Reduced Order Modeling in Multispectral Photoacoustic Tomography

Arvind Saibaba   Sarah Vallélian

Statistical and Applied Mathematical Sciences Institute
&
North Carolina State University

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Outline

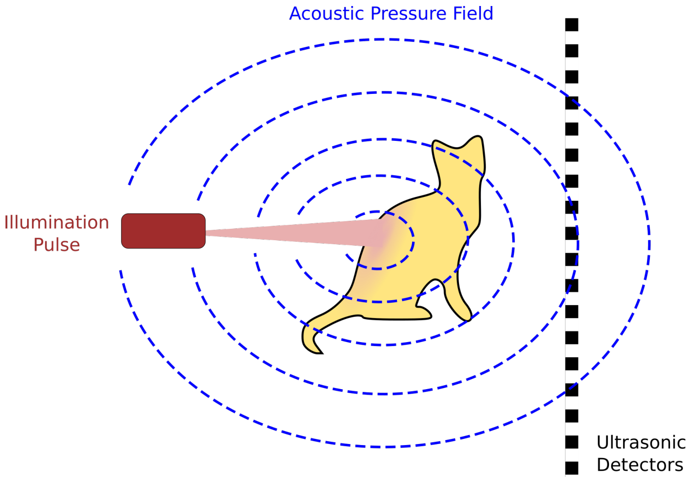


Figure: The photoacoustic tomography experiment.

Advantages

- High contrast and high resolution
- Noninvasive, nondestructive, inexpensive
- Multispectral imaging capability for better reconstructions
- Multiscale imaging capability
- Success in small animals

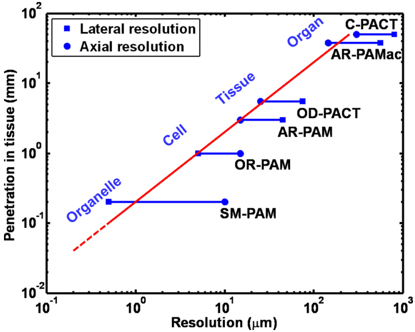
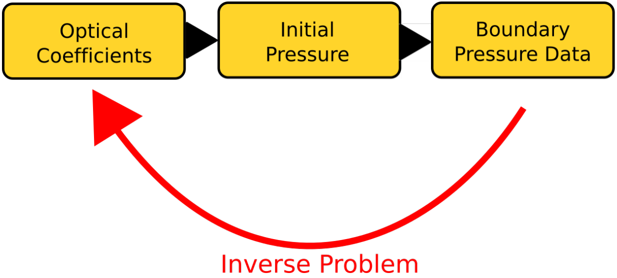


Figure: Photoacoustic tomography on multiple scales. S. Hu and L. Wang, Front. Neuroenergetics 2010.

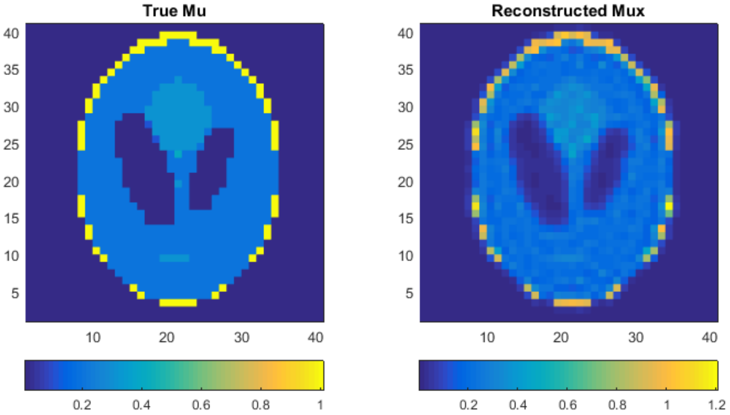
Forward Problem: To compute the ultrasound pressure field at the detector array assuming that we know all properties of the medium.

1. Use the illumination pattern and *optical properties* to determine the absorbed optical energy, converted into pressure by the photoacoustic effect. (diffusion equation)
2. Use the internal pressure as an initial condition and propagate it with the given ultrasound speed to the detectors. (wave equation)

Inverse Problem: To recover the interior optical properties, using the ultrasound pressure measurements at the detector array as data.



One-step reconstruction via non-linear least squares is most flexible in terms of data, enforcing/requiring prior knowledge. T. Ding, K. Ren and S. V., Inverse Problems 2015.



