Nontrivial Dynamics in the Forced Navier-Stokes Equations: A Computer-Assisted Proof

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SIAM Conference on Applications of Dynamical Systems
Snowbird, UTAH
5.21.2017

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Periodic orbits in the 3D Navier-Stokes equations

The incompressible Navier-Stokes equations on the 3D torus $\mathbb{T}^3$ are given by

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \dfrac{1}{\rho} \nabla p = f, & \text{on } \mathbb{T}^3 \times \mathbb{R} \\ \nabla \cdot u = 0, & \text{on } \mathbb{T}^3 \times \mathbb{R}, \end{cases}$$

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where

- $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t)) \in \mathbb{R}^3$ is the velocity field
- $p = p(x,t) \in \mathbb{R}$ is the pressure
- $x = (x_1, x_2, x_3) \in \mathbb{T}^3$ and $t \geq 0$
- ρ is the density of the fluid and ν is the kinematic viscosity
- $[(u \cdot \nabla) u]_k = u_1 \dfrac{\partial u_k}{\partial x_1} + u_2 \dfrac{\partial u_k}{\partial x_2} + u_3 \dfrac{\partial u_k}{\partial x_3}$, for $k = 1, 2, 3$
- $f = f(x,t)$ is the external forcing term.

Goal: prove the existence (constructively) of periodic orbits

Periodic orbits in the 3D Navier-Stokes equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \dfrac{1}{\rho} \nabla p = f & \text{on } \mathbb{T}^3 \times \mathbb{R} \\ \nabla \cdot u = 0 & \text{on } \mathbb{T}^3 \times \mathbb{R} \end{cases}$$

Does not lead to a diagonal dominant derivative in Fourier space

→ We consider the vorticity equation

$$\nabla \times \left(\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \dfrac{1}{\rho} \nabla p = f & \text{on } \mathbb{T}^3 \times \mathbb{R} \\ \nabla \cdot u = 0 & \text{on } \mathbb{T}^3 \times \mathbb{R} \end{cases} \right)$$

$\nabla \cdot u = 0 \rightarrow 0$

Periodic orbits in the 3D Navier-Stokes equations

Let the vorticity $\omega \stackrel{\text{def}}{=} \nabla \times u$. Using $(u \cdot \nabla)u = \nabla \left(\frac{u^2}{2}\right) - u \times \omega$:

$$\begin{aligned}\nabla \times ((u \cdot \nabla)u) &= \nabla \times (\omega \times u) \\ &= (u \cdot \nabla)\,\omega - (\omega \cdot \nabla)\,u + \omega\,(\nabla \cdot u) - u\,(\nabla \cdot \omega)\,,\end{aligned}$$

and since u and ω are divergence free :

$$\nabla \times ((u \cdot \nabla)u) = (u \cdot \nabla)\,\omega - (\omega \cdot \nabla)\,u.$$

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$$\nabla \times ((u \cdot \nabla)u) = (u \cdot \nabla)\,\omega - (\omega \cdot \nabla)\,u.$$

The vorticity equation is then given by

$$\boxed{\partial_t \omega + (u \cdot \nabla)\,\omega - (\omega \cdot \nabla)\,u - \nu \Delta \omega = g} \quad \text{on } \mathbb{T}^3 \times \mathbb{R},$$

where $g \stackrel{\text{def}}{=} \nabla \times f$.

Periodic orbits in the 3D Navier-Stokes equations

$$\partial_t \omega + \boxed{(u \cdot \nabla)\,\omega - (\omega \cdot \nabla)\,u} - \nu \Delta \omega = g$$

still depends on the velocity

We express u in term of ω by solving

$$\begin{cases} \nabla \times u = \omega \\ \nabla \cdot u = 0. \end{cases}$$

Applying a curl to the first equation, and using that $\nabla \cdot u = 0$, we get

$$-\Delta u = \nabla \times \omega,$$

and so

$$u = -\Delta^{-1} \nabla \times \omega.$$

Periodic orbits in the vorticity equation

$$\boxed{\partial_t \omega - \nu \Delta \omega} - \left(\left(\Delta^{-1} \nabla \times \omega \right) \cdot \nabla \right) \omega + (\omega \cdot \nabla) \left(\Delta^{-1} \nabla \times \omega \right) = g$$

diagonal dominant linear part in Fourier space

Periodic orbits in the vorticity equation

$$\partial_t \omega - \nu \Delta \omega - \left(\left(\left(\Delta^{-1} \nabla \times \omega \right) \cdot \nabla \right) \omega + \left(\omega \cdot \nabla \right) \left(\Delta^{-1} \nabla \times \omega \right) \right) = g$$

nonlinear terms

Periodic orbits in the vorticity equation

$$\partial_t \omega - \nu \Delta \omega - \left(\left(\Delta^{-1} \nabla \times \omega \right) \cdot \nabla \right) \omega + (\omega \cdot \nabla) \left(\Delta^{-1} \nabla \times \omega \right) = g$$

Describes the "dynamics" of the vorticity as time evolves

Periodic orbits in the vorticity equation

$$\partial_t \omega - \nu \Delta \omega - \left(\left(\Delta^{-1} \nabla \times \omega\right) \cdot \nabla\right) \omega + (\omega \cdot \nabla)\left(\Delta^{-1} \nabla \times \omega\right) = g$$

