A Linearization Technique for Nonlinear Parabolic Problems in Porous Media

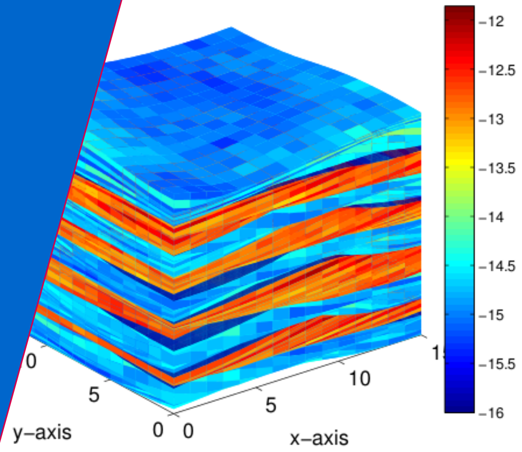
Koondanibha Mitra
(TU Eindhoven & Hasselt University)

joint work with

Iuliu Sorin Pop
(Hasselt University & University of Bergen)

and

Xiulei Cao (York University)



Technische Universiteit
Eindhoven
University of Technology

Houston, March 2019

Contents

Introduction

To solve $f(x) = 0$ Newton scheme

Introduction

To solve $f(x) = 0$ Newton scheme

- Takes initial guess x_0

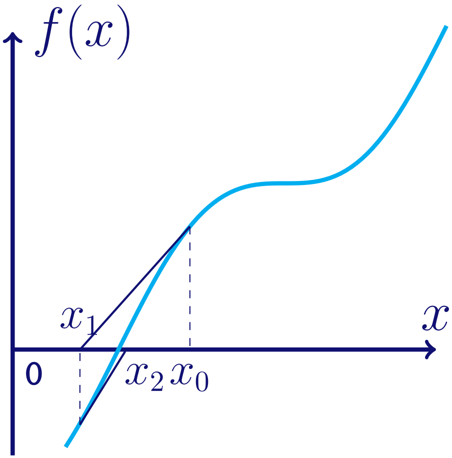
Introduction

To solve $f(x) = 0$ Newton scheme

- Takes initial guess x_0
- Updates for all $i \in \mathbb{N}$

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}$$

- The solution being $\lim x_i = \bar{x}$



Introduction

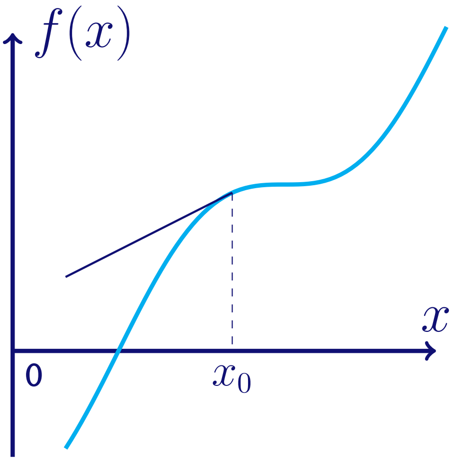
To solve $f(x) = 0$ Newton scheme

- Takes initial guess x_0
- Updates for all $i \in \mathbb{N}$

$$x_i = x_{i-1} - \frac{f(x_{i-1})}{f'(x_{i-1})}$$

- The solution being $\lim x_i = \bar{x}$

However, if x_0 is not close to $\bar{x}$ then the scheme might not converge



Introduction

If instead one uses the iteration

$$Lx_i = Lx_{i-1} - f(x_{i-1})$$

for $L > \max_{x \in \mathbb{R}}\{f'(x)\}$, then

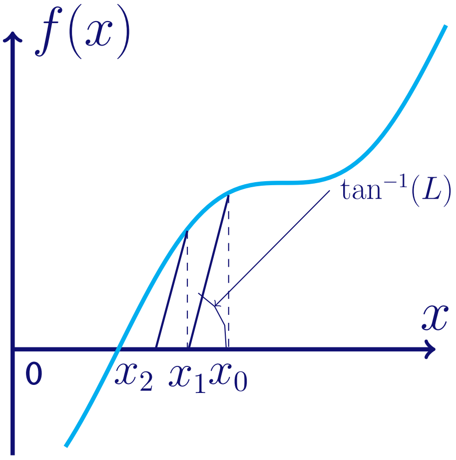
Introduction

If instead one uses the iteration

$$Lx_i = Lx_{i-1} - f(x_{i-1})$$

for $L > \max_{x\in\mathbb{R}}\{f'(x)\}$, then

- Iterations converge irrespective of initial guess
- Errors decrease monotonically
- However, the convergence is slower (linear)



Introduction

Learning from above we propose

$$L^i x_i = L^i x_{i-1} - f(x_{i-1})$$

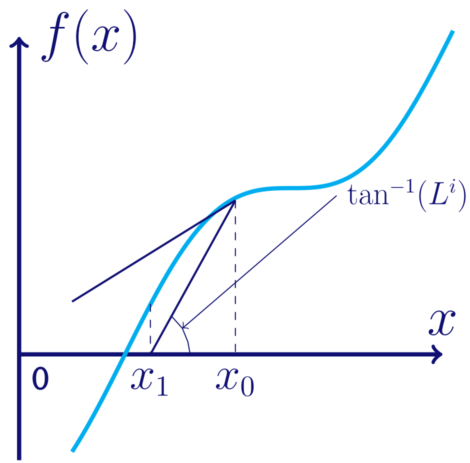
with $L^i = f'(x_{i-1}) + \mathfrak{M}$, $\mathfrak{M} > 0$ being a tolerance.

Introduction

Learning from above we propose

$$L^i x_i = L^i x_{i-1} - f(x_{i-1})$$

with $L^i = f'(x_{i-1}) + \mathfrak{M}$, $\mathfrak{M} > 0$ being a tolerance.



Does their exists an $\mathfrak{M}$ such that

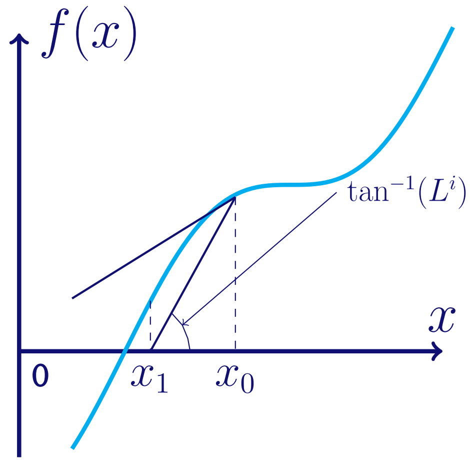
- The errors decrease monotonically
- The convergence is faster than L-scheme

Introduction

Learning from above we propose

$$L^i x_i = L^i x_{i-1} - f(x_{i-1})$$

with $L^i = f'(x_{i-1}) + \mathfrak{M}$, $\mathfrak{M} > 0$ being a tolerance.



Does their exists an $\mathfrak{M}$ such that

- The errors decrease monotonically
- The convergence is faster than L-scheme

We look for such a scheme for nonlinear PDEs in the study of porous flows

Equations

Richards Equation

$$\partial_t S_w = \nabla \cdot [k_w(S_w)(\nabla p - \rho_w \hat{g})], \quad -p = P_c(S_w)$$

Equations

- Richards Equation

$$\partial_t S_w = \nabla \cdot [k_w(S_w)(\nabla p - \rho_w \hat{g})], \quad -p = P_c(S_w)$$

- The two-phase porous media equation

$$\partial_t S_\alpha = \nabla \cdot [k_\alpha(S_\alpha)(\nabla p_\alpha - \rho_\alpha \hat{g})],\ \alpha \in \{o, w\}$$
$$S_o + S_w = 1,\ p_o - p_w = P_c(S_w)$$

Equations

- Richards Equation

$$\partial_t S_w = \nabla \cdot [k_w(S_w)(\nabla p - \rho_w \hat{g})], \quad -p = P_c(S_w)$$

