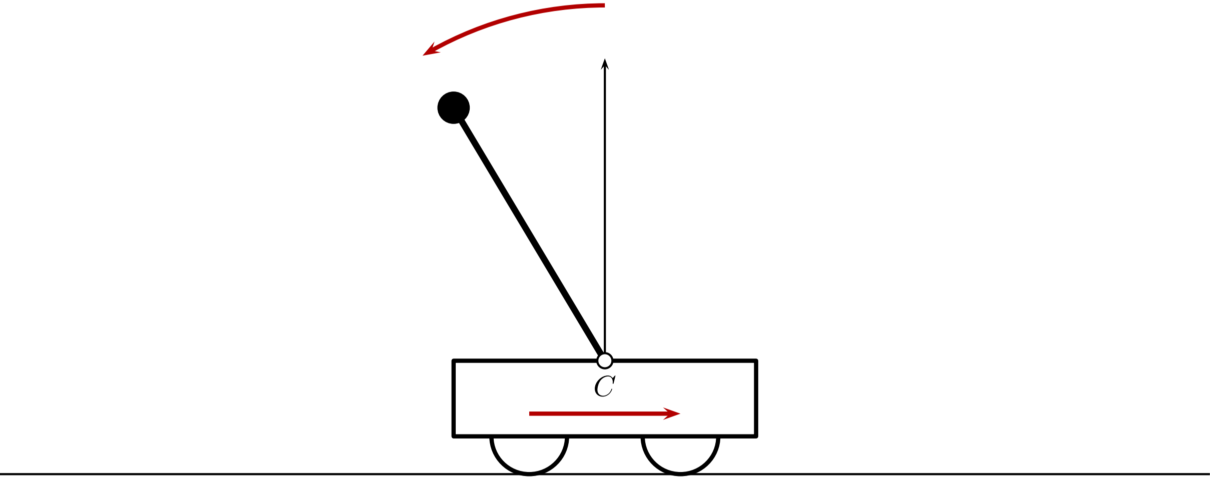
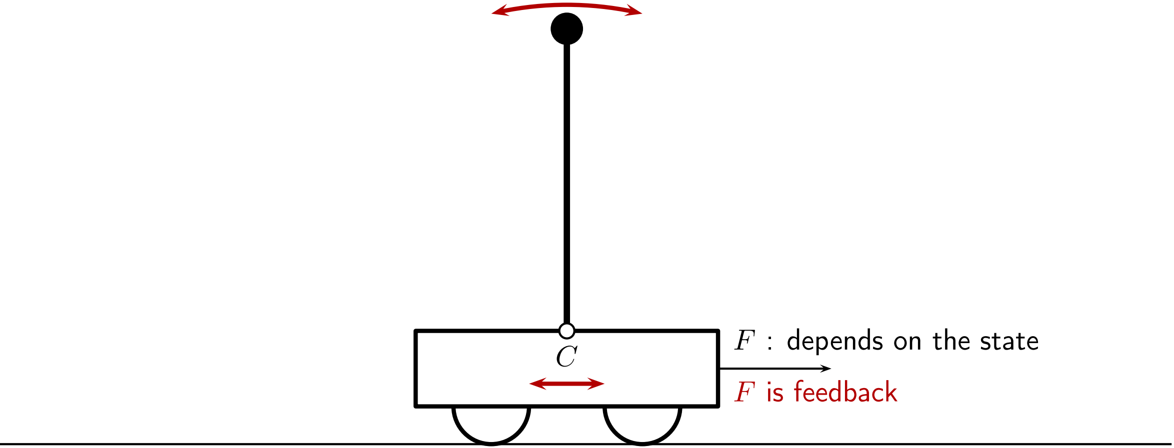
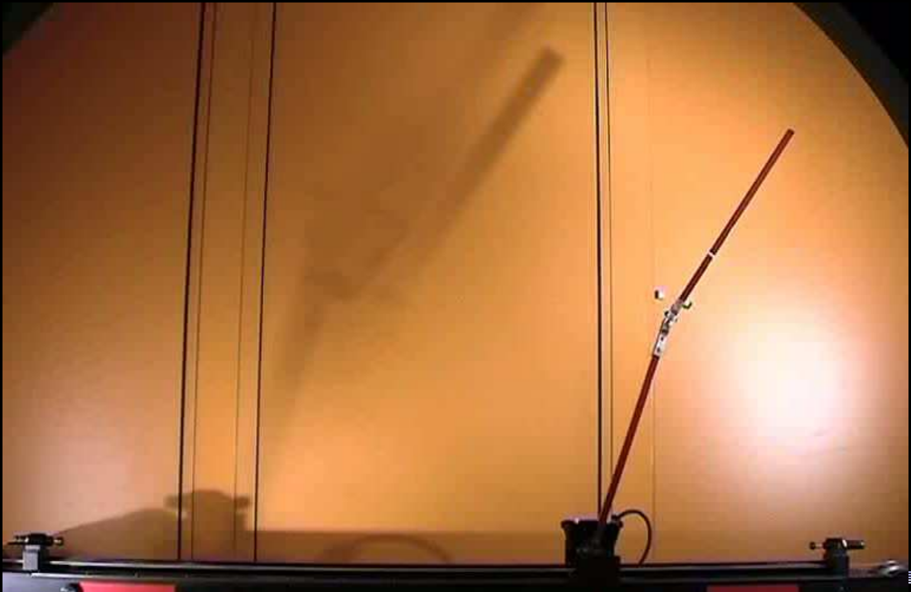


Stabilization of the equilibrium



Doubleinverted pendulum (CAS, ENSMP/La Villette)



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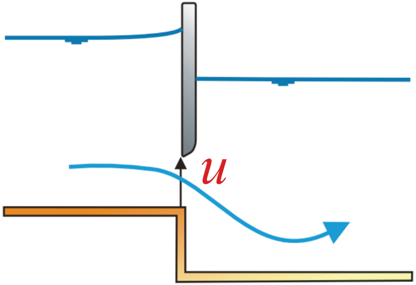
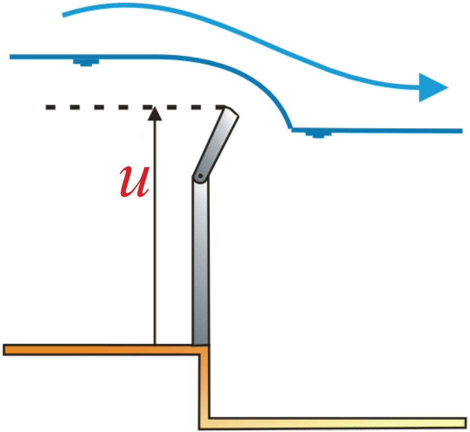
Two types of gates

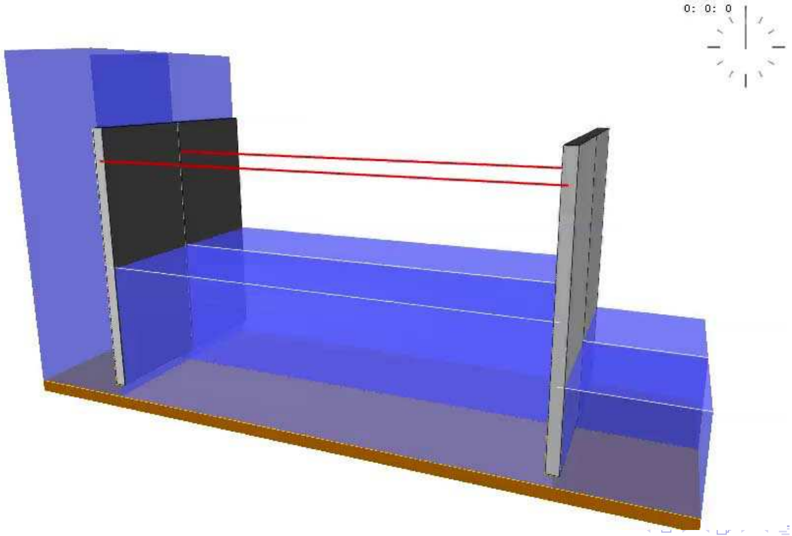
Underflow (sluice)



Overflow (spillway)



La Sambre (B. d'Andréa-Novel, G. Bastin, JMC, V. Dos Santos, J. de Halleux, L. Moens and C. Prieur (2003-...))



La Sambre (B. d'Andréa-Novel, G. Bastin, JMC, V. Dos Santos, J. de Halleux, L. Moens and C. Prieur (2003-...))

