Stability of Periodic and Quasiperiodic Traveling Wave Solutions

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Joint work with
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Introduction

- The generalized Korteweg de-Vries (gKdV) equation is given by

$$u_t = u_{xxx} + (f(u))_x$$

for some "nice" nonlinearity f. Some examples:

  - Surface Waves: $f(u) = u^2$
  - Internal Waves: $f(u) = \alpha u^3 + \beta u^2$
  - Plasmas: $f(u) = u^{r+\frac{1}{2}} \qquad r \geq 0.$

- Interested in the stability of traveling wave solutions of form $u(x,t) = u(x+ct)$ with wave-speed $c > 0$.
- Describes *stationary* solutions in the traveling coordinate system $\xi = x + ct$.

Introduction

Profile of traveling wave satisfies

$$u_{xxx} + f(u)_x - cu_x = 0.$$

Integrating twice gives the nonlinear oscillator:

$$\frac{1}{2}u_x^2 = E + au + cu^2/2 - F(u)$$
$$\frac{du}{\sqrt{2(E + au + cu^2/2 - F(u))}} = dx$$

with a, E constants of integration, c wavespeed and F the antiderivative of f.

Alternative Point of View

Alternative Point of View Two conserved Quantities mass and momentum (M, P) plus spatial period (T)

$$T = \int dx$$
$$M = \int u dx$$
$$P = \frac{1}{2} \int u^2 dx$$

Solitary wave equation equivalent to

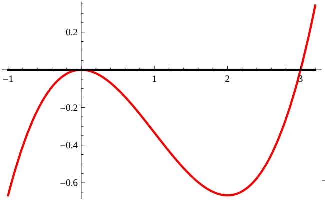
$$\partial_u H + a\partial_u M + c\partial_u P + E\partial_u T = 0$$

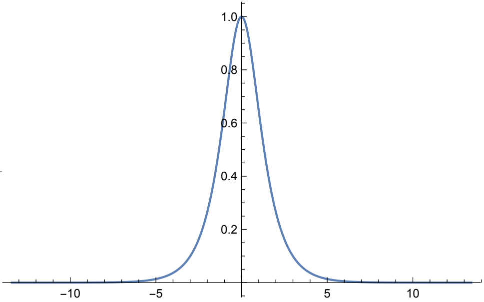
- c is a Lagrange multiplier enforcing constraint $P =$ constant.
- a is a Lagrange multiplier enforcing constraint $M =$ constant.
- Morally E is a Lagrange multiplier enforcing constraint $T =$ constant.

Introduction

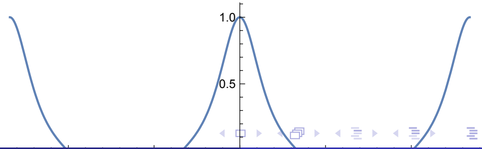
Exists two classes of bounded solutions to traveling wave ODE:

(1) Asymptotically constant (solitary wave solutions):





(2) Periodic (periodic traveling wave solutions):



Example: (Critical) KdV-4 ($f(u) = u^5$)

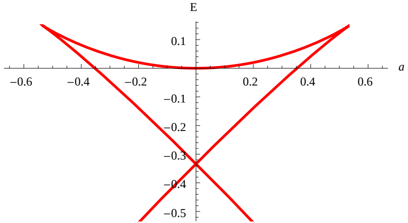
The effective nonlinear oscillator is given by

$$\frac{u_x^2}{2} = E + au + c\frac{u^2}{2} - \frac{u^6}{6}$$

The discriminant of this sixth degree polynomial is

$$\Delta_{KdV-4} = -48a^2 - 3125a^6 + 11250a^4E - 10800a^2E^2 + 1728E^3 + 7776E^5$$

The zero set of which gives the familiar swallowtail cusp:



Identities

Define the classical action for the traveling wave

$$K = \int p dq = \int \sqrt{2(E + au + cu^2/2 + F(u))} dx$$

This is a generating function for the conserved quantities

$$T = \frac{\partial K}{\partial E}$$
$$M = \frac{\partial K}{\partial a}$$
$$P = \frac{\partial K}{\partial c}$$

These are the Maxwell relations from thermodynamics. They hold **VERY** generally.