- The two-phase porous media equation

$$\partial_t S_\alpha = \nabla \cdot [k_\alpha(S_\alpha)(\nabla p_\alpha - \rho_\alpha \hat{g})], \; \alpha \in \{o, w\}$$
$$S_o + S_w = 1, \; p_o - p_w = P_c(S_w)$$

- Non-equilibrium effects: hysteresis and dynamic capillarity

$$-p \text{ or } p_o - p_w \in P_c(S_w) - \gamma(S_w)\mathrm{sign}(\partial_t S_w) - \mathcal{T}(S_w)\partial_t S_w$$

Equations

- Richards Equation

$$\partial_t S_w = \nabla \cdot [k_w(S_w)(\nabla p - \rho_w \hat{g})], \quad -p = P_c(S_w)$$

- The two-phase porous media equation

$$\partial_t S_\alpha = \nabla \cdot [k_\alpha(S_\alpha)(\nabla p_\alpha - \rho_\alpha \hat{g})], \; \alpha \in \{o, w\}$$
$$S_o + S_w = 1, \; p_o - p_w = P_c(S_w)$$

- Non-equilibrium effects: hysteresis and dynamic capillarity

$$-p \text{ or } p_o - p_w \in P_c(S_w) - \gamma(S_w)\text{sign}(\partial_t S_w) - \mathcal{T}(S_w)\partial_t S_w$$

- Domain decomposition schemes for unsaturated and two-phase cases (Seus *et al.* (2018))

Technische Universiteit **Eindhoven** University of Technology

Time-discrete solutions

Let us talk about the nonlinear advection diffusion equation

$$\partial_t b(u) + \nabla \cdot \mathbf{F}(\mathbf{x}, u) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u)\nabla u] + r(\mathbf{x}, t, u)$$

How do we solve it numerically??

Time-discrete solutions

Let us talk about the nonlinear advection diffusion equation

$$\partial_t b(u) + \nabla \cdot \mathbf{F}(\mathbf{x}, u) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u)\nabla u] + r(\mathbf{x}, t, u)$$

How do we solve it numerically??

- Let $\Omega \in \mathbb{R}^d$ be a Lipschitz domain and $t \in [0, T]$, $T > 0$

Time-discrete solutions

Let us talk about the nonlinear advection diffusion equation

$$\partial_t b(u) + \nabla \cdot \mathbf{F}(\mathbf{x}, u) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u)\nabla u] + r(\mathbf{x}, t, u)$$

How do we solve it numerically??

- Let $\Omega \in \mathbb{R}^d$ be a Lipschitz domain and $t \in [0, T]$, $T > 0$
- For $N \in \mathbb{N}$ define timestep size $\tau = T/N$

Time-discrete solutions

Let us talk about the nonlinear advection diffusion equation

$$\partial_t b(u) + \nabla \cdot \mathbf{F}(\mathbf{x}, u) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u)\nabla u] + r(\mathbf{x}, t, u)$$

How do we solve it numerically??

- Let $\Omega \in \mathbb{R}^d$ be a Lipschitz domain and $t \in [0, T]$, $T > 0$
- For $N \in \mathbb{N}$ define timestep size $\tau = T/N$
- Let $u = u_0$ at $t = 0$ be the initial condition and $u = g$ at $\partial\Omega$ be the boundary condition

Time-discrete solutions

Let us talk about the nonlinear advection diffusion equation

$$\partial_t b(u) + \nabla \cdot \mathbf{F}(\mathbf{x}, u) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u)\nabla u] + r(\mathbf{x}, t, u)$$

How do we solve it numerically??

- Let $\Omega \in \mathbb{R}^d$ be a Lipschitz domain and $t \in [0, T]$, $T > 0$
- For $N \in \mathbb{N}$ define timestep size $\tau = T/N$
- Let $u = u_0$ at $t = 0$ be the initial condition and $u = g$ at $\partial\Omega$ be the boundary condition
- For any $n \in \{1, .., N\}$ use backward Euler scheme for time discretization. This leads to the following system of equation

$$b(u_n) - b(u_{n-1}) + \tau\nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \tau\nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \tau r(\mathbf{x}, n\tau, u_n) \text{ in } \Omega$$

Time-discrete solutions

Let us talk about the nonlinear advection diffusion equation

$$\partial_t b(u) + \nabla \cdot \mathbf{F}(\mathbf{x}, u) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u)\nabla u] + r(\mathbf{x}, t, u)$$

How do we solve it numerically??

- Let $\Omega \in \mathbb{R}^d$ be a Lipschitz domain and $t \in [0, T]$, $T > 0$
- For $N \in \mathbb{N}$ define timestep size $\tau = T/N$
- Let $u = u_0$ at $t = 0$ be the initial condition and $u = g$ at $\partial\Omega$ be the boundary condition
- For any $n \in \{1, .., N\}$ use backward Euler scheme for time discretization. This leads to the following system of equation

  $$b(u_n) - b(u_{n-1}) + \tau\nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \tau\nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \tau r(\mathbf{x}, n\tau, u_n) \text{ in } \Omega$$
- Solve using some linearization technique

The Linearization Techniques

The equation

$$b(u_n) - b(u_{n-1}) + \tau\nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \tau\nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \tau r(\mathbf{x}, n\tau, u_n)$$

for a fixed n is an elliptic equation of the form

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

and consequently can be solved by following iterative linearization techniques

The Linearization Techniques

The equation

$$b(u_n) - b(u_{n-1}) + \tau\nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \tau\nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \tau r(\mathbf{x}, n\tau, u_n)$$

for a fixed n is an elliptic equation of the form

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

and consequently can be solved by following iterative linearization techniques

Local schemes

The equation

$$b(u_n) - b(u_{n-1}) + \tau \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \tau \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n) \nabla u_n] + \tau r(\mathbf{x}, n\tau, u_n)$$

for a fixed n is an elliptic equation of the form

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n) \nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

and consequently can be solved by following iterative linearization techniques

Local schemes

For example

- Newton
- Picard or modified-Picard
- Jäger and Kačur

The Linearization Techniques

The equation

$$b(u_n) - b(u_{n-1}) + \tau\nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \tau\nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \tau r(\mathbf{x}, n\tau, u_n)$$

for a fixed n is an elliptic equation of the form

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

and consequently can be solved by following iterative linearization techniques

Local schemes

For example

- Newton
- Picard or modified-Picard
- Jäger and Kačur

Global schemes

For example

- L-scheme

The Local Schemes

To solve

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

the local schemes use

- Approximation of the nonlinearities using the last iteration
- Generally they converge if the initial guess u_n^0 is close enough to u_n

TU/e Technische Universiteit Eindhoven University of Technology

The Local Schemes

To solve

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

the local schemes use

- Approximation of the nonlinearities using the last iteration
- Generally they converge if the initial guess u_n^0 is close enough to u_n

Sufficient condition for convergence

For the original parabolic problem the schemes converge if $u_n^0 = u_{n-1}$ and

The Local Schemes

To solve

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

the local schemes use

- Approximation of the nonlinearities using the last iteration
- Generally they converge if the initial guess u_n^0 is close enough to u_n

Sufficient condition for convergence

For the original parabolic problem the schemes converge if $u_n^0 = u_{n-1}$ and[a]

$$\tau < Ch^d$$

for some constant $C > 0$ and meshsize h

Technische Universiteit Eindhoven University of Technology

The Local Schemes

To solve

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

the local schemes use

- Approximation of the nonlinearities using the last iteration
- Generally they converge if the initial guess u_n^0 is close enough to u_n

Sufficient condition for convergence

For the original parabolic problem the schemes converge if $u_n^0 = u_{n-1}$ and[a]

$$\tau < Ch^d$$

for some constant $C > 0$ and meshsize h

• A severe restriction: for $d \geq 2$, for processes that involve large time scales or fine mesh-resolution, e.g. reservoir modelling

[a] Radu *et al.* (2006)

The Global Scheme: L-scheme

To solve

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

L-scheme uses iterations

TU/e Technische Universiteit Eindhoven University of Technology

The Global Scheme: L-scheme

To solve

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

L-scheme uses iterations

$$Lu_n^i - \nabla \cdot [\mathcal{D}_n^{i-1}\nabla u_n^i] = Lu_n^{i-1} - \mathcal{B}(u_n^{i-1}) + [-\nabla \cdot \mathbf{F}_n^{i-1} + \mathcal{R}_n^{i-1}]$$

where L is constant[a]