1. Examples of feedback laws

2. Some general results on stabilization in finite dimension
   - Stabilizability problem
   - Linear systems and applications to nonlinear systems
   - Obstruction to the stabilizability
   - Time-varying feedback laws

3. Stabilization of 1-D quasilinear hyperbolic systems

The stabilizability problem

We consider the control system $\dot{y} = f(y, u)$ where y in $\mathbb{R}^n$ is the state and u in $\mathbb{R}^m$ is the control. We assume that $f(0, 0) = 0$.

Problem

Does there exists $u : \mathbb{R}^n \to \mathbb{R}^m$ vanishing at 0 such that $0 \in \mathbb{R}^n$ is (locally) asymptotically stable for $\dot{y} = f(y, u(y))$? (If the answer is yes, one says that the control system is locally asymptotically stabilizable.)

Remark

The map $u : y \in \mathbb{R}^n \mapsto \mathbb{R}^m$ is called a feedback (or feedback law). The dynamical system $\dot{y} = f(y, u(y))$ is called the closed loop system.

Regularity of the feedback laws

The regularity of $y \mapsto u(y)$ is an important point. With u continuous, asymptotic stability implies the existence of a smooth strict Lyapunov function and, therefore, one has robustness with respect to small actuator errors as well as small measurement errors.

If u is discontinuous, one needs to define the notion of solution of the closed loop system $\dot{y} = f(y, u(y))$ and study carefully the robustness of the closed loop system. In this talk we assume that the feedback laws are continuous.

Controllability

Let $T > 0$. Given two states y^0 and y^1, does there exist a control $t \in [0, T] \mapsto u(t)$ which steers the control system from y^0 to y^1, i.e. such that

(1) $$\left(\dot{y} = f(y, u(t)),\ y(0) = y^0\right) \Rightarrow \left(y(T) = y^1\right)?$$

If the answer is yes, the control system is said to be controllable on $[0, T]$.

Controllability of linear control systems

The control system is

(1) $$\dot{y} = Ay + Bu,\ y \in \mathbb{R}^n,\ u \in \mathbb{R}^m,$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$.

Theorem (Kalman's rank condition (1960))

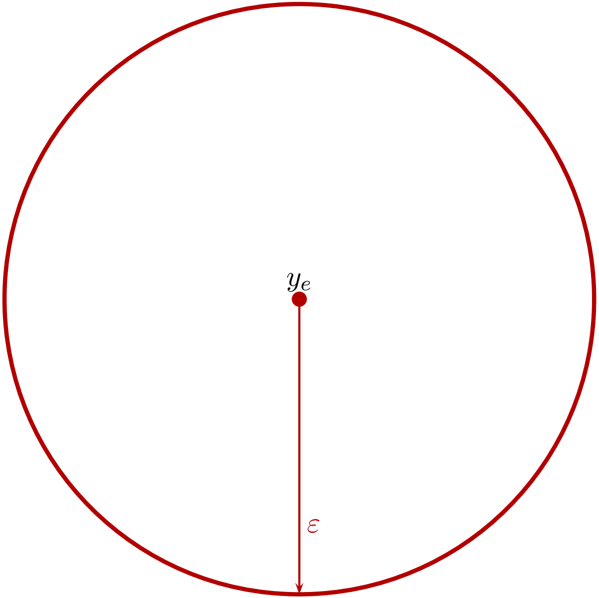
The linear control system $\dot{y} = Ay + Bu$ is controllable on $[0, T]$ if and only if

(2) $$\text{Span}\ \{A^i Bu; u \in \mathbb{R}^m,\ i \in \{0, 1, \dots, n-1\}\} = \mathbb{R}^n.$$

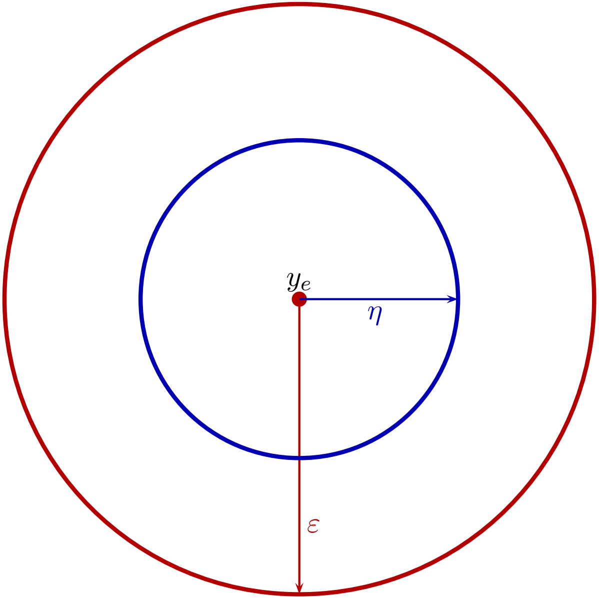
Small-time local controllability (STLC)

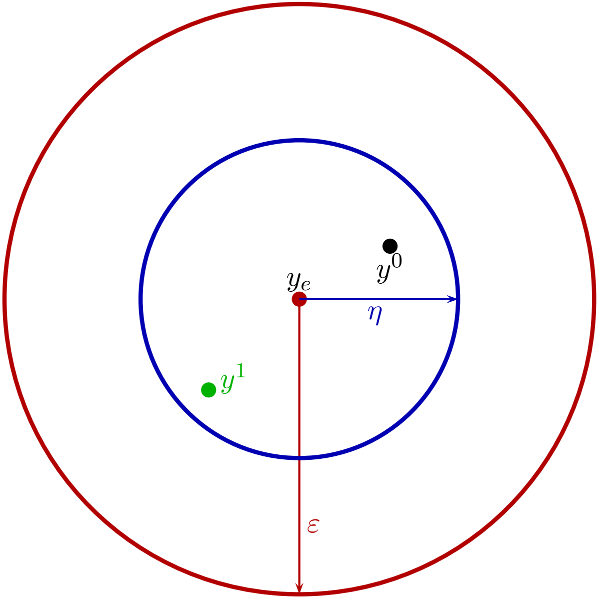
We assume that (y_e, u_e) is an equilibrium, i.e., $f(y_e, u_e) = 0$. Many possible choices for natural definitions of local controllability. The most popular one is **Small-Time Local Controllability (STLC)**: the state remains close to y_e, the control remains close to u_e and the time is small.

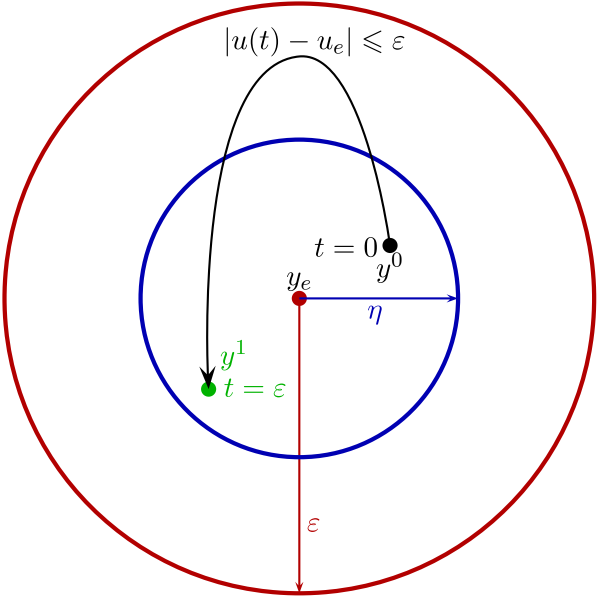

y_e



y_e
ε


y_e
η
ε


y_e
y^0
η
y^1
ε



$|u(t) - u_e| \leqslant \varepsilon$
$t = 0$
y^0
y_e
η
y^1
$t = \varepsilon$
ε

The linear test

We consider the control system $\dot{y} = f(y, u)$ and assume that $f(y_e, u_e) = 0$. How to check the small-time local controllability at (y_e, u_e)? We use Nirenberg's advice to depressed researchers:

The linear test

We consider the control system $\dot{y} = f(y, u)$ and assume that $f(y_e, u_e) = 0$. How to check the small-time local controllability at (y_e, u_e)? We use Nirenberg's advice to depressed researchers:

Have you tried to linearize?

The linearized control system at (y_e, u_e) is the linear control system $\dot{y} = Ay + Bu$ with

(1)
$$A := \frac{\partial f}{\partial y}(y_e, u_e), \; B := \frac{\partial f}{\partial u}(y_e, u_e).$$

If the linearized control system $\dot{y} = Ay + Bu$ is controllable, then $\dot{y} = f(y, u)$ is small-time locally controllable at (y_e, u_e).
What to do if the linearized control system at (y_e, u_e) is not controllable? Answer: use iterated Lie brackets.