Goals:

- Study stability of solutions to periodic as well as long wave-length perturbations
- Develop geometric criteria for understanding instability.
- As motivation, we briefly recall the stability theory of solitary wave solutions of gKdV.

Solitary Wave Stability

- Recall traveling wave solutions satisfy
$$\frac{1}{2}u_x^2 = E - F(u) + \frac{c}{2}u^2 + au$$
Up to translation, gKdV admits a three parameter family of bounded solitary wave solutions of the form
$$u(x,t) = u_c(x+ct), \quad c > 0.$$

- gKdV admits three conserved quantities:
$$T = \int dx$$
$$M = \int u dx$$
$$P = \int u^2 dx$$
Solitary wave *one-parameter* $(E = 0, a = 0)$ submanifold.

Solitary Wave Stability

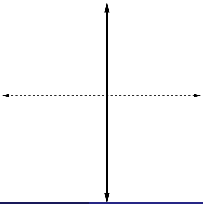
Theorem (Benjamin, Bona, Grillakis-Shatah-Strauss, Bona-Souganidis-Strauss, Pego-Weinstein,...)

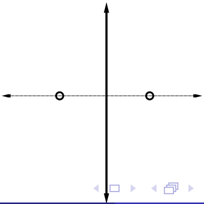
Let u be a solitary wave solution of gKdV of wave speed $c_0 > 0$. *Then u is orbitally stable if*

$$\frac{\partial}{\partial c} P(c)\Big|_{c=c_0} > 0$$

and spectrally unstable if

$$\frac{\partial}{\partial c} P(c)\Big|_{c=c_0} < 0.$$





Facts: Periodic Stability Problem

Linearized spectral problem takes the form

$$\partial_x \mathcal{L} v = \mu v$$

With the operator $\mathcal{L}$ a periodic Schrödinger operator.

- Determining essential spectrum hard part of problem.
- Behavior near the origin (in spectral plane) can be computed analytically (Whitham Theory).
  - Third order operator - Three parameter family of periodic waves.
  - Basis to tangent space of manifold of traveling waves generates (generalized) kernel of $\partial_x \mathcal{L}$
- Spectral information near origin related to geometric information about underlying classical mechanics

Periodic (Spectral) Stability Theory

Recall traveling waves are reducible to quadrature:

$$\frac{1}{2}u_x^2 = E + au + \frac{c}{2}u^2 - F(u).$$

Thus, (up to translation) $\exists$ three parameter family of periodic traveling wave solutions of gKdV

$$u(x; a, E, c), \quad \text{period} \quad T = T(a, E, c)$$

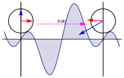
Conserved quantities:

$$T(a, E, c) = \int_0^T dx = \oint \frac{du}{\sqrt{E + au + cu^2/2 - F(u)}},$$

$$M(a, E, c) = \int_0^T u(x; a, E, c)dx = \oint \frac{udu}{\sqrt{E + au + cu^2/2 - F(u)}},$$

$$P(a, E, c) = \int_0^T u(x; a, E, c)^2 dx = \oint \frac{u^2 du}{\sqrt{E + au + cu^2/2 - F(u)}}$$

Periodic Stability Theory: Some results



- Given the monodromy map $\mathbf{M}(\mu)$ define the periodic Evans function:

$$D(\mu,\kappa) = \det\left(\mathbf{M}(\mu) - e^{i\kappa}\mathbf{I}\right)$$

  then $D(\mu, 0)$ detects periodic eigenvalues of $\partial_x \mathcal{L}[u]$ in $L^2_{\mathrm{per}}([0, T])$.
- **Notation**: We use the following Poisson bracket style notation for Jacobian determinants:

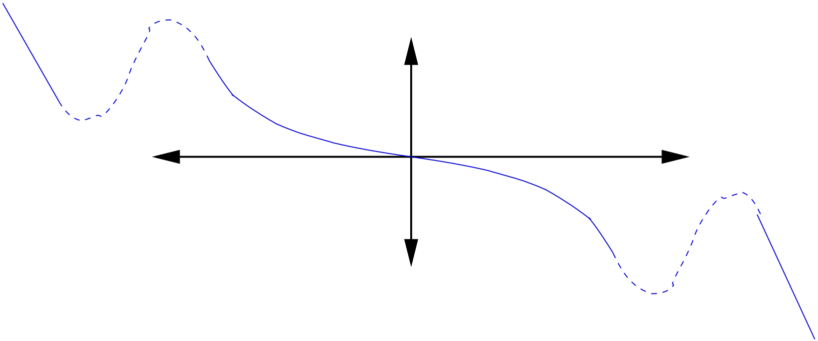
$$\{f, g\}_{x,y} = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}$$

$$\{f, g, h\}_{x,y,z} = \begin{vmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{vmatrix}$$

Orientation Index

Theorem (J. C. B. & Mathew Johnson 2008)

Let $u = u(\,\cdot\,; a_0, E_0, c_0)$ be a periodic traveling wave solution of gKdV such that $\{T, M, P\}_{a,E,c}$ is non-zero at (a_0, E_0, c_0). The number of real positive periodic eigenvalues is even if $\{T, M, P\}_{a,E,c} > 0$ and odd if $\{T, M, P\}_{a,E,c} < 0$.



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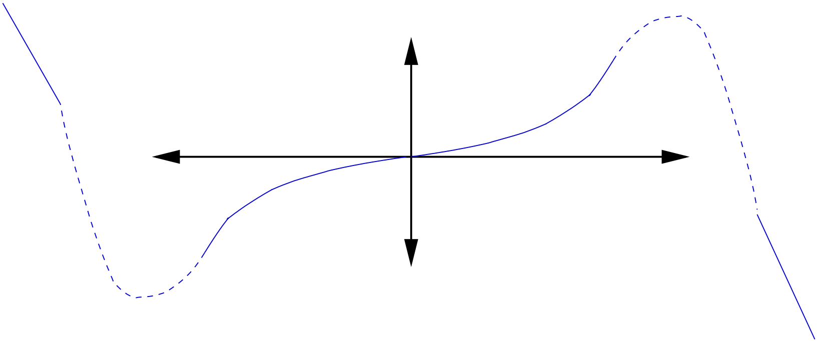


Remarks:

- The quantity $\{T, M, P\}_{a,E,c}$ is the natural analog of the quantity studied in the solitary wave case.
- This quantity can be interpreted as the derivative of the momentum P along the curve defined by M and T constant.
- Can also be expressed in terms of Hamiltonian: $E\{T, M, P\}_{a,E,c} = -\{H, M, P\}_{a,E,c}$.
- Natural from point of view of Whitham theory: Think of conserved quantities as parameterizing manifold of solutions.

An Index Theorem

Theorem (J. C. B. & Mathew Johnson & Todd Kapitula 2009)

Consider the operator $\partial_x \mathcal{L}$ acting on $L^2(\mathbb{R}/(kT\mathbb{Z}))$ -In other words look at perturbations of period k times the fundamental period. Define $n_{\mathbb{R}}$ to be the number of real eigenvalues in open positive half-line, $n_{\mathbb{C}}$ to be the number of complex (not purely real) eigenvalues in open right half-plane, and $n_{\mathbb{I}}^-$ to be the number of purely imaginary eigenvalues of negative Krein signature, and $P(\partial^2 K)$ to be the number of positive eigenvalues of the Hessian of the classical action K of the traveling wave. Then one has the following count:

$$n_{\mathbb{R}} + n_{\mathbb{C}} + n_{\mathbb{I}}^- = 2k - 1 - P(\partial^2 K)$$