The Global Scheme: L-scheme

To solve

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

L-scheme uses iterations

$$L u_n^i - \nabla \cdot [\mathcal{D}_n^{i-1} \nabla u_n^i] = L u_n^{i-1} - \mathcal{B}(u_n^{i-1}) + [-\nabla \cdot \mathbf{F}_n^{i-1} + \mathcal{R}_n^{i-1}]$$

where L is constant[a]

- If $\mathcal{B}' \geq 0$, L scheme converges unconditionally for $L \geq \frac{1}{2}\max\{\mathcal{B}'\}$

TU/e Technische Universiteit Eindhoven University of Technology

The Global Scheme: L-scheme

To solve

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

L-scheme uses iterations

$$L u_n^i - \nabla \cdot \left[\mathcal{D}_n^{i-1} \nabla u_n^i\right] = L u_n^{i-1} - \mathcal{B}(u_n^{i-1}) + [-\nabla \cdot \mathbf{F}_n^{i-1} + \mathcal{R}_n^{i-1}]$$

where L is constant[a]

- If $\mathcal{B}' \geq 0$, L scheme converges unconditionally for $L \geq \frac{1}{2}\max\{\mathcal{B}'\}$

Theorem

If $\mathcal{B}' \geq 0$; $\partial_u \mathcal{R} \leq 0$; $\mathcal{D}, \mathbf{F}_i \in C^1(\Omega \times \mathbb{R})$; $0 < \mathcal{D}_m \leq \mathcal{D} \leq \mathcal{D}_M$ then there exists a τ_0 and L_0 (independent of meshsize) s.t. for all $\tau < \tau_0$ and $L > L_0$, L-scheme converges linearly in $H^1(\Omega)$ irrespective of the initial guess.

The Global Scheme: L-scheme

To solve

$$\mathcal{B}(u_n) + \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \mathcal{R}(\mathbf{x}, u_n)$$

L-scheme uses iterations

$$L u_n^i - \nabla \cdot [\mathcal{D}_n^{i-1} \nabla u_n^i] = L u_n^{i-1} - \mathcal{B}(u_n^{i-1}) + [-\nabla \cdot \mathbf{F}_n^{i-1} + \mathcal{R}_n^{i-1}]$$

where L is constant[a]

- If $\mathcal{B}' \geq 0$, L scheme converges unconditionally for $L \geq \frac{1}{2}\max\{\mathcal{B}'\}$

Theorem

If $\mathcal{B}' \geq 0$; $\partial_u \mathcal{R} \leq 0$; $\mathcal{D}, \mathbf{F}_i \in C^1(\Omega \times \mathbb{R})$; $0 < \mathcal{D}_m \leq \mathcal{D} \leq \mathcal{D}_M$ then there exists a τ_0 and L_0 (independent of meshsize) s.t. for all $\tau < \tau_0$ and $L > L_0$, L-scheme converges linearly in $H^1(\Omega)$ irrespective of the initial guess.

- The convergence speed is substantially less[b] for $L >> 1$ or τ small

[a] Pop *et al.* (2004)

[b] List and Radu. (2016)

Observation

Observe that

- The local estimates make the schemes faster but less stable

Observation

Observe that

- The local estimates make the schemes faster but less stable
- All the schemes mentioned above are designed for elliptic problems. $u_n^0 = u_{n-1}$ remains relatively unused

Observation

Observe that

- The local estimates make the schemes faster but less stable
- All the schemes mentioned above are designed for elliptic problems. $u_n^0 = u_{n-1}$ remains relatively unused

Questions??

Can we design the $\mathfrak{M}$-scheme in this case so that it is both

TU/e Technische Universiteit Eindhoven University of Technology

Observation

Observe that

- The local estimates make the schemes faster but less stable
- All the schemes mentioned above are designed for elliptic problems. $u_n^0 = u_{n-1}$ remains relatively unused

Questions??

Can we design the $\mathfrak{M}$-scheme in this case so that it is both

- fast

Observation

Observe that

- The local estimates make the schemes faster but less stable
- All the schemes mentioned above are designed for elliptic problems. $u_n^0 = u_{n-1}$ remains relatively unused

Questions??

Can we design the $\mathfrak{M}$-scheme in this case so that it is both

- fast
- and stable

TU/e Technische Universiteit Eindhoven University of Technology

Modified L-scheme

Assumptions:

A1. The associated functions are smooth up to second derivative

Modified L-scheme

Assumptions:

A1. The associated functions are smooth up to second derivative

A2.
- $b' \geq m \geq 0$
- $\partial_u r \leq 0$
- $0 < \mathcal{D}_m \leq \mathcal{D} \leq \mathcal{D}_M$
- $u_0 \in H^1(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$, $u_0 = g$ at $\partial\Omega$

Modified L-scheme

Assumptions:

A1. The associated functions are smooth up to second derivative

A2.
- $b' \geq m \geq 0$
- $\partial_u r \leq 0$
- $0 < \mathcal{D}_m \leq \mathcal{D} \leq \mathcal{D}_M$
- $u_0 \in H^1(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$, $u_0 = g$ at $\partial\Omega$

A3. $\|u_n - u_{n-1}\|_{L^\infty(\Omega)} < \Lambda\tau$ for some $\Lambda > 0$

Modified L-scheme

Assumptions:

A1. The associated functions are smooth up to second derivative

A2.
- $b' \geq m \geq 0$
- $\partial_u r \leq 0$
- $0 < \mathcal{D}_m \leq \mathcal{D} \leq \mathcal{D}_M$
- $u_0 \in H^1(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$, $u_0 = g$ at $\partial\Omega$

A3. $\|u_n - u_{n-1}\|_{L^\infty(\Omega)} < \Lambda\tau$ for some $\Lambda > 0$

- Translates to $\|\partial_t u\|_{L^\infty(\Omega\times(0,T])} < \infty$

Modified L-scheme

Assumptions:

A1. The associated functions are smooth up to second derivative

A2.
- $b' \geq m \geq 0$
- $\partial_u r \leq 0$
- $0 < \mathcal{D}_m \leq \mathcal{D} \leq \mathcal{D}_M$
- $u_0 \in H^1(\Omega)$, $g \in H^{\frac{1}{2}}(\partial\Omega)$, $u_0 = g$ at $\partial\Omega$

A3. $\|u_n - u_{n-1}\|_{L^\infty(\Omega)} < \Lambda\tau$ for some $\Lambda > 0$

- Translates to $\|\partial_t u\|_{L^\infty(\Omega\times(0,T])} < \infty$
- This holds for sufficiently regular domains, ICs and BCs: e.g. if $u_0 \in \mathcal{C}^2(\Omega)$

Modified L-scheme

Consider the equation $\partial_t b(u) - \nabla \cdot (\mathcal{D}(\mathbf{x})\nabla u) = r(\mathbf{x}, t, u)$

Modified L-scheme

Consider the equation $\partial_t b(u) - \nabla \cdot (\mathcal{D}(\mathbf{x})\nabla u) = r(\mathbf{x}, t, u)$ discretized into

$$b(u_n) - b(u_{n-1}) - \tau\nabla \cdot (\mathcal{D}(\mathbf{x})\nabla u_n) = \tau r(\mathbf{x}, n\tau, u_n)$$

Modified L-scheme

Consider the equation $\partial_t b(u) - \nabla \cdot (\mathcal{D}(\mathbf{x})\nabla u) = r(\mathbf{x}, t, u)$ discretized into

$$b(u_n) - b(u_{n-1}) - \tau\nabla \cdot (\mathcal{D}(\mathbf{x})\nabla u_n) = \tau r(\mathbf{x}, n\tau, u_n)$$

We propose the following scheme

$$L_n^i(\mathbf{x})u_n^i - \tau\nabla \cdot (\mathcal{D}\nabla u_n^i) = L_n^i(\mathbf{x})u_n^{i-1} - (b(u_n^{i-1}) - b(u_{n-1})) + \tau r(\mathbf{x}, n\tau, u_n^i)$$