Lie bracket and iterated Lie brackets

Definition (Lie bracket)

(1) $$[X,Y](y) := Y'(y)X(y) - X'(y)Y(y).$$

Iterated Lie brackets: $[X,[X,Y]]$, $[[Y,X],[X,[X,Y]]]$ etc. For simplicity, from now on we assume that

(2) $$f(y,u) = f_0(y) + \sum_{i=1}^{m} u_i f_i(y) \text{ with } f_0(0) = 0.$$

Drift: f_0. **Driftless control systems:** $f_0 = 0$. We denote by Lie $\{f_0, f_1, \ldots, f_m\}$ the smallest vector subspace $\mathcal{E}$ of $C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ containing f_0, f_1,..., f_m which is stable for the Lie bracket: if $X \in \mathcal{E}$ and $Y \in \mathcal{E}$, then $[X,Y] \in \mathcal{E}$.

The Lie algebra rank condition and STLC

One says that the control system $\dot{y} = f_0(y) + \sum_{i=1}^{m} u_i f_i(y)$ satisfies the **Lie algebra rank condition** at $0 \in \mathbb{R}^n$ if

$$\{h(0);\ h \in \text{Lie}\,\{f_0, f_1, \ldots, f_m\}\} = \mathbb{R}^n. \tag{1}$$

One has the following theorems.

Theorem (R. Hermann (1963) and T. Nagano (1966))

If the f_i's are analytic in a neighborhood of $0 \in \mathbb{R}^n$ and if the control system $\dot{y} = f_0(y) + \sum_{i=1}^{m} u_i f_i(y)$ is small-time locally controllable at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$, then this control system satisfies the Lie algebra rank condition at $0 \in \mathbb{R}^n$.

Theorem (P. Rashevski (1938), W.-L. Chow (1939))

If $\dot{y} = \sum_{i=1}^{m} u_i f_i(y)$ satisfies the Lie algebra rank condition at $0 \in \mathbb{R}^n$, then it is small-time locally controllable at $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$.

Stabilizability of linear controllable systems

Notations. For a matrix $M \in \mathbb{R}^{n\times n}$, P_M denotes the characteristic polynomial of M: $P_M(z) := \det\,(zI - M)$.
Let us denote by $\mathcal{P}_n$ the set of polynomials of degree n in z such that the coefficients are all real numbers and such that the coefficient of z^n is 1. One has the following theorem.

Theorem (Pole shifting theorem, M. Wonham (1967))

Let us assume that the linear control system $\dot{y} = Ay + Bu$ is controllable. Then

$$\left\{P_{A+BK};\, K \in \mathbb{R}^{m\times n}\right\} = \mathcal{P}_n. \tag{1}$$

Corollary

If the linear control system $\dot{y} = Ay + Bu$ is controllable, there exists a linear feedback $y \mapsto u(y) = Ky$ such that $0 \in \mathbb{R}^n$ is (globally) asymptotically stable for the closed loop system $\dot{y} = Ay + Bu(y)$.

Application to nonlinear controllable systems

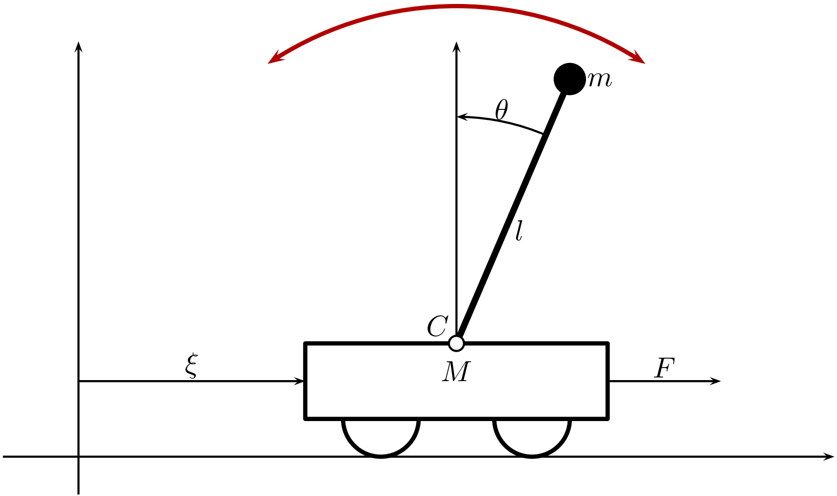
We assume that $f(0,0) = 0$. Let us consider the linearized control system $\dot{y} = Ay + Bu$ of $\dot{y} = f(y,u)$ at $(0,0) \in \mathbb{R}^n \times \mathbb{R}^m$:

(1) $$A := \frac{\partial f}{\partial y}(0,0),\ B := \frac{\partial f}{\partial u}(0,0).$$

Let us assume that the linearized control system $\dot{y} = Ay + Bu$ is controllable. Then, by the pole-shifting theorem, there exists $K \in \mathbb{R}^{m \times n}$ such that $\sigma(A + BK) = \{-1\}$. Let us consider the feedback $u(y) = Ky$. Then, if $X(y) := f(y, u(y))$, $X'(0) = A + BK$. Hence, by Lyapunov's first theorem, $0 \in \mathbb{R}^n$ is locally asymptotically stable for the closed loop system $\dot{y} = f(y, u(y))$. In conclusion, if the linearized control system is controllable, then

- The control system $\dot{y} = f(y,u)$ is small-time locally controllable at $(0,0)$,
- The control system $\dot{y} = f(y,u)$ is locally asymptotically stabilizable (at the equilibrium $(0,0)$).

An example: Cart-inverted pendulum



Cart-inverted pendulum: The equations

Let

(1) $$y_1 := \xi,\ y_2 := \theta,\ y_3 := \dot{\xi},\ y_4 := \dot{\theta},\ u := F,$$

The dynamics of the cart-inverted pendulum system is $\dot{y} = f(y,u)$, with $y = (y_1, y_2, y_3, y_4)^{\text{tr}}$ and

(2) $$f := \begin{pmatrix} y_3 \\ y_4 \\ \dfrac{mly_4^2 \sin(y_2) - mg\sin(y_2)\cos(y_2)}{M + m\sin^2(y_2)} + \dfrac{u}{M + m\sin^2(y_2)} \\ \dfrac{-mly_4^2 \sin(y_2)\cos(y_2) + (M+m)g\sin(y_2)}{(M + m\sin^2(y_2))l} - \dfrac{u\cos(y_2)}{(M + m\sin^2(y_2))l} \end{pmatrix}.$$

Stabilization of the cart-inverted pendulum

For the cart-inverted pendulum, the linearized control system around $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$ is $\dot{y} = Ay + Bu$ with

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\dfrac{mg}{M} & 0 & 0 \\ 0 & \dfrac{(M+m)g}{Ml} & 0 & 0 \end{pmatrix}, \ B = \frac{1}{Ml}\begin{pmatrix} 0 \\ 0 \\ l \\ -1 \end{pmatrix}. \tag{1}$$

One easily checks that this linearized control system satisfies the Kalman rank condition and therefore is controllable. Hence the cart-inverted pendulum is small-time locally controllable at $(0,0) \in \mathbb{R}^4 \times \mathbb{R}$ and is locally asymptotically stabilizable (at the equilibrium $(0,0)$).

Obstruction to the stabilizability

In 1979, H. Sussmann gave the first example of a controllable system which cannot be stabilized by means of continuous feedback laws.

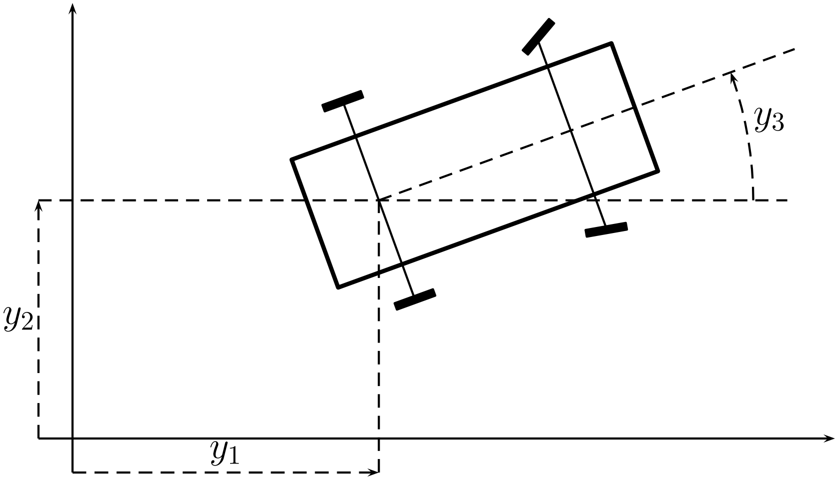
Theorem (R. Brockett (1983))