Index Theorem: Ideas of Proof

Use a formula of Hărăguș and Kapitula

$$n_{\mathbb{R}} + n_{\mathbb{C}} + n_{\mathbb{I}}^{-} = \mathbf{N}(\mathcal{L}|_{\mathrm{Ran}(\partial_x)}) - \mathbf{N}(\mathcal{L}|_{g-Ker(\partial_x \mathcal{L})})$$

Both of these things can be computed in terms of geometric quantities (determinants/Jacobians of maps).

- $\mathbf{N}(\mathcal{L}) = 2k - 1 + \begin{Bmatrix} 0 & T_E > 0 \\ 1 & T_E < 0 \end{Bmatrix}$ follows from Sturm Oscillation theorem
- $\mathbf{N}(\mathcal{L}|_{\mathrm{Ran}(\partial_x)})$ can only differ from $\mathbf{N}(\mathcal{L})$ by at most one - Courant minimax principle.
- $\mathbf{N}(\mathcal{L}|_{g-Ker(\partial_x \mathcal{L})})$ amounts to determining sign of particular inner product.

Additional Modes of Instability in Periodic Case

- It is well understood that periodic waves admit additional instability mechanisms.
- A periodic wave can be stable to perturbations of the same period, but unstable to perturbations of a multiple of the period - modulational or Benjamin-Feir instability mechanism.
- Hărăguș and Kapitula showed that small amplitude periodic waves to KdV-p go unstable at $p = 2$ (Modified KdV).
- Want to find a way to distinguish $n_{\mathbb{I}}^{-}$ and $n_{\mathbb{C}}$ in index formula.

Modulational Instability Index

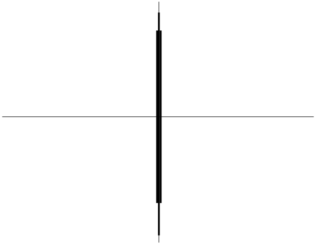
Theorem (J. C. Bronski & M.J. 2008)

Define the following quantity

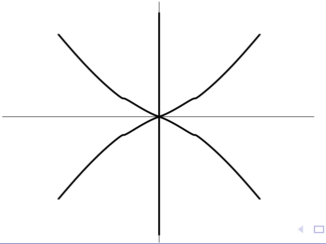
$$\Delta = \frac{1}{2}(\{T, P\}_{E,c} + 2\{M, P\}_{a,E})^3 - \frac{27}{4}(\{T, M, P\}_{a,E,c})^2$$

- *If* $\Delta > 0$ *then in the neighborhood of the origin the spectrum of* $\partial_x \mathcal{L}$ *considered on* $L^2(\mathbb{R})$ *consists of the imaginary axis with multiplicity three.*
- *If* $\Delta < 0$ *then in the neighborhood of the origin the spectrum of* $\partial_x \mathcal{L}$ *considered on* $L^2(\mathbb{R})$ *consists of the imaginary axis together with two curves intersecting the origin transversely to the imaginary axis, all with multiplicity one.*

Modulationally Stable Case : $\Delta > 0$



Modulationally Unstable Case: $\Delta < 0$



Ideas of Proof:

- Explicit Computation: Compute $\mathbf{M}(0)$ in terms of tangent plane.
- Local Normal form calculation (Weierstrauss preparation theorem): Compute

  $$\det(\mathbf{M}(\mu) - e^{i\kappa}\mathbf{I}) = D(\mu,\kappa)$$

  for κ, μ small.
- Normal form homogeneous cubic in κ, μ. Discriminant of cubic tells the story.
- Note: symmetries force non-generic bfiurcation. $\mathbf{M}(0)$ has a non-trivial Jordan block but eigenvalues bifurcate analytically!