Technische Universiteit Eindhoven University of Technology

Modified L-scheme

Consider the equation $\partial_t b(u) - \nabla \cdot (\mathcal{D}(\mathbf{x})\nabla u) = r(\mathbf{x}, t, u)$ discretized into

$$b(u_n) - b(u_{n-1}) - \tau\nabla \cdot (\mathcal{D}(\mathbf{x})\nabla u_n) = \tau r(\mathbf{x}, n\tau, u_n)$$

We propose the following scheme

$$L_n^i(\mathbf{x})u_n^i - \tau\nabla \cdot (\mathcal{D}\nabla u_n^i) = L_n^i(\mathbf{x})u_n^{i-1} - (b(u_n^{i-1}) - b(u_{n-1})) + \tau r(\mathbf{x}, n\tau, u_n^i)$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(\mathbf{x}, n\tau, u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Modified L-scheme

Consider the equation $\partial_t b(u) - \nabla \cdot (\mathcal{D}(\mathbf{x})\nabla u) = r(\mathbf{x}, t, u)$ discretized into

$$b(u_n) - b(u_{n-1}) - \tau\nabla \cdot (\mathcal{D}(\mathbf{x})\nabla u_n) = \tau r(\mathbf{x}, n\tau, u_n)$$

We propose the following scheme

$$L_n^i(\mathbf{x})u_n^i - \tau\nabla \cdot (\mathcal{D}\nabla u_n^i) = L_n^i(\mathbf{x})u_n^{i-1} - (b(u_n^{i-1}) - b(u_{n-1})) + \tau r(\mathbf{x}, n\tau, u_n^i)$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(\mathbf{x}, n\tau, u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Lemma 1.1 With $u_n^0 = u_{n-1}$ and (A1)-(A3)

$$\|u_n^i - u_n\|_{L^\infty(\Omega)} < \Lambda\tau$$

for all $i \in \mathbb{N}$

Modified L-scheme

The scheme

$$L_n^i(\mathbf{x})u_n^i - \tau\nabla\cdot(\mathcal{D}\nabla u_n^i) = L_n^i(\mathbf{x})u_n^{i-1} - (b(u_n^{i-1}) - b(u_{n-1})) + \tau r(\mathbf{x}, n\tau, u_n^i)$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(\mathbf{x}, n\tau, u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Theorem 1.1 With $u_n^0 = u_{n-1}$ and $\mathfrak{M} > \mathfrak{M}_0 = \Lambda \max_{u\in\mathbb{R}}\{|b''| + \tau|\partial_{uu} r|\}$

Modified L-scheme

The scheme

$$L_n^i(\mathbf{x})u_n^i - \tau\nabla\cdot(\mathcal{D}\nabla u_n^i) = L_n^i(\mathbf{x})u_n^{i-1} - (b(u_n^{i-1}) - b(u_{n-1})) + \tau r(\mathbf{x}, n\tau, u_n^i)$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(\mathbf{x}, n\tau, u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Theorem 1.1 With $u_n^0 = u_{n-1}$ and $\mathfrak{M} > \mathfrak{M}_0 = \Lambda \max_{u\in\mathbb{R}}\{|b''| + \tau|\partial_{uu} r|\}$

The $\mathfrak{M}$-scheme converges linearly in $H^1(\Omega)\cap L^\infty(\Omega)$ for all $\tau > 0$, $m \geq 0$ with convergence rate

$$\alpha = \sup \frac{\|u_n^i - u_n\|_\chi}{\|u_n^{i-1} - u_n\|_\chi} \leq \sqrt{\frac{2\mathfrak{M}}{2\mathfrak{M} + C_\Omega \mathcal{D}_m}}, \quad \chi \in \{H^1(\Omega), L^\infty(\Omega)\}$$

Modified L-scheme

The scheme

$$L_n^i(\mathbf{x})u_n^i - \tau\nabla\cdot(\mathcal{D}\nabla u_n^i) = L_n^i(\mathbf{x})u_n^{i-1} - (b(u_n^{i-1}) - b(u_{n-1})) + \tau r(\mathbf{x}, n\tau, u_n^i)$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(\mathbf{x}, n\tau, u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Theorem 1.1 With $u_n^0 = u_{n-1}$ and $\mathfrak{M} > \mathfrak{M}_0 = \Lambda \max_{u\in\mathbb{R}}\{|b''| + \tau|\partial_{uu}r|\}$

- The $\mathfrak{M}$-scheme converges linearly in $H^1(\Omega)\cap L^\infty(\Omega)$ for all $\tau > 0$, $m \geq 0$ with convergence rate

$$\alpha = \sup \frac{\|u_n^i - u_n\|_\chi}{\|u_n^{i-1} - u_n\|_\chi} \leq \sqrt{\frac{2\mathfrak{M}}{2\mathfrak{M} + C_\Omega \mathcal{D}_m}}, \quad \chi \in \{H^1(\Omega), L^\infty(\Omega)\}$$

- If $m > 0$ and $\tau < \tau_0 = \frac{m}{2\mathfrak{M}}$ then the convergence rate is $\mathcal{O}(\tau)$

General Problem

Time-discrete equation

$$b(u_n) - b(u_{n-1}) + \tau \nabla \cdot \mathbf{F}(\mathbf{x}, u_n) = \tau \nabla \cdot [\mathcal{D}(\mathbf{x}, u_n) \nabla u_n] + \tau r(\mathbf{x}, n\tau, u_n)$$

General Problem

Time-discrete equation

$$b(u_n) - b(u_{n-1}) + \tau\nabla\cdot\mathbf{F}(\mathbf{x}, u_n) = \tau\nabla\cdot[\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \tau r(\mathbf{x}, n\tau, u_n)$$

The scheme reads

$$L_n^i(u_n^i - u_{n-1}^i) - \tau\nabla\cdot\left(\mathcal{D}_n^{i-1}\nabla u_n^i\right) = -(b(u_n^{i-1}) - b(u_{n-1})) + \tau[r_n^{i-1} - \nabla\cdot\mathbf{F}_n^{i-1}]$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Technische Universiteit Eindhoven University of Technology

General Problem

Time-discrete equation

$$b(u_n) - b(u_{n-1}) + \tau\nabla\cdot\mathbf{F}(\mathbf{x},u_n) = \tau\nabla\cdot[\mathcal{D}(\mathbf{x},u_n)\nabla u_n] + \tau r(\mathbf{x},n\tau,u_n)$$

The scheme reads

$$L_n^i(u_n^i - u_{n-1}^i) - \tau\nabla\cdot(\mathcal{D}_n^{i-1}\nabla u_n^i) = -(b(u_n^{i-1}) - b(u_{n-1})) + \tau[r_n^{i-1} - \nabla\cdot\mathbf{F}_n^{i-1}]$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Assumptions:

A4. $\|\nabla u_n\|_{L^\infty(\Omega)} \leq \Lambda_1$ for some $\Lambda_1 > 0$

General Problem

Time-discrete equation

$$b(u_n) - b(u_{n-1}) + \tau\nabla\cdot\mathbf{F}(\mathbf{x}, u_n) = \tau\nabla\cdot[\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \tau r(\mathbf{x}, n\tau, u_n)$$

The scheme reads

$$L_n^i(u_n^i - u_{n-1}^i) - \tau\nabla\cdot\left(\mathcal{D}_n^{i-1}\nabla u_n^i\right) = -(b(u_n^{i-1}) - b(u_{n-1})) + \tau[r_n^{i-1} - \nabla\cdot\mathbf{F}_n^{i-1}]$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Assumptions:

A4. $\|\nabla u_n\|_{L^\infty(\Omega)} \leq \Lambda_1$ for some $\Lambda_1 > 0$

- Required also for proving convergence of L-scheme

Technische Universiteit Eindhoven University of Technology

General Problem

Time-discrete equation

$$b(u_n) - b(u_{n-1}) + \tau\nabla\cdot\mathbf{F}(\mathbf{x}, u_n) = \tau\nabla\cdot[\mathcal{D}(\mathbf{x}, u_n)\nabla u_n] + \tau r(\mathbf{x}, n\tau, u_n)$$