If the control system $\dot{y} = f(y, u)$ *is locally asymptotically stabilizable then*

(B) *the image by* f *of every neighborhood of* $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^m$ *is a neighborhood of* $0 \in \mathbb{R}^n$*.*

The baby stroller

The baby stroller: The dynamic equations of motion



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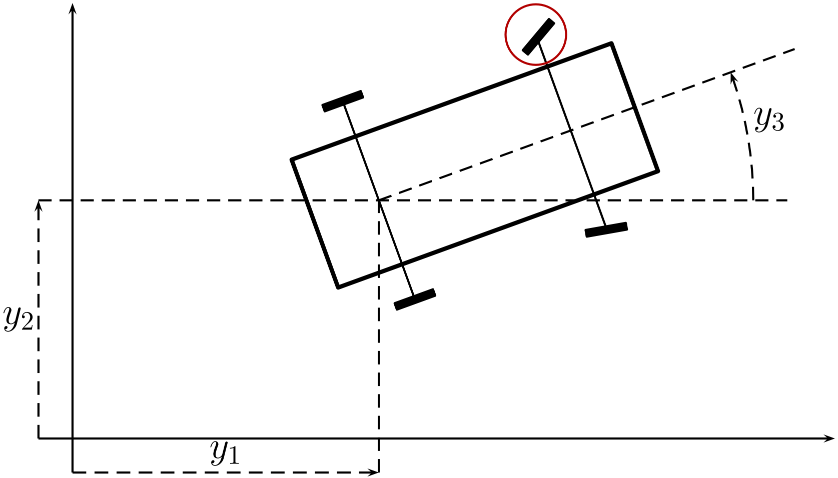

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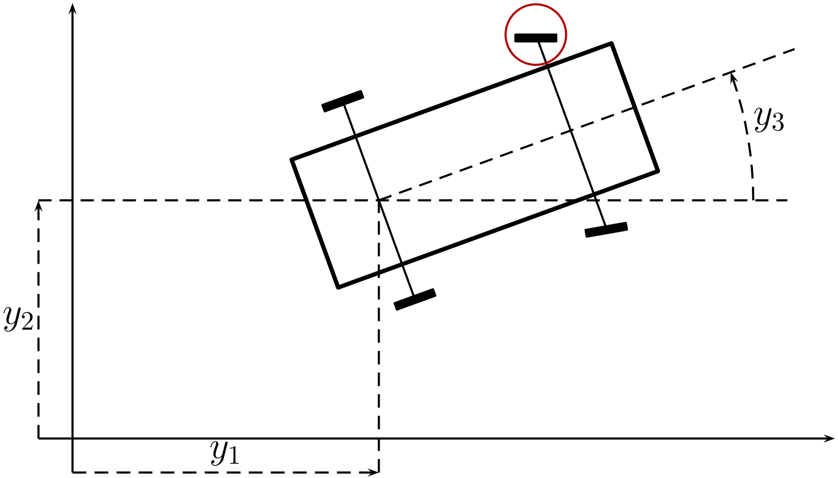

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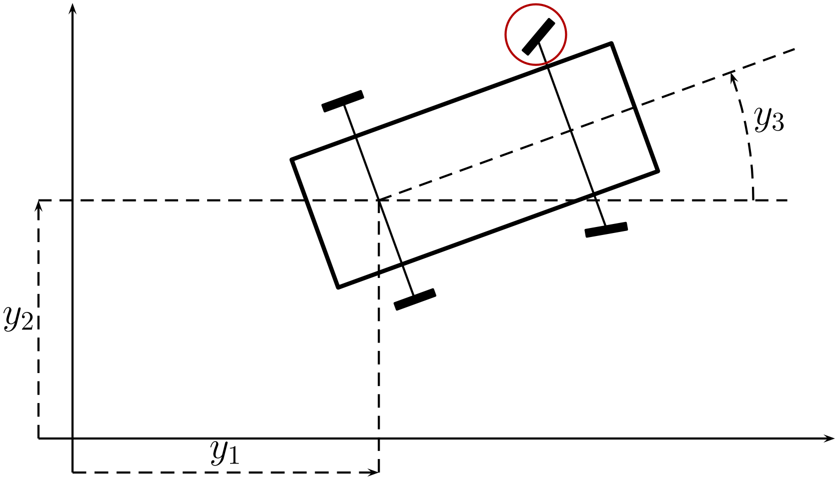

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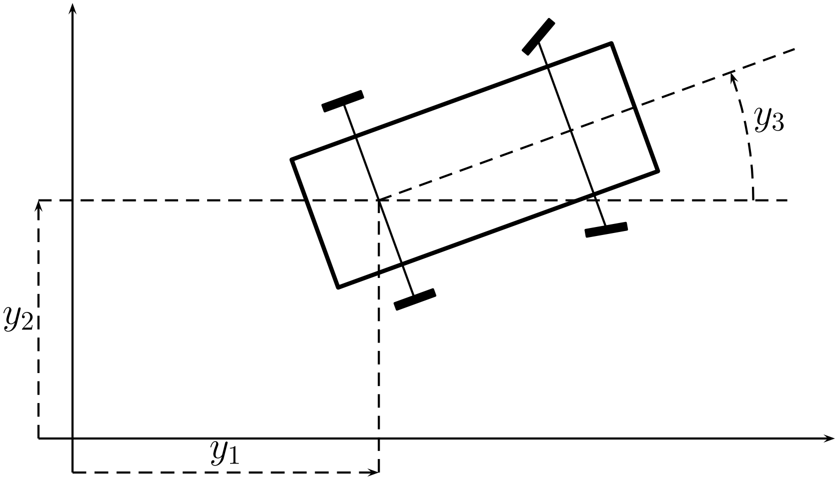


(1) $$\dot{y}_1 = u_1 \cos(y_3),\ \dot{y}_2 = u_1 \sin(y_3), \dot{y}_3 = u_2,\ n = 3,\ m = 2.$$

The baby stroller system: Controllability

(1) $$\dot{y}_1 = u_1 \cos(y_3),\ \dot{y}_2 = u_1 \sin(y_3), \dot{y}_3 = u_2,\ n = 3,\ m = 2.$$

Note that the linearized control system around $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^2$ is

(2) $$\dot{y}_1 = u_1,\ \dot{y}_2 = 0, \dot{y}_3 = u_2,$$

which is not controllable. The baby stroller control system can be written as $\dot{y} = u_1 f_1(y) + u_2 f_2(y)$, with

(3) $$f_1(y) = (\cos(y_3), \sin(y_3), 0)^{\text{tr}},\ f_2(y) = (0,0,1)^{\text{tr}}.$$

One has

(4) $$[f_1, f_2](y) = (\sin(y_3), -\cos(y_3), 0)^{\text{tr}}.$$

Hence $f_1(0)$, $f_2(0)$ and $[f_1, f_2](0)$ all together span all of $\mathbb{R}^3$. This implies the small-time local controllability of the baby stroller at $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^2$.

The baby stroller and the Brockett condition

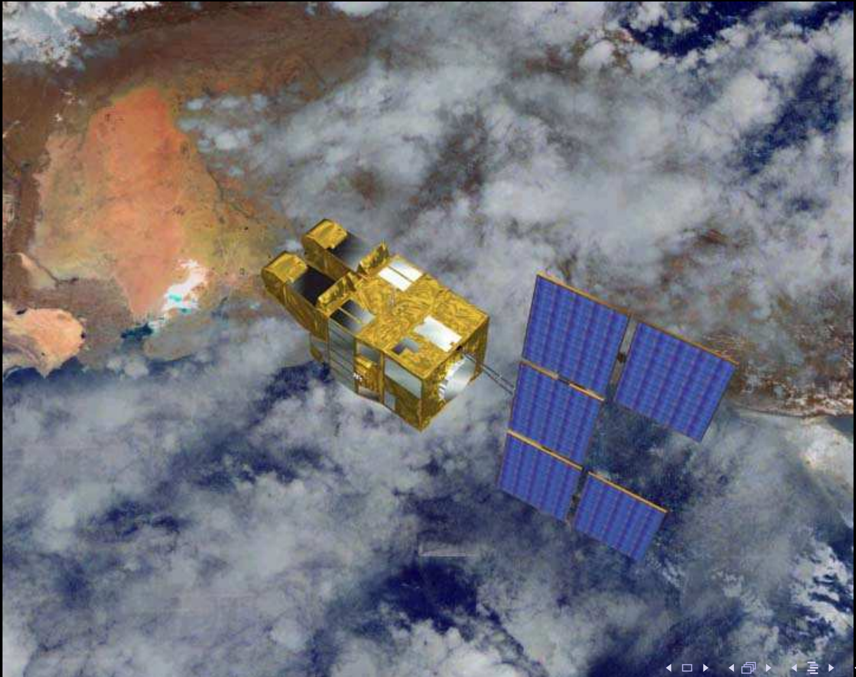
As we just saw the baby stroller control system