Quasiperiodic Waves (w. Johnson/Maragell)

Equations such as the Nonlinear Scrodinger equation

$$i\phi_t = -\frac{1}{2}\phi_{xx} + v(|\phi|^2)\phi$$

have quasi-periodic solutions

$$\phi(x) = A(x)e^{i\theta(x)}$$
$$A(x+T) = A(x)$$
$$\theta(x+T) = \theta(x) + s$$

where s is the quasi-momentum. Spectral theory for quasi-periodic potentials is difficult but modulational viewpoint goes through in a similar way.

Variational Structure

Generic NLS has three conserved quantities

$$M = \int |\phi|^2(x)dx$$

$$P = \int i\left(\phi_x\phi^* - \phi_x^*\phi\right)dx$$

$$H = \int \frac{1}{2}|\phi_x|^2 + V(|\phi|^2)dx$$

Add to these two additional quantities, the period and the quasi-momentum

$$T = \int dx$$

$$s = \int i\frac{\phi_x\phi^* - \phi_x^*\phi}{|\phi|^2}dx$$

Note that the last is well-defined since ϕ cannot vanish for quasiperiodic solutions due to angular momentum barrier.

Maxwell Relations

The quasi-periodic solutions are constrained minimizers of a free energy and thus satisfy Maxwell relations, Defining the action $\mathcal{A}$ by a period integral

$$\mathcal{A} = \oint \sqrt{2E - 2A^2\omega - c^2 A^2 + 2V(A^2) - \frac{\kappa^2}{A^2}} dA$$

we have the Maxwell relations

$$\frac{\partial \mathcal{A}}{\partial E} = T$$
$$\frac{\partial \mathcal{A}}{\partial \omega} = -M$$
$$\frac{\partial \mathcal{A}}{\partial \kappa} = s$$

The integration constants E, ω, κ are Lagrange multipliers enforcing the constraints of constant period, mass and quasi-momentum resepctively.

Kernel of the Linearized Operator

The linearized operator takes the form

$$\mathcal{L} = \begin{pmatrix} S & L_- \\ -L_+ & -S \end{pmatrix}$$

where S is skew-adjoint. For generic quasi-periodic waves the structure of the kernel is as follows:

$$\dim(\ker(L)) = 2$$
$$\dim(\ker(L^2)/\ker(L)) = 2$$

so the Jordan form consists of two 2×2 Jordan blocks. This reflects the action-angle variables: the two elements of $\ker(L)$ correspond to the two angle variables, the two elements of $\ker(L^2)$ to the actions.

Breakup of Spectrum under Perturbation - Local Normal Form

- Under generic pertubations a 2×2 Jordan block does not break analytically
- *However..* perturbation very non-generic.
- Normal form: eigenvalue $\lambda(\mu)$ with quasi-momentum $s + \mu$ leads to eigenvalue condition

$$\lambda^4 + A\lambda^2\mu^2 + \mu^4 = 0$$

- Quantity A completely expressible in terms of period integrals.

Lessons from NLS

- In some ways structure is simpler than KdV. Structure of kernel related to Hamiltonian structure of traveling wave equation.
- Stability can be related to information on the structure of the set of traveling waves: Classical mechanics.
- Maxwell relations hold very generally - don't require quadrature, etc. (Nonlocal equations, etc.)

Herglotz Eigenproblems:

In stability analysis for nonlinear systems stability often reduces to studying an eigenvalue pencil Consider a degenerate reaction-diffusion system where only one species diffuses

$$\mathbf{u}_t = \mathbf{u}_{xx} + F_1(\mathbf{u}, \mathbf{v}, \mathbf{w}, \ldots)$$
$$\mathbf{v}_t = F_2(\mathbf{u}, \mathbf{v}, \mathbf{w}, \ldots) \ldots$$

The stability problem for a stationary solution takes the form

$$\lambda \mathbf{p_1} = \mathbf{p}_{1xx} + \sum \partial_i \mathbf{F} \mathbf{p}_i$$
$$\lambda \mathbf{p_2} = \sum \partial_i F_2 \mathbf{p}_i \ldots$$

The equations for non-diffusing species can be algebraically eliminated. Surprisingly there is a structure that occurs reasonable often in applications that guarantees that all of the eigenvalues are real and simple.