The scheme reads

$$L_n^i(u_n^i - u_{n-1}^i) - \tau\nabla\cdot(\mathcal{D}_n^{i-1}\nabla u_n^i) = -(b(u_n^{i-1}) - b(u_{n-1})) + \tau[r_n^{i-1} - \nabla\cdot\mathbf{F}_n^{i-1}]$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Assumptions:

A4. $\|\nabla u_n\|_{L^\infty(\Omega)} \leq \Lambda_1$ for some $\Lambda_1 > 0$

- Required also for proving convergence of L-scheme
- Holds if $u_0 \in W^{2,2q}(\Omega),\ q \in \mathbb{N},\ 2q > d$

General Problem

$$L_n^i(u_n^i - u_{n-1}^i) - \tau\nabla\cdot\left(\mathcal{D}_n^{i-1}\nabla u_n^i\right) = -(b(u_n^{i-1}) - b(u_{n-1})) + \tau[r_n^{i-1} - \nabla\cdot\mathbf{F}_n^{i-1}]$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Theorem 2.1

For $u_n^0 = u_{n-1}$, $\mathfrak{M} > \mathfrak{M}_0$ and $\tau < \tau_0$ assume (A1)-(A4)*. Then

General Problem

$$L_n^i(u_n^i - u_{n-1}^i) - \tau\nabla\cdot(\mathcal{D}_n^{i-1}\nabla u_n^i) = -(b(u_n^{i-1}) - b(u_{n-1})) + \tau[r_n^{i-1} - \nabla\cdot\mathbf{F}_n^{i-1}]$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Theorem 2.1

For $u_n^0 = u_{n-1}$, $\mathfrak{M} > \mathfrak{M}_0$ and $\tau < \tau_0$ assume (A1)-(A4)*. Then

The $\mathfrak{M}$-scheme converges in $H^1(\Omega)$ for all $m \geq 0$

General Problem

$$L_n^i(u_n^i - u_{n-1}^i) - \tau\nabla\cdot(\mathcal{D}_n^{i-1}\nabla u_n^i) = -(b(u_n^{i-1}) - b(u_{n-1})) + \tau[r_n^{i-1} - \nabla\cdot\mathbf{F}_n^{i-1}]$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Theorem 2.1

For $u_n^0 = u_{n-1}$, $\mathfrak{M} > \mathfrak{M}_0$ and $\tau < \tau_0$ assume (A1)-(A4)*. Then

- The $\mathfrak{M}$-scheme converges in $H^1(\Omega)$ for all $m \geq 0$
- The $\mathfrak{M}$-scheme converges linearly in $H^1(\Omega)$ if $m > 0$

General Problem

$$L_n^i(u_n^i - u_{n-1}^i) - \tau\nabla\cdot(\mathcal{D}_n^{i-1}\nabla u_n^i) = -(b(u_n^{i-1}) - b(u_{n-1})) + \tau[r_n^{i-1} - \nabla\cdot\mathbf{F}_n^{i-1}]$$

with

$$L_n^i := \max([b'(u_n^{i-1}) - \tau\partial_u r(u_n^{i-1}) + \mathfrak{M}\tau], 2\mathfrak{M}\tau)$$

Theorem 2.1

For $u_n^0 = u_{n-1}$, $\mathfrak{M} > \mathfrak{M}_0$ and $\tau < \tau_0$ assume (A1)-(A4)*. Then

- The $\mathfrak{M}$-scheme converges in $H^1(\Omega)$ for all $m \geq 0$
- The $\mathfrak{M}$-scheme converges linearly in $H^1(\Omega)$ if $m > 0$
- For $m > 0$ the convergence rate is $\alpha = \mathcal{O}(\sqrt{\tau})$

Technische Universiteit Eindhoven University of Technology

Numerical Study

Richards equation in 2-D

Numerical Study

Richards equation in 2-D

$$\partial_t S_w(p) = \nabla \cdot [k_w(S_w(p))(\nabla p - \rho_w \hat{g})] + f \text{ on } (0,1) \times (0,1)$$

Numerical Study

Richards equation in 2-D

$$\partial_t S_w(p) = \nabla \cdot [k_w(S_w(p))(\nabla p - \rho_w \hat{g})] + f \text{ on } (0,1) \times (0,1)$$

Take van Genuchten parameters[a]: for $m = \frac{2}{3}, n = \frac{1}{1-m}$

$$S_w(p) = \begin{cases} \dfrac{1}{(1+(-p)^n)^m} & \text{if } p < 0 \\ 1 & \text{if } p \geq 0 \end{cases}$$

$$k_w(S) = \sqrt{S}(1-(1-S^{\frac{1}{m}})^m)^2$$

Richards equation in 2-D

$$\partial_t S_w(p) = \nabla \cdot [k_w(S_w(p))(\nabla p - \rho_w \hat{g})] + f \text{ on } (0,1)\times(0,1)$$

Take van Genuchten parameters[a]: for $m = \frac{2}{3}, n = \frac{1}{1-m}$

$$S_w(p) = \begin{cases} \dfrac{1}{(1+(-p)^n)^m} & \text{if } p < 0 \\ 1 & \text{if } p \geq 0 \end{cases}$$

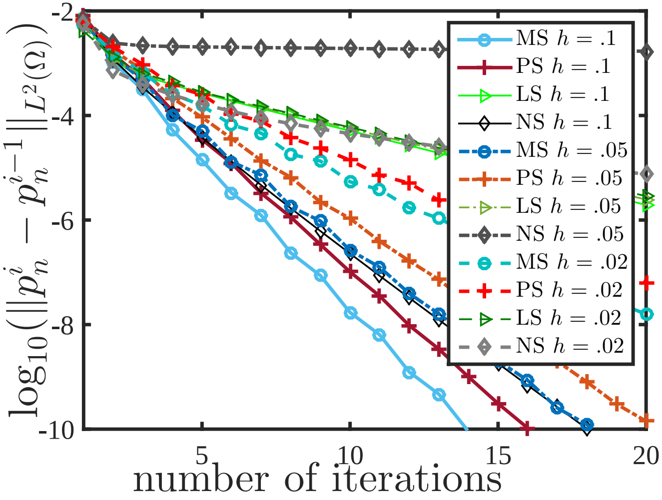
$$k_w(S) = \sqrt{S}(1-(1-S^{\frac{1}{m}})^m)^2$$

Assumed initial and boundary conditions with $\tilde{p}(x,y,t) = 1-(1+t^2)(1+x^2+y^2)$,

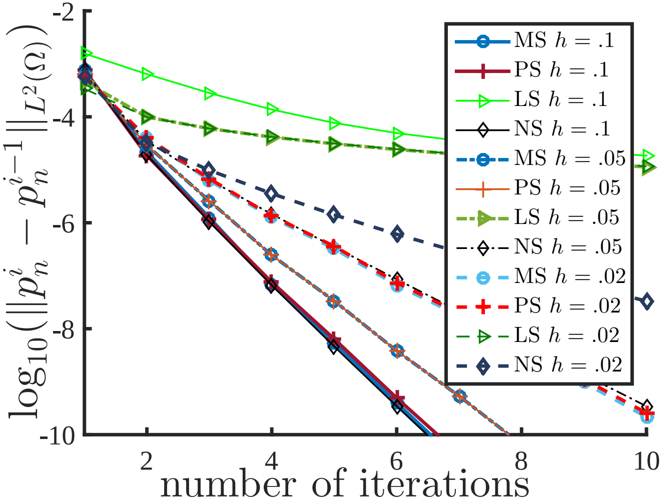
IC	$t = 0$	$p(x,y,0) = \tilde{p}(x,y,0)$	on Ω	
BC	$x = 0:$	$p(0,y,t) = \tilde{p}(0,y,t),$	$x = 1:$	$p(1,y,t) = \tilde{p}(1,y,t),$
	$y = 0:$	$\partial_y p = 0,$	$y = 1:$	$k(S(p))\partial_y p = k(S(\tilde{p}(x,1,t))\partial_y \tilde{p}(x,1,t).$