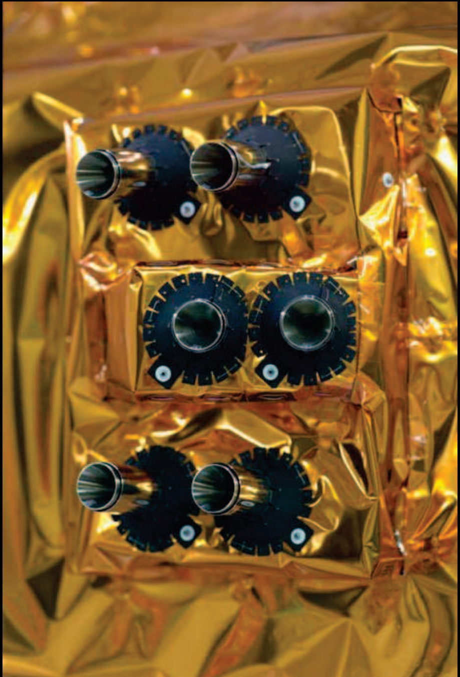
(1) $$\dot{y}_1 = u_1 \cos(y_3),\ \dot{y}_2 = u_1 \sin(y_3),\ \dot{y}_3 = u_2$$

is small-time locally controllable at $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^2$. However the Brockett condition (B) does not hold for the baby stroller control system: try to solve

(2) $$u_1 \cos(y_3) = 0,\ u_1 \sin(y_3) = \delta,\ u_2 = 0.$$

Hence the baby stroller control system cannot be locally asymptotically stabilized.





Control of the attitude of the satellite: Notations

- $\eta = (\phi, \theta, \psi)^{\text{tr}} \in \mathbb{R}^3$ are the Euler angles of a frame attached to the satellite representing rotations with respect to a fixed reference frame,
- $\omega = (\omega_1, \omega_2, \omega_3)^{\text{tr}} \in \mathbb{R}^3$ is the angular velocity of the frame attached to the satellite with respect to the reference frame, expressed in the frame attached to the satellite,
- J is the inertia matrix of the satellite,
- The $b_1, \ldots, b_m$ are m fixed independent vectors in $\mathbb{R}^3$ and $u_i b_i \in \mathbb{R}^3$, $1 \leqslant i \leqslant m$, are the torques applied to the satellite, the $u_i \in \mathbb{R}$, $1 \leqslant i \leqslant m$, are the controls.

Control of the attitude of the satellite

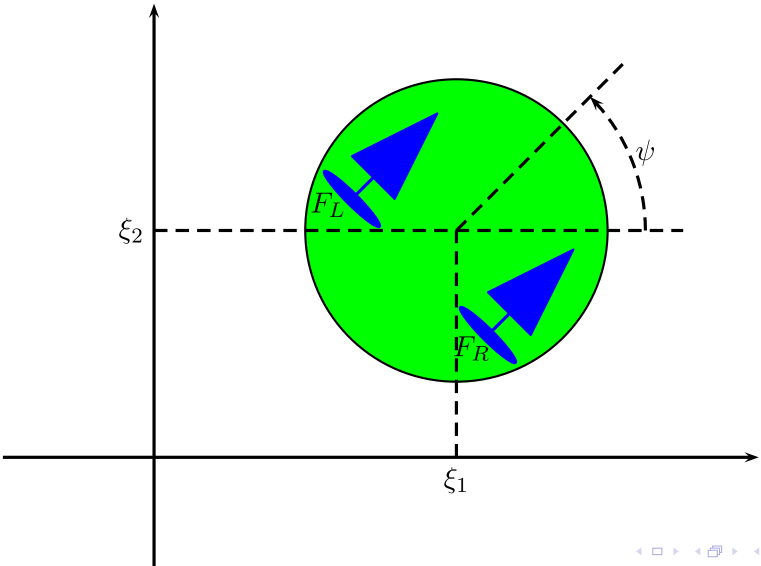
(1) $$\dot{\omega} = J^{-1}S(\omega)J\omega + \sum_{i=1}^{m} u_i J^{-1} b_i,\ \dot{\eta} = A(\eta)\omega,$$

with $S(\omega)y := y \wedge \omega$. One has $A(0) = \text{Id}$. The vectors $b_1, \ldots, b_m$ are independent. If $m = 3$, the linearized control system around the equilibrium $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^3$ is controllable and the control system is locally asymptotically stabilizable. We now turn to the case where $m = 2$. One easily sees that (B) never holds. However, if

(2) $$\text{Span } \{b_1, b_2, S(\omega)J^{-1}\omega;\ \omega \in \text{ Span } \{b_1, b_2\}\} = \mathbb{R}^3,$$

then, by a general sufficient condition for local controllability proved by H. Sussmann in 1987, the control system (1) is small-time locally controllable at $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^2$. However, if $m < 3$, (1) does not satisfy the Brockett condition (B).

The slider



Equations for the slider

The slider is actuated by two propellers producing forces F_L and F_R. The sum of these two forces is directly linked to the acceleration of the vehicle, whereas the difference acts on the angular dynamics. Let us denote $\tau_1 = F_L + F_R$ and $\tau_2 = F_R - F_L$, the dynamics can be written:

$$\left\{ \begin{array}{rcl} m\ddot{\xi}_1 & = & \cos(\psi)\tau_1, \\ m\ddot{\xi}_2 & = & \sin(\psi)\tau_1, \\ I\ddot{\psi} & = & \tau_2, \end{array} \right. \tag{1}$$

where m is the slider mass and I is the moment of inertia of the slider about its center of mass.

Equations for the slider in the form $\dot{y} = f(y, u)$

Let

(1)
$$\begin{cases} y_1 = \xi_1,\ y_2 = \dot{\xi}_1,\ y_3 = \xi_2,\ y_4 = \dot{\xi}_2, \\ y_5 = \psi,\ y_6 = \dot{\psi},\ u_1 = \dfrac{\tau_1}{m},\ u_2 = \dfrac{\tau_2}{I}. \end{cases}$$

Then the dynamics of the slider can be written in the form $\dot{y} = f(y, u)$ with

(2)
$$f(y, u) := (y_2, u_1 \cos(y_5), y_4, u_1 \sin(y_5), y_6, u_2)^{\mathrm{tr}}.$$

Slider: Controllability and Brockett's condition

One has the following theorem.

Theorem

The slider control system is small-time locally controllable at the equilibrium $(0,0) \in \mathbb{R}^6 \times \mathbb{R}^2$ *but does not satisfy the Brockett condition.*

For the Brockett condition, consider the equation

$$(y_2, u_1 \cos(y_5), y_4, u_1 \sin(y_5), y_6, u_2)^{\text{tr}} = (0, 0, 0, \delta, 0, 0)^{\text{tr}}. \tag{1}$$

Time-varying feedback laws

Strategy to overcome the obstruction to stabilization: Use time-varying feedback laws. Instead of $u(y)$, use $u(t, y)$. Note that asymptotic stability for time-varying feedback laws is also robust (there exists again a strict Lyapunov function). Pioneer works : $n = 1$: E. Sontag and H. Sussmann (1980); the baby stroller control system: C. Samson (1992).
Let us give some general results with this strategy.

Time-varying feedback laws for driftless control systems

Theorem (JMC (1992))

Assume that $\dot{y} = \sum_{i=1}^{m} u_i f_i(y)$ *satisfies the Lie algebra rank condition at* $0 \in \mathbb{R}^n$*. Then, for every* $T > 0$*, there exists* u *in* $C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ *such that*

(1) $$u(t, 0) = 0, \; \forall t \in \mathbb{R},$$

(2) $$u(t + T, y) = u(t, y), \; \forall y \in \mathbb{R}^n, \; \forall t \in \mathbb{R},$$

(3) $$0 \text{ is asymptotically stable for } \dot{y} = \sum_{i=1}^{m} u_i(t, y) f_i(y).$$

Construction of explicit stabilizing time-varying feedback laws for various driftless controllable systems (including the baby stroller): J.-B. Pomet (1992), C. Canudas de Wit and O. J. Sørdalen (1992), ..., A. Zuyev (2016), J.-P. Guilleron, B. d'Andréa-Novel, JMC and W. Perruquetti (2016).

Sketch of proof

Sketch of the proof of the theorem. Let $T > 0$. Assume that there exists $\bar{u}$ in $C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ T-periodic with time, vanishing for $y = 0$, and such that, if $\dot{y} = f(y, \bar{u}(t, y))$, then

(i) $y(T) = y(0)$,

(ii) If $y(0) \neq 0$, then the linearized control system around the trajectory $t \in [0, T] \mapsto (y(t), \bar{u}(t, y(t)))$ is controllable on $[0, T]$.