Plugging the space-time expansion $\omega(x,t) = \sum_{n=(\tilde{n},n_4)\in\mathbb{Z}^4} \omega_n e^{i(\tilde{n}\cdot x + n_4 \Omega t)}$ in the vorticity equation leads to $F(W) = (F_n(W))_{n\in\mathbb{Z}^4} = 0$, where

$$F_n(W) = \begin{cases} \omega_0, & n = 0 \\ (i\Omega n_4 + \nu \tilde{n}^2)\omega_n + i\left[M\omega \cdot \left(\tilde{D} \otimes \omega\right)\right]_n - i\left[\omega \cdot \left(\tilde{D} \otimes M\omega\right)\right]_n - g_n, & n \neq 0 \end{cases}$$

with $W \stackrel{\text{def}}{=} (\Omega, (\omega_n)_{n\in\mathbb{Z}^4})$, $M\omega = (M_n \omega_n)_{n\in\mathbb{Z}^4}$ and

$$M_n \stackrel{\text{def}}{=} \begin{cases} \dfrac{i}{\tilde{n}^2} \begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} & \tilde{n} \neq 0, \\ 0 & \tilde{n} = 0. \end{cases}$$

leads to a diagonal dominant derivative

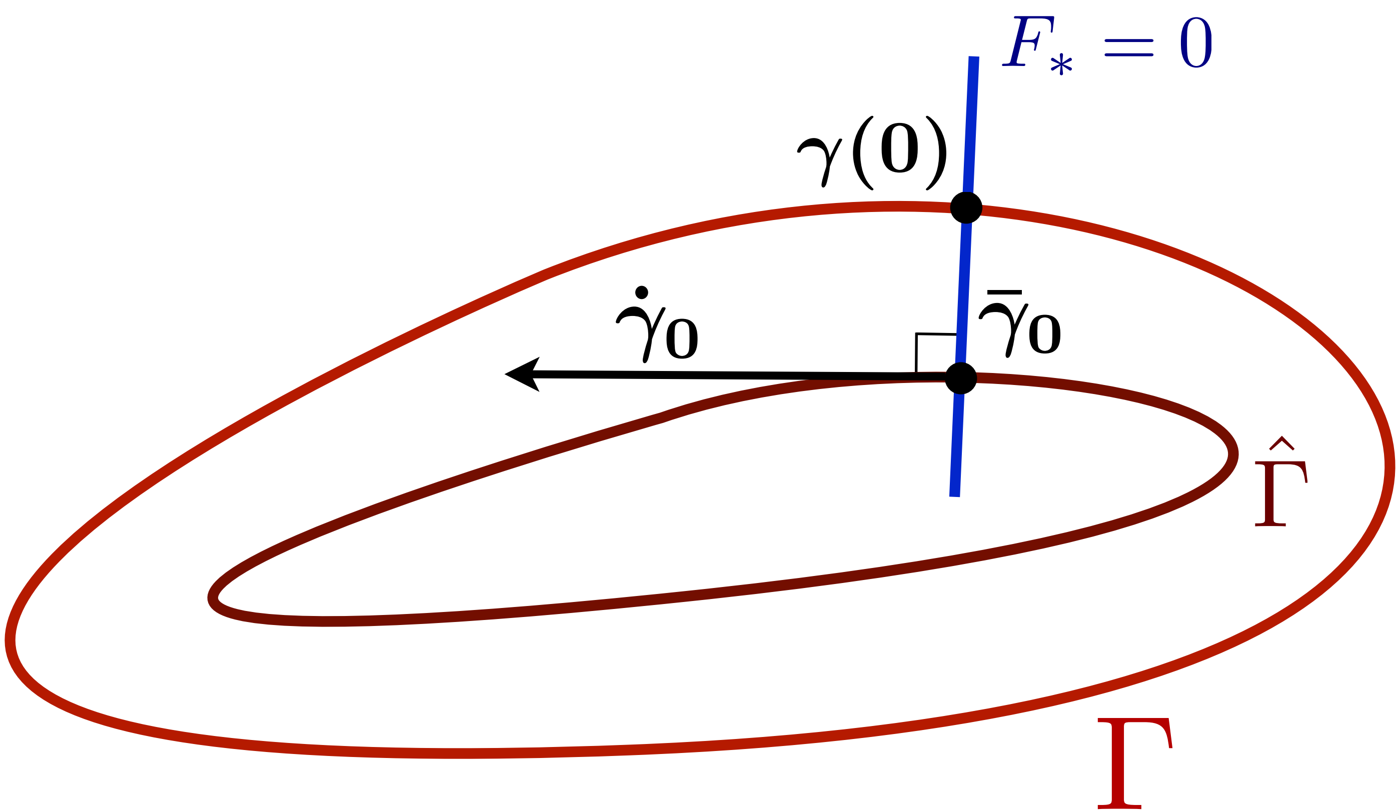
$$u = -\Delta^{-1} \nabla \times \omega$$

Periodic orbits in the vorticity equation

$$\partial_t\omega - \nu\Delta\omega - \left(\left(\Delta^{-1}\nabla \times \omega\right)\cdot\nabla\right)\omega + (\omega\cdot\nabla)\left(\Delta^{-1}\nabla \times \omega\right) = g$$

The periodic orbits need to be isolated fixed points

To eliminate arbitrary time shift, we impose a *Poincaré phase condition*.