[a] van Genuchten. (1980)

- For $t = 0.5$, $\mathfrak{M} = 10$, $L = 1$



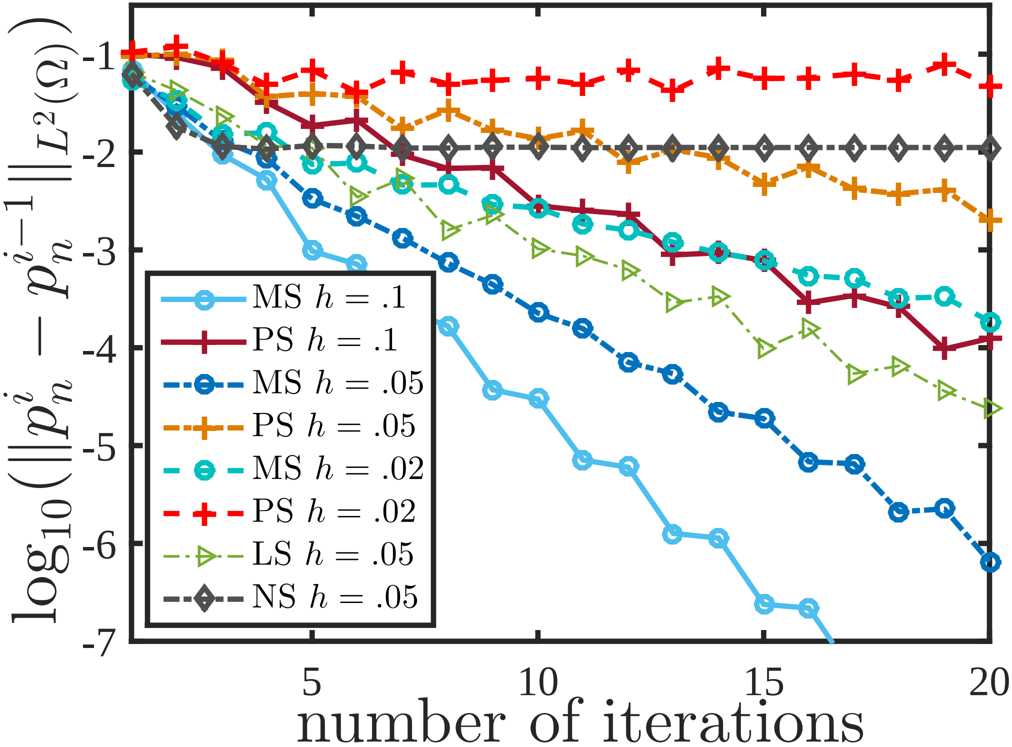
$\tau = 0.01$



$\tau = 0.001$

Mesh Study

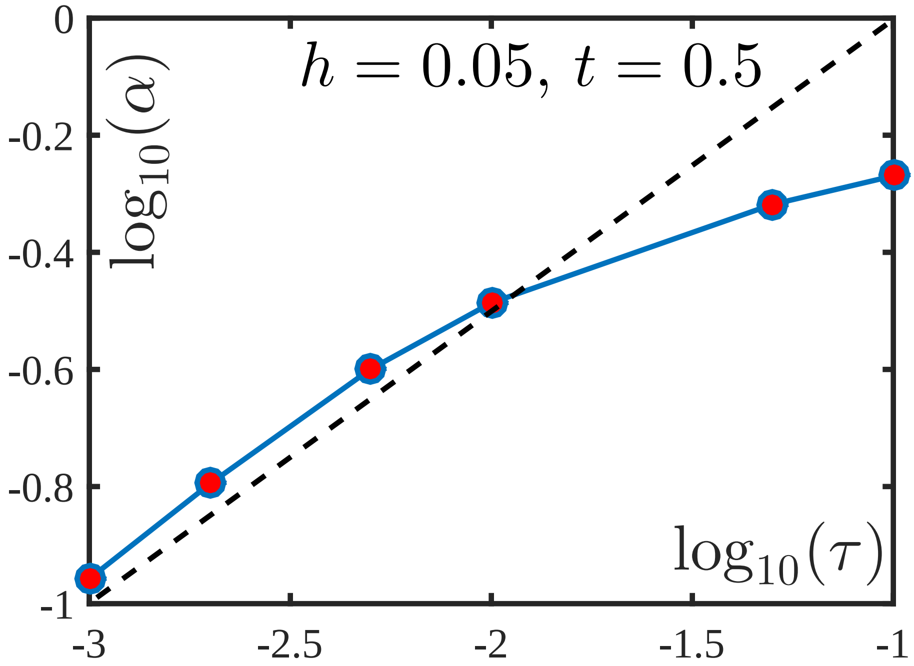
- For $t = 0.5$, $\mathfrak{M} = 10$, $L = 1$