Using (i) and (ii) one easily sees that one can construct a “small" feedback v in $C^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^m)$ T-periodic with time and vanishing for $y = 0$ such that, if $\dot{y} = f(y, (\bar{u} + v)(t, y))$ and $y(0) \neq 0$, then

$$|y(T)| < |y(0)|, \tag{1}$$

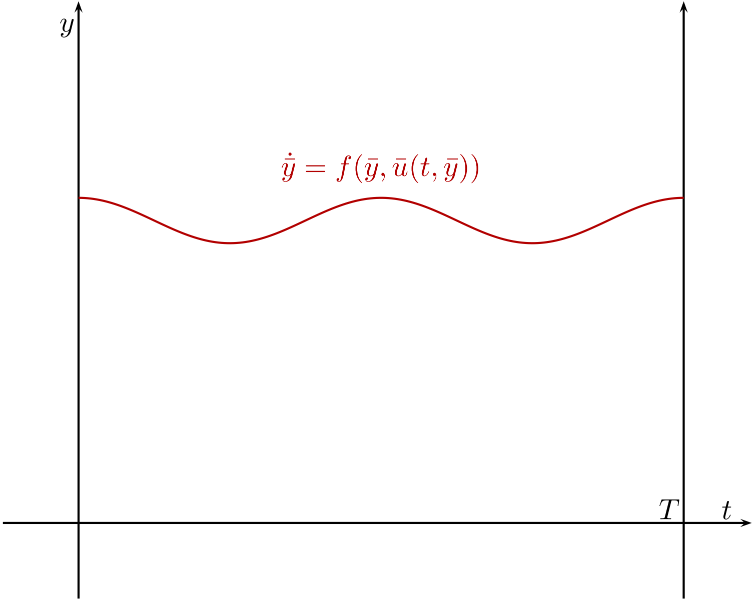
which implies that 0 is globally asymptotically stable for $\dot{y} = f(y, (\bar{u} + v)(t, y))$.

Construction of $\bar{u}$

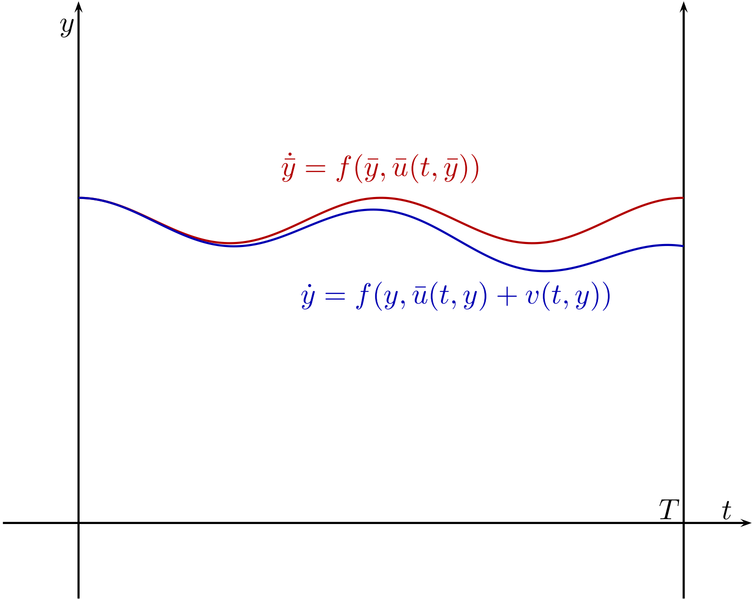
In order to get (i), one just imposes on $\bar{u}$ the condition

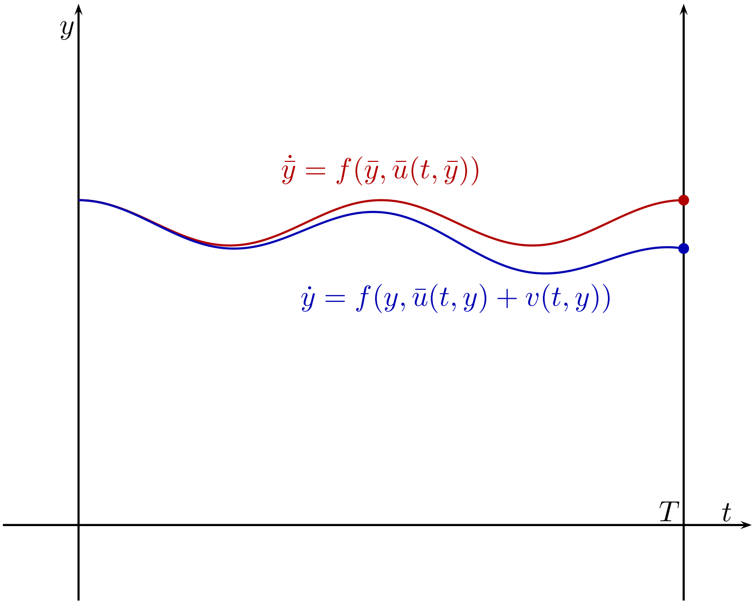
$$(1) \qquad \bar{u}(t,y) = -\bar{u}(T-t,y), \, \forall (t,y) \in \mathbb{R} \times \mathbb{R}^n,$$

which implies that $y(t) = y(T-t), \forall t \in [0,T]$, for every solution of $\dot{y} = f(y, u(t,y))$, and therefore gives $y(0) = y(T)$. Finally, one proves that (ii) holds for “many" $\bar{u}$'s (this is the difficult part of the proof).



y
$\dot{y} = f(\bar{y}, \bar{u}(t, \bar{y}))$
T
t


y
$\dot{\bar{y}} = f(\bar{y}, \bar{u}(t, \bar{y}))$
$\dot{y} = f(y, \bar{u}(t, y) + v(t, y))$
T
t



y
$\dot{\bar{y}} = f(\bar{y}, \bar{u}(t, \bar{y}))$
$\dot{y} = f(y, \bar{u}(t, y) + v(t, y))$
T
t

A method to avoid Lie brackets: The return method (JMC (1992))

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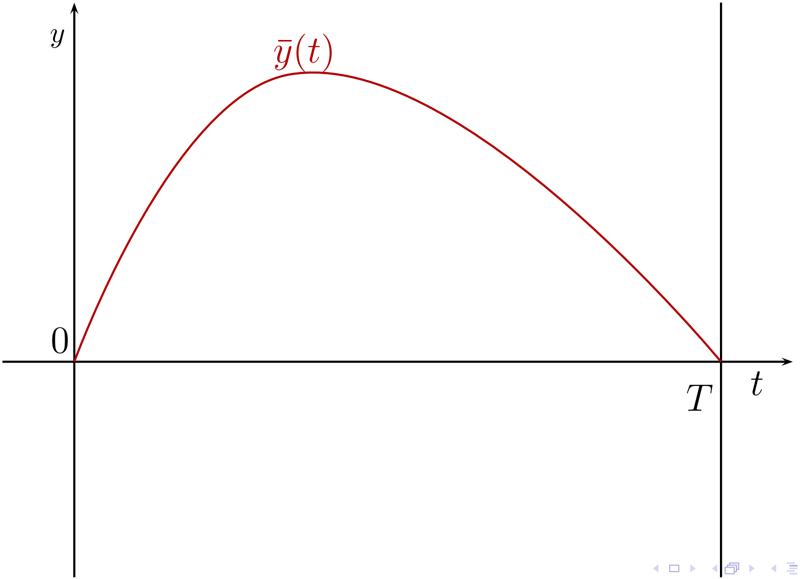
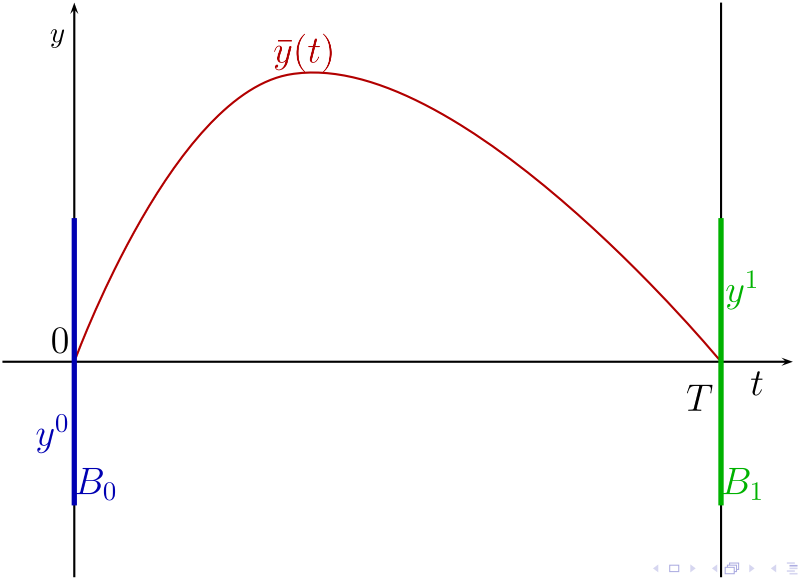
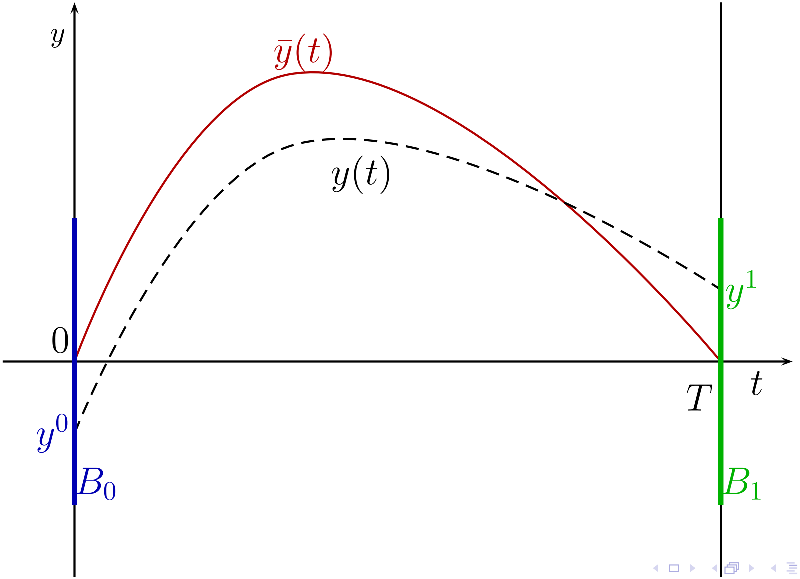