Naive approach: 40×40 image, single wavelength, 5 mins. Data contains 1% random noise. Relative $L2$ error: 0.19

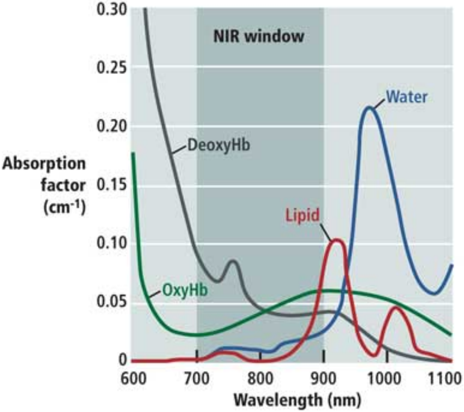


Figure: Chromophores for biomedical optical imaging. T.G. Phan and A. Bullen, Immunol. Cell Biol. 2010.

Wavelength Dependence

- $D(\mathbf{x}, \lambda) = \alpha(\lambda) D(\mathbf{x})$
- $\Upsilon(\mathbf{x}, \lambda) = \gamma(\lambda) \Upsilon(\mathbf{x})$
- $\mu(\mathbf{x}, \lambda) = \sum_{k=1}^{K} \sigma_k(\lambda) \mu_k(\mathbf{x})$

Diffusion Equation

$$
\begin{aligned}
-\nabla \cdot D(\mathbf{x},\lambda)\nabla\phi + \mu(\mathbf{x},\lambda)\phi &= 0, && \text{in } \Omega \times \mathcal{L} \\
\phi &= g(\mathbf{x},\lambda), && \text{on } \partial\Omega \times \mathcal{L}.
\end{aligned}
$$

Wave Equation

$$
\begin{aligned}
\frac{1}{c^2(\mathbf{x})}\frac{\partial^2 p}{\partial t^2} - \Delta p &= 0, && \text{in } \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{L} \\
p(0,\mathbf{x},\lambda) &= \Upsilon(\mathbf{x},\lambda)\mu(\mathbf{x},\lambda)\phi\chi_\Omega(\mathbf{x}), && \text{in } \mathbb{R}^d \times \mathcal{L} \\
\frac{\partial p}{\partial t}(0,\mathbf{x},\lambda) &= 0, && \text{in } \mathbb{R}^d \times \mathcal{L}.
\end{aligned}
$$

Measurements

$m_\lambda := p(t,\mathbf{x},\lambda)$ at certain times $(0,\tau)$, locations Σ, wavelengths $\mathcal{L}$

Given
$\Upsilon(\mathbf{x},\lambda), D(\mathbf{x},\lambda)$, and $\{\sigma_k(\lambda)\}_{k=1}^K$

Unknown
$\boldsymbol{\mu} = \{\mu_k(\mathbf{x})\}_{k=1}^K$

Solution

$$\boldsymbol{\mu}^* = \arg\min J(\boldsymbol{\mu}) := \frac{1}{2} \sum_{\lambda \in \mathcal{L}} \|m_\lambda - \Lambda(\boldsymbol{\mu};\lambda)\|^2 + \alpha \mathcal{R}(\boldsymbol{\mu})$$

Solve via e.g. quasi-Newton method, dimension is a problem:

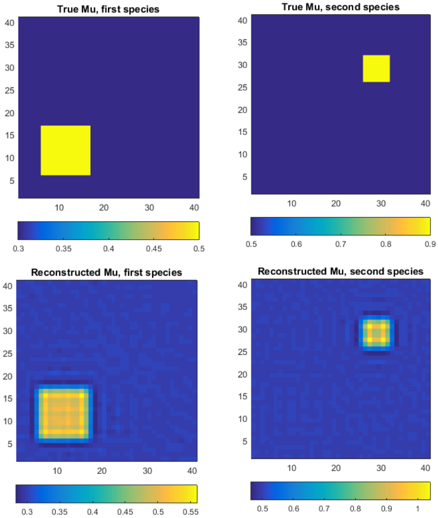
- Compute $\Lambda(\boldsymbol{\mu};\lambda)$ for a given unknown $\boldsymbol{\mu}$ at each iteration (solving 2 PDEs)
  - diffusion: solve linear system of size N # pixels
  - wave: dominant cost $\sim \mathcal{O}(N^2)$ in $2d$
- Compute the gradient for a given unknown $\boldsymbol{\mu}$ at each iteration (solving 2 adjoint PDEs)

At each iteration, must solve at all wavelengths!