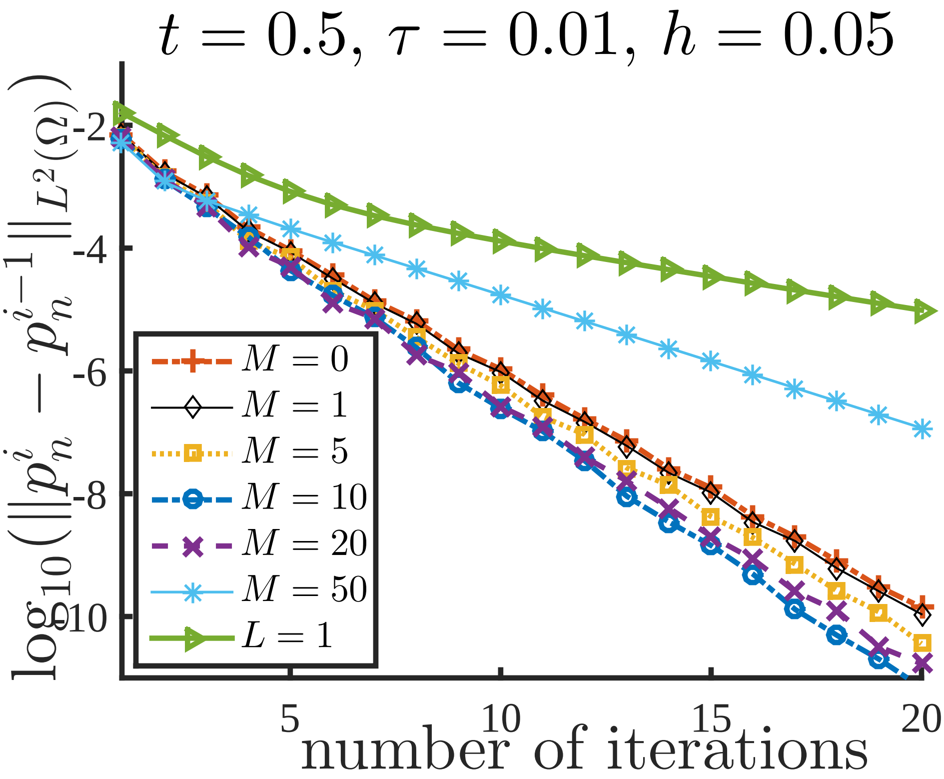
$\tau = 0.1$

Timestep Study

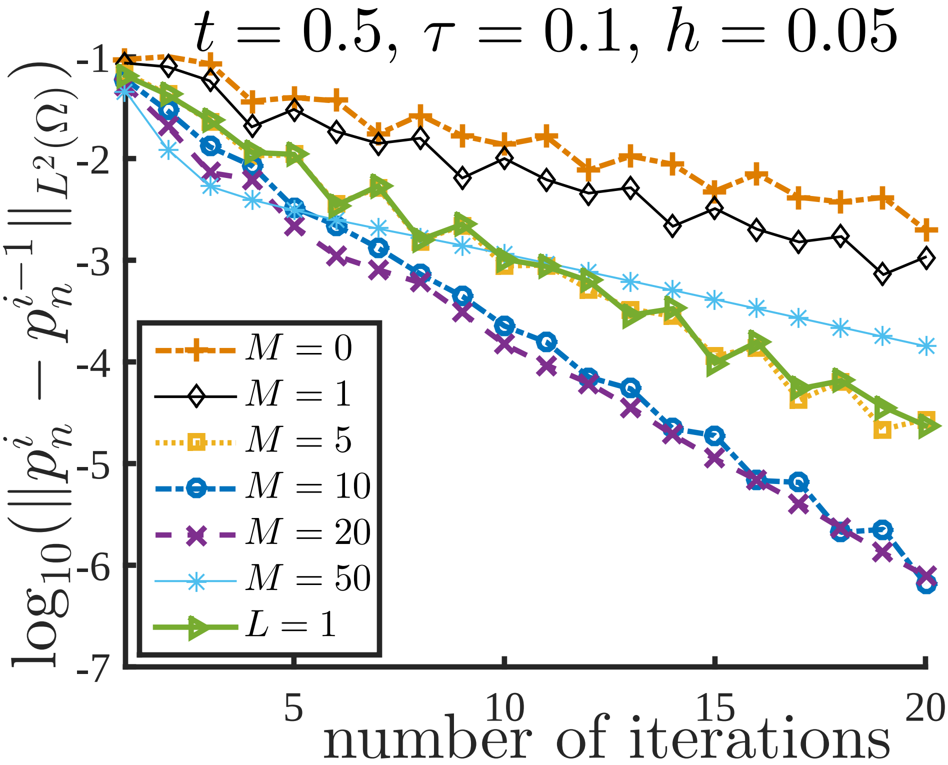
- For $t = .5$, $h = 0.05$, $\mathfrak{M} = 10$



Effect of $\mathfrak{M}$



Effect of $\mathfrak{M}$



Technische Universiteit Eindhoven University of Technology

For details

see

Mitra, K. & Pop, I. S. (2018). A modified L-scheme for nonlinear parabolic equations. *Computers & Mathematics With Applications*.

TU/e Technische Universiteit Eindhoven University of Technology

Other Problems

Two Phase Equation: The $\mathfrak{M}$-scheme given as

$$-(S^i_{w,n} - S_{w,n-1}) = \tau\nabla\cdot[k_o(1-S^{i-1}_{w,n})(\nabla p^i_{o,n} - \rho_o\hat{g})]$$
$$(S^i_{w,n} - S_{w,n-1}) = \tau\nabla\cdot[k_w(S^{i-1}_{w,n})(\nabla p^i_{w,n} - \rho_w\hat{g})]$$
$$p^i_{o,n} - p^i_{w,n} = P_c(S^{i-1}_{w,n}) - L^i_n(S^i_{w,n} - S^{i-1}_{w,n})$$

with $L^i_n := -P_c'(S^{i-1}_{w,n}) + \mathfrak{M}\tau$

Theorem 3.1 With $(p^0_{o,n}, p^0_{w,n}) = (p_{o,n-1}, p_{w,n-1})$ define

$$e^i_n = \|p^i_{w,n} - p_{w,n}\|_{H^1(\Omega)} + \|p^i_{o,n} - p_{o,n}\|_{H^1(\Omega)} + \|S^i_{w,n} - S_{w,n}\|_{L^2(\Omega)}.$$

Assume for $i\in\mathbb{N}$, $p_n \in W^{1,\infty}(\Omega)$ and $\|S^i_n - S_n\|_{L^\infty(\Omega)} < \Lambda\tau$ for some $\Lambda > 0$. Then $e^i_n \to 0$ as $i\to\infty$ for τ small enough and $\mathfrak{M}$ large enough. Moreover, if $P_c'(S) < 0$ and $P_c \in C^2(\mathbb{R})$ then for small enough τ

$$\frac{e^i_n}{e^{i-1}_n} = \mathcal{O}(\sqrt{\tau}).$$

Other Problems

Capillary Hysteresis (play-type) and Dynamic Capillarity

Richards equation: $\partial_t S_w = \nabla \cdot [k(S_w)(\nabla p - \rho_w \hat{g})]$

Other Problems

Capillary Hysteresis (play-type) and Dynamic Capillarity

Richards equation: $\partial_t S_w = \nabla \cdot [k(S_w)(\nabla p - \rho_w \hat{g})]$

Closure relation: $-p = P_c(S_w) - \gamma(S_w)\text{sign}(\partial_t S_w) - \mathcal{T}(S_w)\partial_t S_w$

Other Problems

Capillary Hysteresis (play-type) and Dynamic Capillarity

Richards equation: $\partial_t S_w = \nabla \cdot [k(S_w)(\nabla p - \rho_w \hat{g})]$

Closure relation: $-p = P_c(S_w) - \gamma(S_w)\text{sign}(\partial_t S_w) - \mathcal{T}(S_w)\partial_t S_w$

This can be simplified to

$$\partial_t S_w = \mathcal{F}(S_w, p) := \frac{1}{\mathcal{T}(S_w)} \begin{cases} P_c(S_w) - \gamma(S_w) + p & \text{if } p < P_c(S_w) - \gamma(S_w) \\ 0 & \text{if } p \in [P_c - \gamma, P_c + \gamma](S_w) \\ P_c(S_w) + \gamma(S_w) + p & \text{if } p > P_c(S_w) + \gamma(S_w) \end{cases}$$

Other Problems

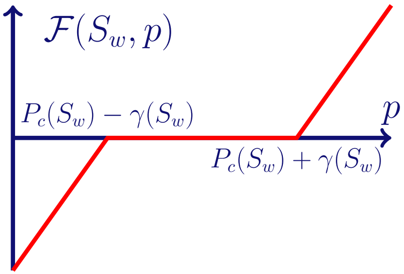
Capillary Hysteresis (play-type) and Dynamic Capillarity

Richards equation: $\partial_t S_w = \nabla \cdot [k(S_w)(\nabla p - \rho_w \hat{g})]$

Closure relation: $-p = P_c(S_w) - \gamma(S_w)\mathrm{sign}(\partial_t S_w) - \mathcal{T}(S_w)\partial_t S_w$

This can be simplified to

$$\partial_t S_w = \mathcal{F}(S_w, p) := \frac{1}{\mathcal{T}(S_w)} \begin{cases} P_c(S_w) - \gamma(S_w) + p & \text{if } p < P_c(S_w) - \gamma(S_w) \\ 0 & \text{if } p \in [P_c - \gamma, P_c + \gamma](S_w) \\ P_c(S_w) + \gamma(S_w) + p & \text{if } p > P_c(S_w) + \gamma(S_w) \end{cases}$$



Other Problems

Time-discrete version

S-equation: $S_{w,n} = S_{w,n-1} + \tau \mathcal{F}(S_{w,n}, p_n)$

p-equation: $\nabla \cdot [k(S_{w,n})(\nabla p_n - 1)] = \mathcal{F}(S_{w,n}, p_n)$

Other Problems

Time-discrete version

S-equation: $S_{w,n} = S_{w,n-1} + \tau \mathcal{F}(S_{w,n}, p_n)$

p-equation: $\nabla \cdot [k(S_{w,n})(\nabla p_n - 1)] = \mathcal{F}(S_{w,n}, p_n)$

Solution strategy: $\mathfrak{M}$-scheme

Update: $S^i_{w,n} = S_{w,n-1} + \tau \mathcal{F}(S^{i-1}_{w,n}, p^{i-1}_n)$

Solve: $L^i_n p^i_n - \nabla \cdot [k(S^i_{w,n})(\nabla p^i_n - 1)] = L^i_n p^{i-1}_n - \mathcal{F}(S^i_{w,n}, p^{i-1}_n)$

With $L^i_n := \partial_p \mathcal{F}(S^i_{w,n}, p^{i-1}_n) + \mathfrak{M}\tau$

Time-discrete version

S-equation: $S_{w,n} = S_{w,n-1} + \tau \mathcal{F}(S_{w,n}, p_n)$

p-equation: $\nabla \cdot [k(S_{w,n})(\nabla p_n - 1)] = \mathcal{F}(S_{w,n}, p_n)$