Periodic orbits in the vorticity equation

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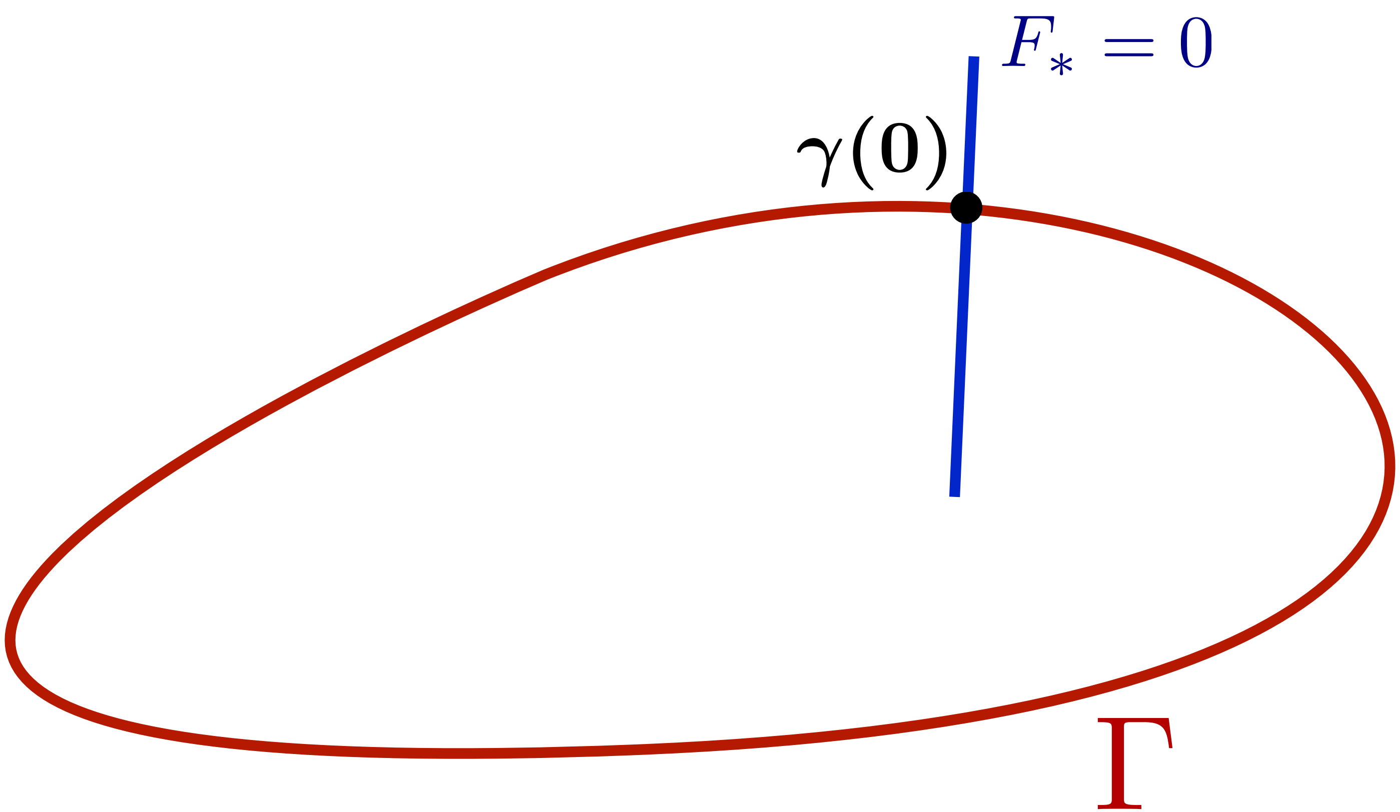


Periodic orbits in the vorticity equation

Looking for periodic orbits of the vorticity equation boils down to solve

$$\mathcal{F}(W) = \begin{pmatrix} F_*(W) \\ (F_n(W))_{n\in\mathbb{Z}^4} \end{pmatrix} = 0, \qquad W \stackrel{\text{def}}{=} \begin{pmatrix} \Omega \\ (\omega_n)_{n\in\mathbb{Z}^4} \end{pmatrix}.$$

we solve using computer-assisted analysis

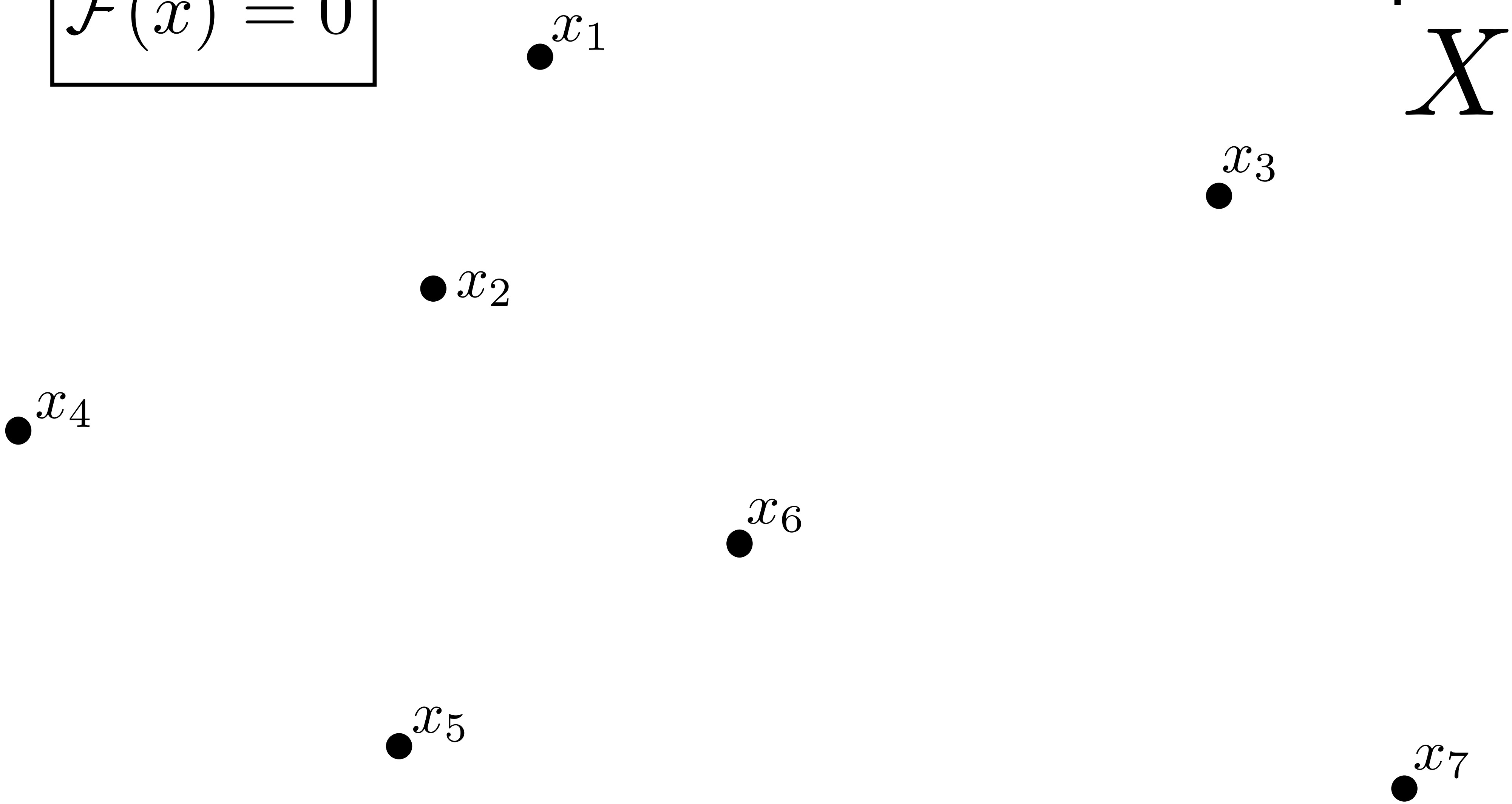
Lemma : Assume that the external forcing term f does not depend on time, that $\mathcal{F}(W)=0$ and that $\nabla\cdot\omega = 0$. Let $u \stackrel{\text{def}}{=} M\omega$. Then there exists a pressure term p such that (u,p) is a $\frac{2\pi}{\Omega}$-periodic solution of the forced incompressible Navier-Stokes equations on the 3D torus $\mathbb{T}^3$