- $N = 1$ million: $3d$ image of size 1 cm^3 (approx. size of mouse brain), 100 μm resolution
- N_λ = 10-200: different wavelengths for optical imaging
- $K = 2 - 4$: chromophores
- N_r = 50-300: # ultrasound receivers
- N_t : # time points measured

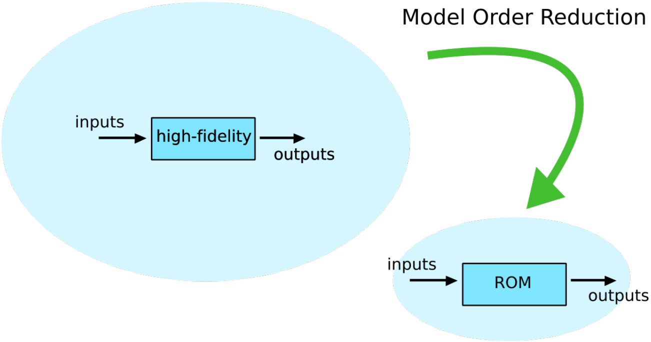
Main Issue
Repeated solution of large-scale systems

Typical Reconstructions



Naive approach: 40×40 images, 100 wavelengths, 4 hours. Data contains 1% random noise. Relative $L2$ error for the pair: 0.03

What is reduced-order modeling (ROM)?



Many ROMs use projection over a reduced-space basis: $x \approx \mathbf{V}x_r$

- $x \in \mathbb{R}^N$ is full-scale output
- $x_r \in \mathbb{R}^R$ is reduced-order output
- Columns of $\mathbf{V} \in \mathbb{R}^{N \times R}$ are O.N. basis vectors, $R \ll N$

Goal: Avoid solving the PDEs for all wavelengths whenever possible

- Photon density $\phi(\mathbf{x}, \lambda)$ is a smooth function of λ
- Diffusion system is parameterized as

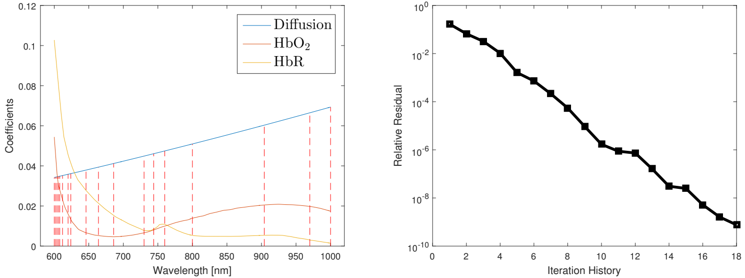
$$\left(\alpha(\lambda)\mathbf{K} + \sum_{k=1}^{K} \sigma_k(\lambda)\mathbf{M}_k\right)\phi = \mathbf{b}(\lambda)$$

- *Optimal* ROM for wavelength-dependence: truncate SVD of $\boldsymbol{\phi} \in \mathbb{R}^{N \times N_\lambda}$; too expensive
- Alternative: greedy basis construction

Given the diffusion model as linear system $\mathbf{A}\boldsymbol{\phi} = \mathbf{b}$, iteratively construct a projection matrix $\mathbf{V} \in \mathbb{R}^{N \times R}$ $(R \ll N_\lambda)$:

1. First column of $\mathbf{V}$ is normalized solution $\boldsymbol{\phi_1}$ to system eqns at an initial wavelength λ_1
2. Galerkin project the linear system with $\mathbf{V}$ and solve for reduced soln $\boldsymbol{\phi_r}$ at all wavelengths
3. Compute the residual norm at all wavelengths
4. Pick the wavelength λ_j with largest residual; add $\boldsymbol{\phi_j}$ to $\mathbf{V}$ after ortho-normalization
5. Stop when residual at all remaining wavelengths is below tolerance

Test - Two Chromophores



$N_\lambda = 200$ wavelengths reduced to $R = 18$. Relative error $\|\mathbf{b}(\lambda) - \mathbf{A}(\lambda)\mathbf{V}\phi\|$ for all wavelengths decreases as R increases. No loss of accuracy.

For the current value of $\boldsymbol{\mu}$,

Operation	# Diffusion Solves	# Wave Solves
Forward map	$N_\lambda \to R$	$N_\lambda \to R$
Gradient	$N_\lambda \to R$	$N_\lambda \to R$

Forward Map:

- Reduce # of diffusion solves
- Compress the initial condition of wave
- Reduce # of wave solves

Gradient:

- Compress the residual $m_\lambda - \Lambda(\boldsymbol{\mu}; \lambda)$
- Reduce # of adjoint wave solves
- Compress the source of adjoint diffusion
- Reduce # of adjoint diffusion solves

- Photoacoustic tomography is an emerging modality with applications to brain imaging
- Reduced-order modeling can approximate the PDE solutions in lower dimensions, cheaply
- We developed a ROM framework for multispectral, multispecies PAT
- Tradeoff: setup cost for the ROMs vs. computational savings in the inverse problem

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Thank you!