Solution strategy: $\mathfrak{M}$-scheme

Update: $S^i_{w,n} = S_{w,n-1} + \tau \mathcal{F}(S^{i-1}_{w,n}, p^{i-1}_n)$

Solve: $L^i_n p^i_n - \nabla \cdot [k(S^i_{w,n})(\nabla p^i_n - 1)] = L^i_n p^{i-1}_n - \mathcal{F}(S^i_{w,n}, p^{i-1}_n)$

With $L^i_n := \partial_p \mathcal{F}(S^i_{w,n}, p^{i-1}_n) + \mathfrak{M}\tau$

Theorem 4.1 For small enough τ, large enough $\mathfrak{M}$, $p_n \in W^{1,\infty}(\Omega)$, there exists a $\alpha = \mathcal{O}(\tau/\mathcal{T})$ such that

$$\|S^i_{w,n} - S_{w,n}\|_{W^{1,\infty}} + \|p^i_n - p_n\|_{W^{1,\infty}} \leq \alpha[\|S^{i-1}_{w,n} - S_{w,n}\|_{W^{1,\infty}} + \|p^{i-1}_n - p_n\|_{W^{1,\infty}}]$$

Time-discrete version

S-equation: $S_{w,n} = S_{w,n-1} + \tau \mathcal{F}(S_{w,n}, p_n)$

p-equation: $\nabla \cdot [k(S_{w,n})(\nabla p_n - 1)] = \mathcal{F}(S_{w,n}, p_n)$

Solution strategy: $\mathfrak{M}$-scheme

Update: $S^i_{w,n} = S_{w,n-1} + \tau \mathcal{F}(S^{i-1}_{w,n}, p^{i-1}_n)$

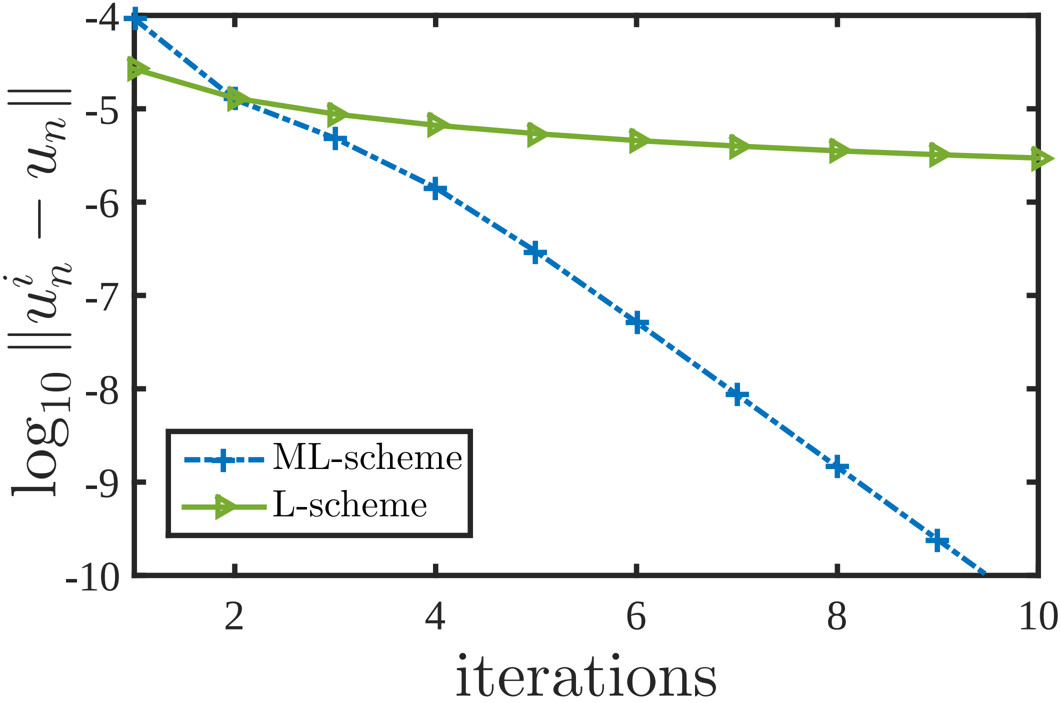
Solve: $L^i_n p^i_n - \nabla \cdot [k(S^i_{w,n})(\nabla p^i_n - 1)] = L^i_n p^{i-1}_n - \mathcal{F}(S^i_{w,n}, p^{i-1}_n)$

With $L^i_n := \partial_p \mathcal{F}(S^i_{w,n}, p^{i-1}_n) + \mathfrak{M}\tau$

Theorem 4.2 If $\mathcal{T} > 0$ then there exists $\hat{\tau} > 0$ independent of $\mathcal{T}$ such that for $\tau < \hat{\tau}$ and large enough $\mathfrak{M}$, $(S^i_{w,n}, p^i_n)$ converges in $H^1(\Omega)$

Numerical Results

- For[a] $\mathcal{T} = .1$, $L = 100$, $\mathfrak{M} = 1$, $h = .1$, $\tau = .001$, $t = 10$



$\mathfrak{M}$-scheme $\alpha \approx .15$

[a] van Duijn, Mitra and Pop. (2018)

Summary

- The linearization schemes are faster if local estimations are taken but are less stable

Summary

- The linearization schemes are faster if local estimations are taken but are less stable
- Using $u_n^0 = u_{n-1}$ and local estimations one can have a scheme that has the following properties

Summary

- The linearization schemes are faster if local estimations are taken but are less stable
- Using $u_n^0 = u_{n-1}$ and local estimations one can have a scheme that has the following properties
  - It is simple and converges unconditionally for small enough timestep sizes independent of meshsize

Summary

- The linearization schemes are faster if local estimations are taken but are less stable
- Using $u_n^0 = u_{n-1}$ and local estimations one can have a scheme that has the following properties
  - It is simple and converges unconditionally for small enough timestep sizes independent of meshsize
  - The convergence rate improves as the timestep size decreases

Summary

- The linearization schemes are faster if local estimations are taken but are less stable
- Using $u_n^0 = u_{n-1}$ and local estimations one can have a scheme that has the following properties
  - It is simple and converges unconditionally for small enough timestep sizes independent of meshsize
  - The convergence rate improves as the timestep size decreases
- The ideas were validated with numerical experiments

Summary

- The linearization schemes are faster if local estimations are taken but are less stable
- Using $u_n^0 = u_{n-1}$ and local estimations one can have a scheme that has the following properties
  - It is simple and converges unconditionally for small enough timestep sizes independent of meshsize
  - The convergence rate improves as the timestep size decreases
- The ideas were validated with numerical experiments
- The ideas are extended to pseudo parabolic equations

TU/e Technische Universiteit Eindhoven University of Technology

References

- W. Jäger and J. Kačur. Solution of doubly nonlinear and degenerate parabolic problems by relaxation schemes. ESAIM: Mathematical Modelling and Numerical Analysis, 29(5), 1995.
- F. List and F.A. Radu. A study on iterative methods for solving richards' equation. Computational Geosciences, 20(2), 2004.
- I. S. Pop, F. Radu, and P. Knabner. Mixed finite elements for the Richards equation: linearization procedure. Journal of Computational and Applied Mathematics, 168(1), 2004.
- F.A. Radu, I.S. Pop, and P. Knabner. Newton-type methods for the mixed finite element discretization of some degenerate parabolic equations. In Numerical Mathematics and Advanced Applications, 2006.
- M. Celia, E.T. Bouloutas, and R. L. Zarba. General mass-conservative numerical solution for the unsaturated ow equation. Water Resources Research, 26(7), 1990.
- D. Seus, K. Mitra, I.S. Pop, F. Radu, and C. Rohde. A linear domain decomposition method for partially saturated flow in porous media. arXiv preprint arXiv:1708.03224, 2017.
- F. Lehmann and P.H. Ackerer. Comparison of iterative methods for improved solutions of the fluid flow equation in partially saturated porous media. Transport in Porous Media, 31(3), 1998.
- L. Bergamaschi and M. Putti. Mixed finite elements and newton-type linearizations for the solution of Richards' equation. International Journal for Numerical Methods in Engineering, 45(8), 1999.

Thanks to

NWO

FOM

UHASSELT

and Thank You for listening