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - \nu\Delta u + \dfrac{1}{\rho}\nabla p = f, & \text{on } \mathbb{T}^3\times\mathbb{R} \\ \nabla\cdot u = 0, & \text{on } \mathbb{T}^3\times\mathbb{R}. \end{cases}$$

A general nonlinear problem

$$\mathcal{F}(x) = 0$$

to solve in a Banach space



Most of the time impossible to compute exactly !

A general nonlinear problem

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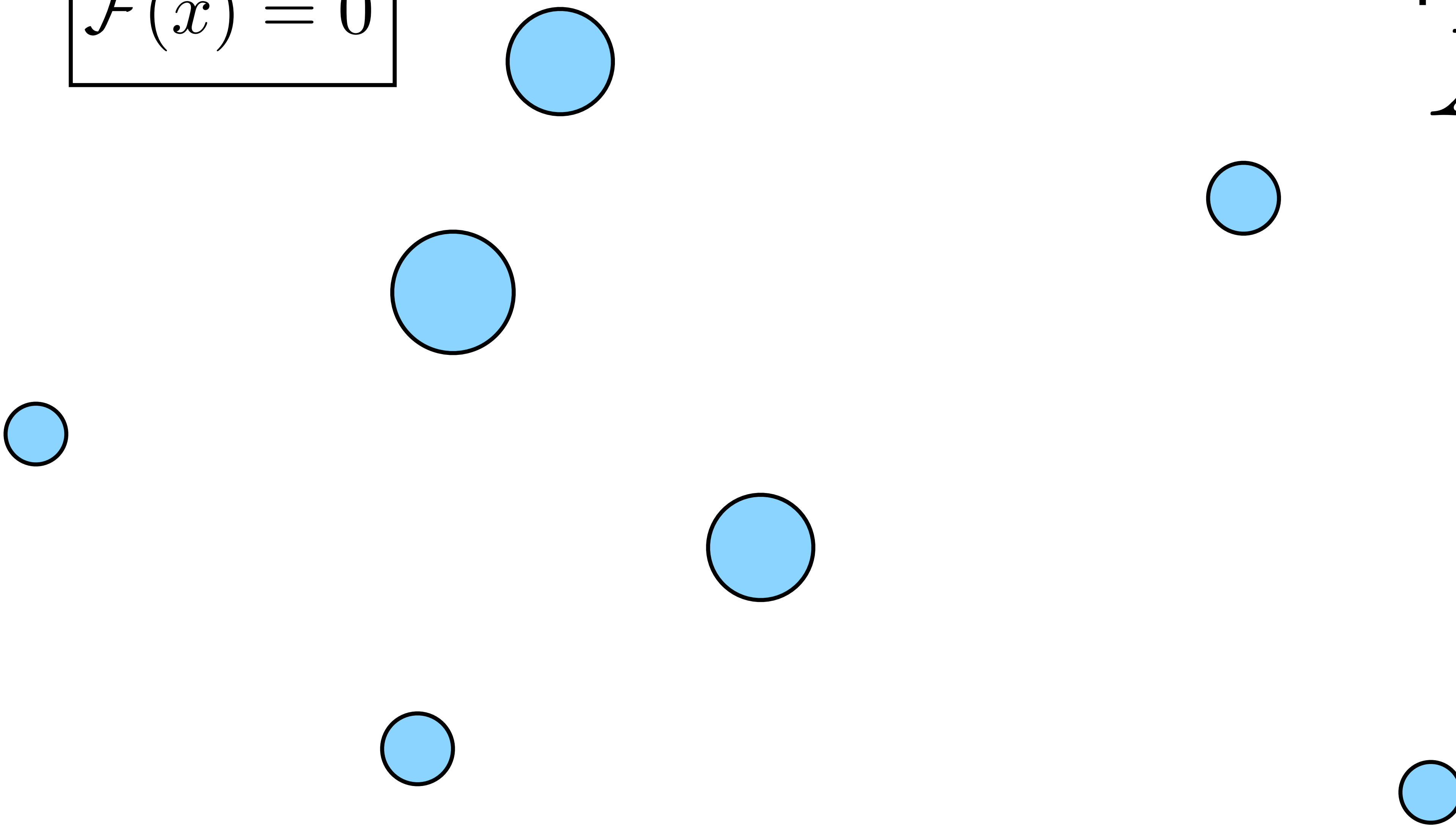
$$X$$



Alternative: find small balls in which it is demonstrated (in a mathematically rigorous sense) that a unique solution exists.

How to find these small isolating balls ?

1. Let $\bar{x}$ a numerical approximation of $\mathcal{F}(x) = 0$ in X computed using a finite dimensional reduction.

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5. Consider $B_{\bar{x}}(r) \subset X$ the closed ball of radius r centered at $\bar{x}$.
6. Find $r > 0$ such that $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction mapping (tool : **radii polynomials**).