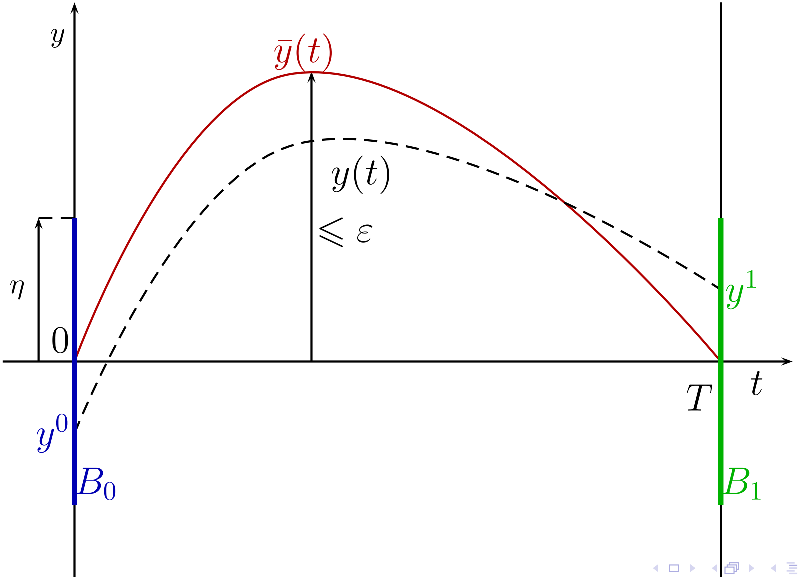
A method to avoid Lie brackets: The return method (JMC (1992))



A method to avoid Lie brackets: The return method (JMC (1992))



A method to avoid Lie brackets: The return method (JMC (1992))



The return method: An example in finite dimension

We go back to the baby stroller control system

(1) $$\dot{y}_1 = u_1 \cos(y_3),\ \dot{y}_2 = u_1 \sin(y_3),\ \dot{y}_3 = u_2.$$

For every $\bar{u} : [0, T] \to \mathbb{R}^2$ such that, for every t in $[0, T]$, $\bar{u}(T - t) = -\bar{u}(t)$, every solution $\bar{y} : [0, T] \to \mathbb{R}^3$ of

(2) $$\dot{\bar{y}}_1 = \bar{u}_1 \cos(\bar{y}_3),\ \dot{\bar{y}}_2 = \bar{u}_1 \sin(\bar{y}_3), \dot{\bar{y}}_3 = \bar{u}_2,$$

satisfies $\bar{y}(0) = \bar{y}(T)$. The linearized control system around $(\bar{y}, \bar{u})$ is

(3) $$\begin{cases} \dot{y}_1 = -\bar{u}_1 y_3 \sin(\bar{y}_3) + u_1 \cos(\bar{y}_3),\ \dot{y}_2 = \bar{u}_1 y_3 \cos(\bar{y}_3) + u_1 \sin(\bar{y}_3), \\ \dot{y}_3 = u_2, \end{cases}$$

which is controllable if (and only if) $\bar{u} \not\equiv 0$. We have got the controllability of the baby stroller system without using Lie brackets. We have only used controllability results for **linear** control systems.

Return method: References

- Stabilization of driftless systems in finite dimension: JMC (1992),
- Euler equations of incompressible fluids: JMC (1993,1996), O. Glass (1997,2000), O. Glass-Th. Horsin (2010, 2012, 2016),
- Control of driftless systems in finite dimension: E.D. Sontag (1995),
- Navier-Stokes equations: JMC (1996), JMC and A. Fursikov (1996), A. Fursikov and O. Imanuvilov (1999), S. Guerrero, O. Imanuvilov and J.-P. Puel (2006), JMC and S. Guerrero (2009), M. Chapouly (2009), JMC and P. Lissy (2014),
- Saint-Venant equations: JMC (2002),
- Vlasov Poisson: O. Glass (2003),

Return method: References (continued)

- Isentropic Euler equations: O. Glass (2006),
- Schrödinger equation: K. Beauchard (2005), K. Beauchard and JMC (2006),
- Hyperbolic/wave equations: JMC, O. Glass and Z. Wang (2009), F. Alabau, JMC and G. Olive (2017), C. Zhang (2017),
- Ensemble controllability of Bloch equations: K. Beauchard, JMC and P. Rouchon (2010),
- Parabolic systems: JMC, S. Guerrero and L. Rosier (2010), JMC and J.-Ph. Guilleron (2017),
- Uniform controllability of scalar conservation laws in the vanishing viscosity limit: M. Léautaud (2010).

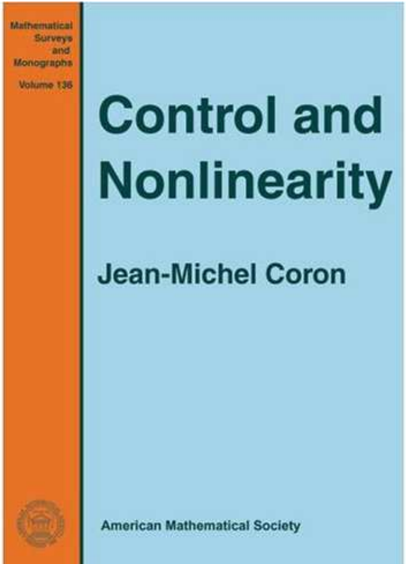
Small-Time local controllability stabilization in finite time

Theorem (JMC (1995))

"Most of the known sufficient conditions" (see JMC SICON 1995 for a precise statement) for small-time local controllability imply the asymptotic stabilizability by means of time-varying feedback laws if $n \notin \{1, 2, 3\}$.