Reminder: Herglotz Functions

If $\mathbb{C}^+$ denotes the open upper half-plane $\mathrm{Re}(\lambda) > 0$ and similar $\mathbf{C}^-$ a meromorphic function f is Herglotz (Nevanlinna, Nevanlinna-Pick, etc) if

$$f(\mathbb{C}^+) \subseteq \mathbb{C}^+ \qquad f(\mathbb{C}^-) \subseteq \mathbb{C}^-$$

An example of a Herglotz function is a function of the form

$$f(z) = Az + B - \sum \frac{C_i}{z - z_i}$$

with A real and positive, B real, C_i real and positive and z_i real. It is well-known (to those that well-know it) that a Herglotz function

- Has all zeroes and poles on the real axis.
- Zeroes and poles alternate on the real axis, and are simple.
- Is monotonically increasing between poles.

Herglotz Pencils:

If $\mathbf{H}(\lambda)$ is an operator pencil then λ^* is an eigenvalue if $\mathbf{H}^{-1}(\lambda^*)$ fails to exist as a bounded operator.
We say an operator pencil is Herglotz if the diagonal matrix elements are Herglotz functions - in other words

$$f(\lambda) = \langle \mathbf{vH}(\lambda)\mathbf{v} \rangle$$

is a Herglotz function for all complex vectors $\mathbf{v} \in \mathrm{dom}(\mathbf{H})$. It is easy to prove the following theorem

Theorem

A Herglotz operator pencil has only real eigenvalues, and the Jordan block structure is trivial.

An Example:

Consider the linear operator pencil

$$\mathbf{H}(\lambda) = \mathbf{A} - \lambda \mathbf{B}$$

It is easy to see (via the polarization identity) that $\mathbf{H}(\lambda)$ is a Herglotz pencil if

- $\mathbf{A}$ is self-adjoint.
- $\mathbf{B}$ is self-adjoint and positive semi-definite.

In this case it is well-known that the eigenvalues are real and semi-simple (trivial Jordan blocks).

A Rational Pencil Example

Consider the degenerate reaction-diffusion equation where one of the reactants does not diffuse:

$$u_t = u_{xx} + F(u, v)$$
$$v_t = G(u) - \alpha v$$

Such examples are extremely common in biology: for instance spatial predator-prey models where one of the species cannot move (plant-herbivore)

The stability of a stationary solution is govern by a second order system

$$\lambda p = p_{xx} + F_1(x)p + F_2(x)q$$
$$\lambda q = G_1(x)p - \alpha q$$

with very minimal algebra this is equivalent to the rational Sturm-Liouville pencil

$$p_{xx} + F_1(x)p = \lambda p - \frac{F_2(x)G_1(x)}{\lambda + \alpha} p$$

This is a Herglotz Pencil!

A Sturm Theorem

Consider the Sturm-Liouville pencil

$$p_{xx} + V(x)p = \lambda p - \sum \frac{\alpha_i(x)}{\lambda - \beta_i} \qquad p(0) = 0 = p(L)$$

with $\alpha_i(x) \geq 0$ and β_i real. Then

- The essential spectrum is $\{\beta_i\}_{i=1}^N$
- Let $\beta_0 = -\infty$ and $\beta_{N+1} = \infty$. In each interval (β_{i-1}, β_i) for $\imath \in (1 \ldots N+1)$ there are a (countably) infinite sequence of eigenvalues indexed by the number of roots of the eigenfunction in $(0, L)$.
- The eigenvalues are simple.

In other words there is a Sturm theorem for each image of the real line.