A Newton-Kantorovich type argument

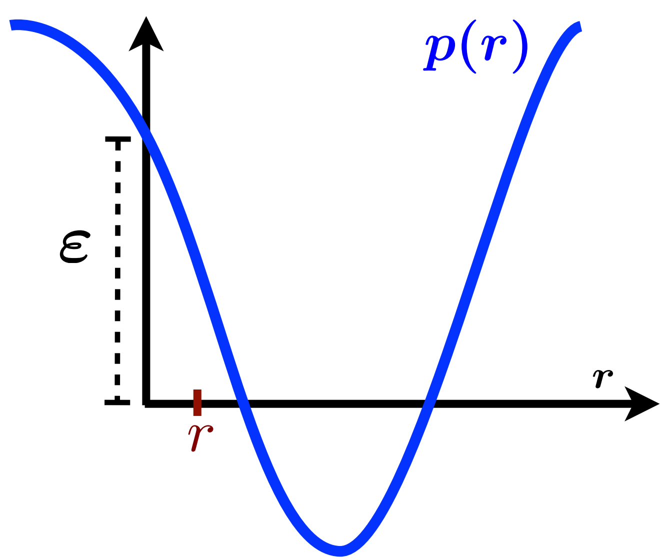
Theorem : Let $T : X \to X$ defined by $T(x) = x - A\mathcal{F}(x)$ with $T \in C^1(X)$. Let $r > 0$ and consider bounds ε and $\kappa = \kappa(r)$ satisfying

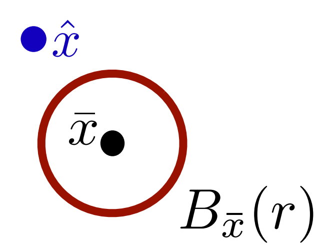
$$\begin{aligned} \|T(\bar{x}) - \bar{x}\|_X &= \|A\mathcal{F}(\bar{x})\|_X \le \varepsilon \\ \sup_{w \in B_{\bar{x}}(r)} \|DT(w)\|_X &= \sup_{w \in B_{\bar{x}}(r)} \|I - A \cdot D\mathcal{F}(w)\|_X \le \kappa(r). \end{aligned}$$

If

$$p(r) \stackrel{\text{def}}{=} \varepsilon + r\kappa(r) - r < 0$$ **(radii polynomial)**

then $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction with Lipschitz constant $\kappa(r) < 1$. Moreover A is injective and therefore $\mathcal{F} = 0$ has a unique solution in $B_{\bar{x}}(r)$.





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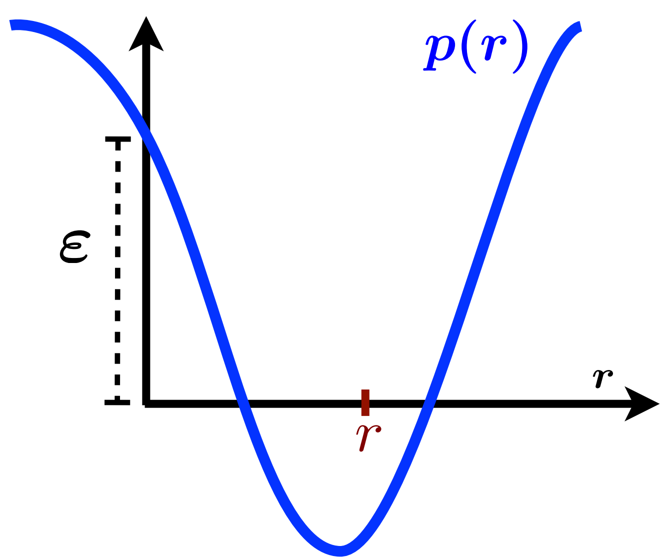

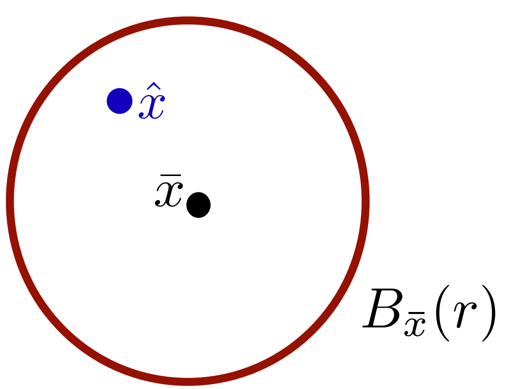

Periodic orbits in the vorticity equation

We consider the Banach space $\mathcal{X}_\eta = \mathbb{C} \times \left(\ell^1_\eta(\mathbb{C})\right)^3$ with the norm

$$\|W\|_{\mathcal{X}_\eta} = \mathbf{max}\left(|\Omega|, \max_{1\le l\le 3} \|\omega^{(l)}\|_{\ell^1_\eta}\right),$$