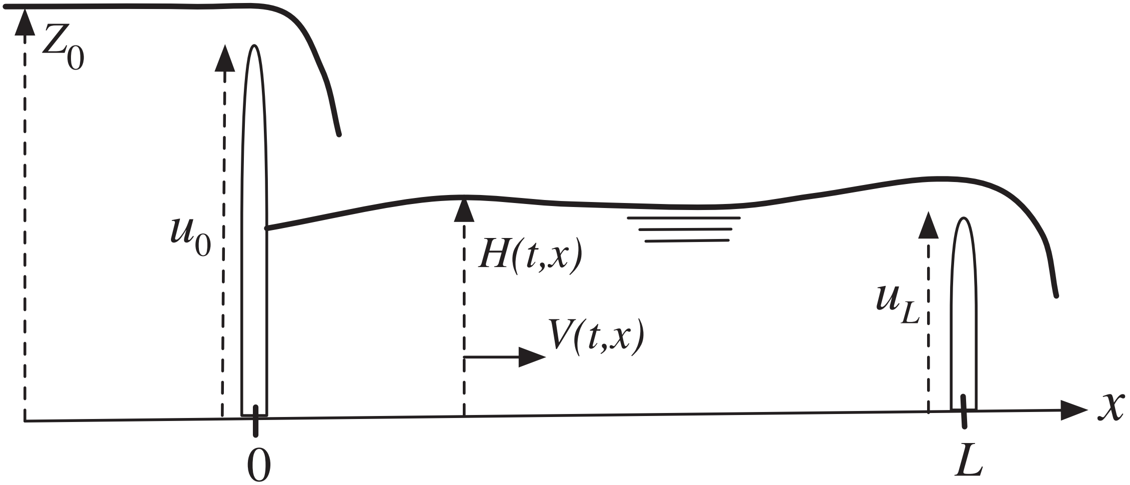
Applications: Underactuated satellite (Explicit stabilizing feedback laws: G. Walsh, R. Montgomery and S. Sastry (1994); P. Morin, C. Samson, J.-B. Pomet and Z.-P. Jiang (1995), JMC and E.-Y. Keraï (1996); P. Morin and C. Samson (1997)), slider (Explicit stabilizing feedback laws: B. d'Andréa-Novel, JMC and W. Perruquetti (2016)).

Tools to design feedback laws (commercial break)



JMC, Control and nonlinearity, Mathematical Surveys and Monographs, 136, 2007, 427 p. Pdf file freely available from my web page.


Meuse
Sambre
8
5
8,4 km
7,7km
7
6
Meuse
11,2km
4
7,6 km
5,2 km
2
6,4 km
1
4,9 km
3
France



Z_0
u_0
$H(t,x)$
$V(t,x)$
u_L
x
0
L

The Saint-Venant equations

The index j is for the j-th pool.
Conservation of mass:

$$H_{jt} + (H_j V_j)_x = 0, \tag{1}$$

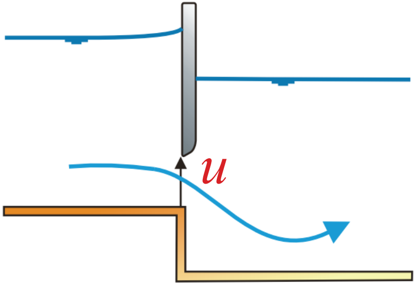
with, as usual, $f_t := \partial_t f$ and $f_x := \partial_x f$.
Conservation of momentum:

$$V_{jt} + \left(gH_j + \frac{V_j^2}{2} \right)_x = 0. \tag{2}$$

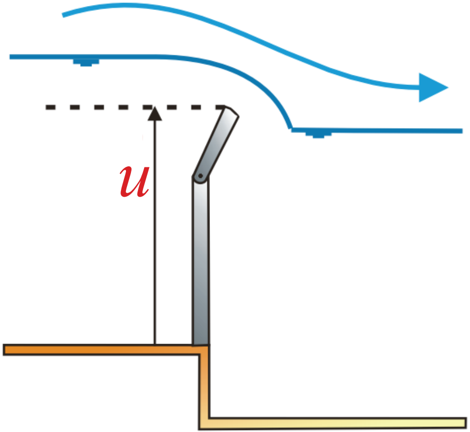
Flow rate: $Q_j = H_j V_j$.

Boundary conditions

Underflow (sluice)



Overflow (spillway)



General 1-D hyperbolic systems

(1) $$y_t + A(y)y_x + B(y)y = 0,\ y \in \mathbb{R}^n,\ x \in [0,1],\ t \in [0,+\infty),$$

- Assumptions on A

(2) $$A(0) = \text{diag}\ (\lambda_1, \lambda_2, \ldots, \lambda_n),$$

(3) $$\lambda_i > 0,\ \forall i \in \{1, \ldots, m\},\ \lambda_i < 0,\ \forall i \in \{m+1, \ldots, n\},$$

(4) $$\lambda_i \neq \lambda_j,\ \forall (i,j) \in \{1, \ldots, n\}^2 \text{ such that } i \neq j.$$

- Boundary conditions on y:

$$\begin{pmatrix} y_+(t,0) \\ y_-(t,1) \end{pmatrix} = G \begin{pmatrix} y_+(t,1) \\ y_-(t,0) \end{pmatrix}, \; t \in [0,+\infty), \tag{1}$$

where

(i) $y_+ \in \mathbb{R}^m$ and $y_- \in \mathbb{R}^{n-m}$ are defined by

$$y = \begin{pmatrix} y_+ \\ y_- \end{pmatrix},$$

(ii) the map $G : \mathbb{R}^n \to \mathbb{R}^n$ vanishes at 0.

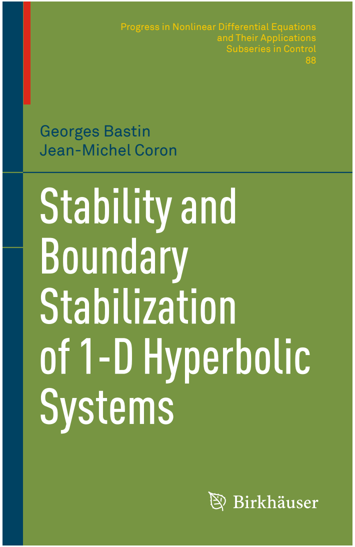
Remark

In the applications, part of G is fixed, part of G is our choice. This last part is the feedback law that we have to design.

Some examples of real life applications

1. navigable rivers,
2. irrigation channels,
3. heat exchangers,
4. tubular plug flow chemical reactors,
5. gas pipe lines,
6. chromatography,
7. road traffic...

New commercial break



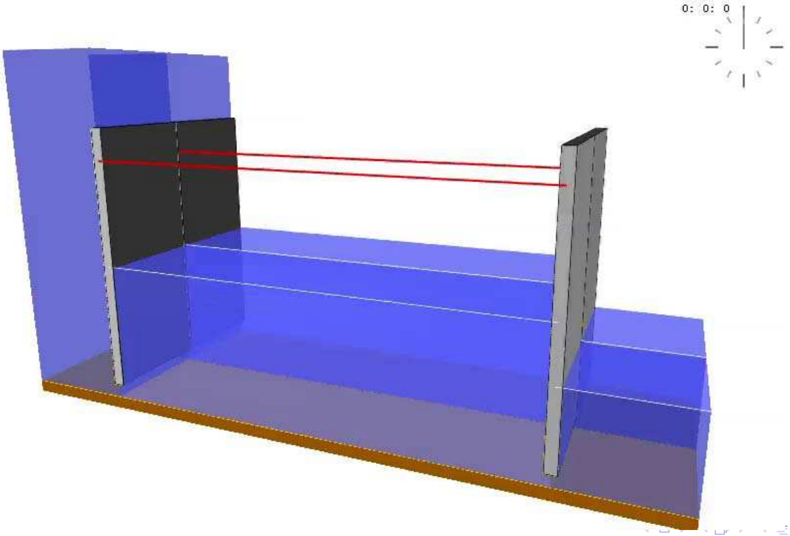
G. Bastin and JMC, Stability and Boundary Stabilization of 1-D Hyperbolic Systems, 2016, PNLDE Subseries in Control, Birkhaüser.

Some results

- Sufficient conditions for asymptotic stability for the C^k-norm, $k \in \mathbb{N} \setminus \{0\}$) (Tie Hu Qin (1985), Yan Chun Zhao (1986), Tatsien Li (1994), G. Bastin and JMC (2014), A. Hayat (2017), A. Hayat and P. Shang (2017).
- Sufficient conditions for asymptotic stability for the H^l-norm, $l \in \mathbb{N} \setminus \{0, 1\}$). This condition is optimal for $n \in \{1, 2, 3, 4, 5\}$ and $B = 0$. (JMC-G. Bastin-B. d'Andréa-Novel (2008), G. Bastin and JMC (2016, 2017)).
- The asymptotic stability for the H^l-norm, $l \in \mathbb{N} \setminus \{0, 1\}$), does not imply the asymptotic stability for the C^1-norm (JMC and Hoai-Minh Nguyen (2014)).
- Real life application to the rivers La Sambre and La Meuse in Belgium (B. d'Andréa-Novel, G. Bastin, JMC, V. Dos Santos, J. de Halleux, L. Moens and C. Prieur (2003-2017)).

However plenty of questions remain open, in particular in the case $B \neq 0$.

La Sambre (B. d'Andréa-Novel, G. Bastin, JMC, V. Dos Santos, J. de Halleux, L. Moens and C. Prieur (2003-...))



La Sambre (B. d'Andréa-Novel, G. Bastin, JMC, V. Dos Santos, J. de Halleux, L. Moens and C. Prieur (2003-...))