where for a complex valued sequence $a \in \mathbb{C}^{\mathbb{Z}^4}$,

$$\|a\|_{\ell^1_\eta} = \sum_{n\in\mathbb{Z}^4} |a_n|\eta^{|n|_1}.$$

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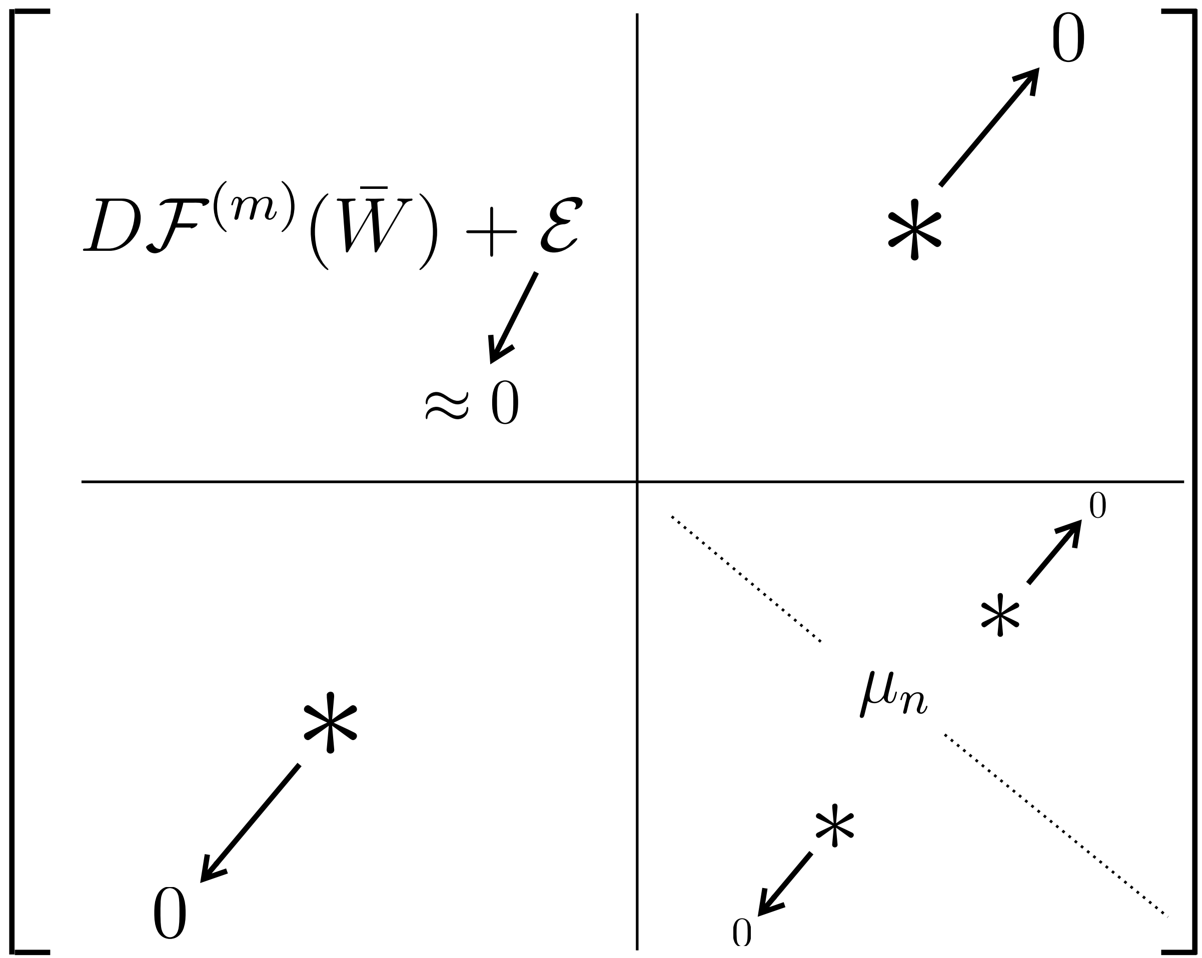
We solve the problem $\mathcal{F}(W) = 0$ in the subspace of $\mathcal{X}_\eta$ of divergence free sequences

$$\mathcal{X}^{div}_\eta \stackrel{\text{def}}{=} \{W \in X_\eta,\ \nabla \cdot \omega = 0\}.$$

The approximate inverse A

$$D\mathcal{F}(\bar{W}) =$$



$$\mathcal{F}_n(W) = \mu_n \omega_n + i\left[M\omega \cdot \left(\tilde{D} \otimes \omega\right)\right]_n - i\left[\omega \cdot \left(\tilde{D} \otimes M\omega\right)\right]_n - g_n, \qquad \mu_n \stackrel{\text{def}}{=} i\Omega n_4 + \nu \tilde{n}^2$$

The approximate inverse A

$$
DF(\bar{W}) \approx \left[\begin{array}{c|c}
DF^{(m)}(\bar{W}) & 0 \\
\hline
0 & \begin{matrix} \ddots & & 0 \\ & \mu_n & \\ 0 & & \ddots \end{matrix}
\end{array}\right]
$$

$$
\mathcal{F}_n(W) = \mu_n \omega_n + i \left[M\omega \cdot \left(\tilde{D} \otimes \omega \right) \right]_n - i \left[\omega \cdot \left(\tilde{D} \otimes M\omega \right) \right]_n - g_n, \qquad \mu_n \stackrel{\text{def}}{=} i\Omega n_4 + \nu \tilde{n}^2
$$

The approximate inverse A

$$D\mathcal{F}(\bar{W})^{-1} \approx \left[\begin{array}{c|c} A_m & 0 \\ \hline 0 & \begin{matrix} \ddots & & 0 \\ & \mu_n^{-1} & \\ 0 & & \ddots \end{matrix} \end{array}\right]$$

$$A_m \approx D\mathcal{F}^{(m)}(\bar{W})^{-1}$$

(Computer-assisted computation)

$$\mu_n \stackrel{\text{def}}{=} i\Omega n_4 + \nu \tilde{n}^2$$

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Final step : prove that $T(W) = W - A\mathcal{F}(W)$ is a contraction on $B_r(\bar{W})$.

Using the symmetries to reduce the dimension

The equation

$$F_n(W) = \begin{cases} \omega_0, & n = 0 \\ (i\Omega n_4 + \nu \tilde{n}^2)\omega_n + i\left[M\omega \cdot \left(\tilde{D} \otimes \omega\right)\right]_n & \\ \qquad -i\left[\omega \cdot \left(\tilde{D} \otimes M\omega\right)\right]_n - g_n, & n \neq 0 \end{cases}$$

has a rather large group of symmetries, generated by the following elements.

— Reflection in the x-direction : $S_x(\omega_n) = \left(\omega^{(x)}_{(-n_x,n_y,n_z)}, -\omega^{(y)}_{(-n_x,n_y,n_z)}, -\omega^{(z)}_{(-n_x,n_y,n_z)}\right)$.

— Reflection in the y-direction : $S_y(\omega_n) = \left(-\omega^{(x)}_{(n_x,-n_y,n_z)}, \omega^{(y)}_{(n_x,-n_y,n_z)}, -\omega^{(z)}_{(n_x,-n_y,n_z)}\right)$.

— Reflection in the z-direction : $S_z(\omega_n) = \left(-\omega^{(x)}_{(n_x,n_y,-n_z)}, -\omega^{(y)}_{(n_x,n_y,-n_z)}, \omega^{(z)}_{(n_x,n_y,-n_z)}\right)$.

— Translation over $d = \frac{2\pi}{l}$ in the vertical direction : $T_l(\omega_n) = \left(e^{\frac{2i\pi}{l}}\right)^{n_z} \omega_n$.

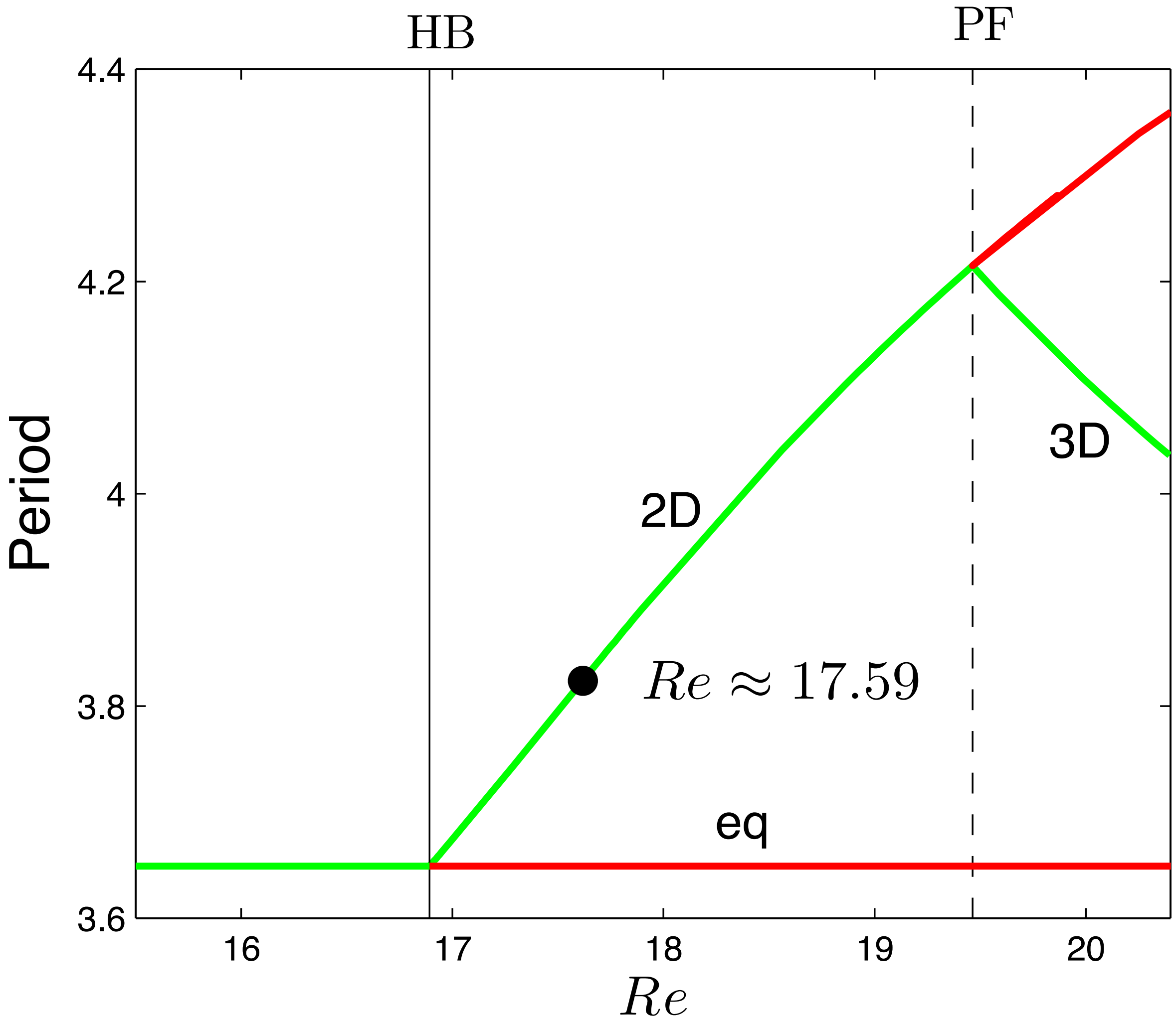
— Translation over $s = \frac{\tau}{l}$ in time (where τ is the period) : $P_l(\omega_n) = \left(e^{\frac{2i\pi}{l}}\right)^{n_t} \omega_n$.

— A shift over π in both the x and y directions $D(\omega_n) = (-1)^{n_x+n_y}\omega_n$.

— Rotation about the axis $x = y = 0$ over $\pi/2$ followed by a shift over π in the x-direction : $R(\omega_n) = (-1)^{n_y}(-\omega^{(y)}_{(-n_y,n_x,n_z)}, \omega^{(x)}_{(-n_y,n_x,n_z)}, \omega^{(z)}_{(-n_y,n_x,n_z)})$.

Result: a 2D periodic orbit in the Navier-Stokes equations

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \frac{1}{\rho} \nabla p = f & \text{on } \mathbb{T}^3 \times \mathbb{R} \\ \nabla \cdot u = 0 & \text{on } \mathbb{T}^3 \times \mathbb{R} \end{cases}$$

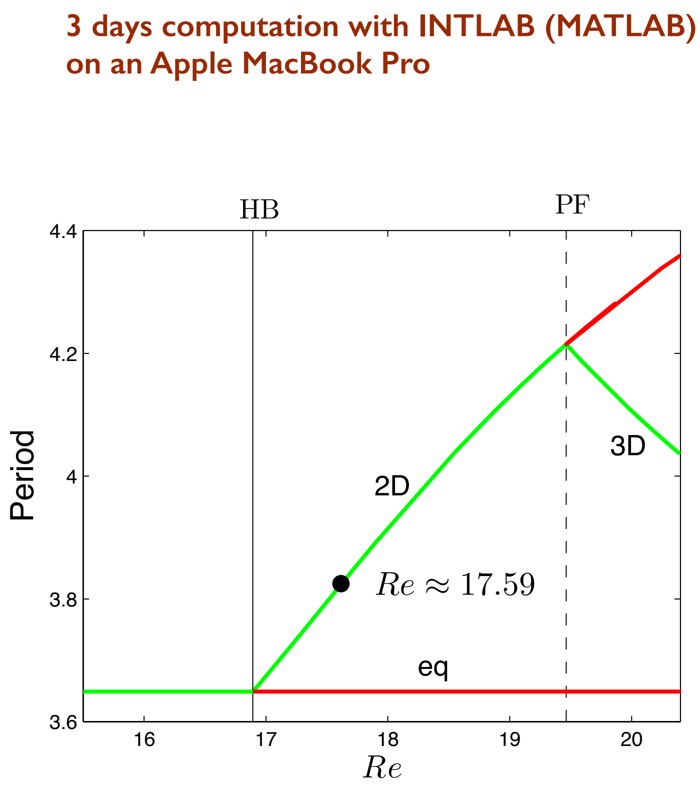


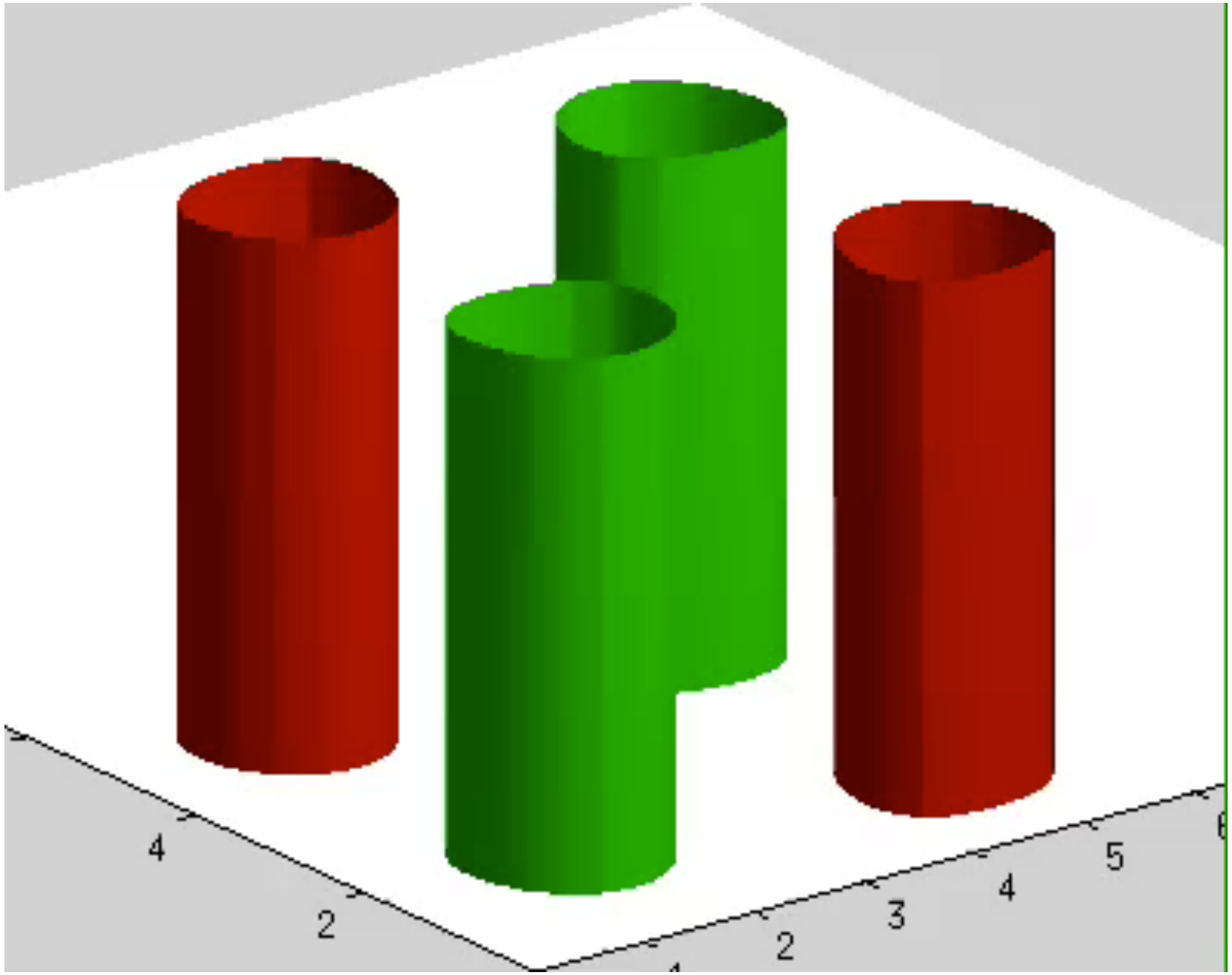
$$f(x_1, x_2, x_3) = \begin{pmatrix} -\sin(x_1)\cos(x_2)/2 \\ \cos(x_1)\sin(x_2)/2 \\ 0 \end{pmatrix}$$

$$(N_x, N_y, N_z, N_t) = (17, 17, 0, 9)$$
$$n = 12803$$

$$\nu = \frac{\sqrt{8\pi}}{Re} \approx 0.285$$

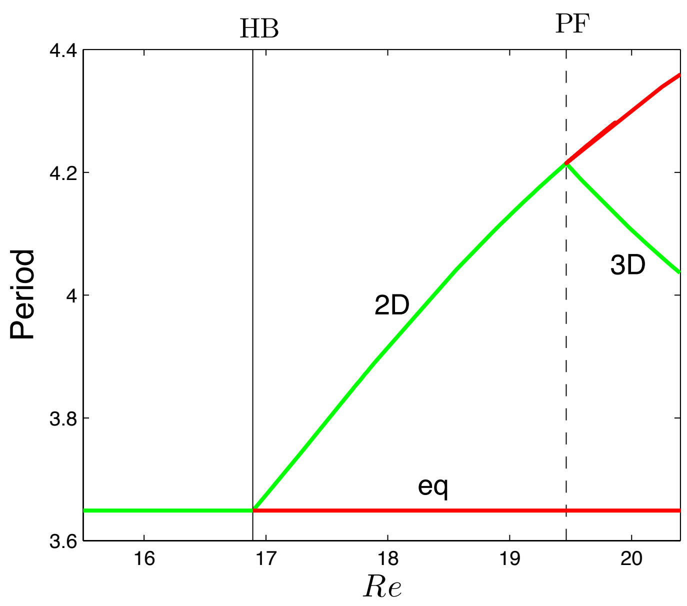
Periodic orbits in the 2D Navier-Stokes equations

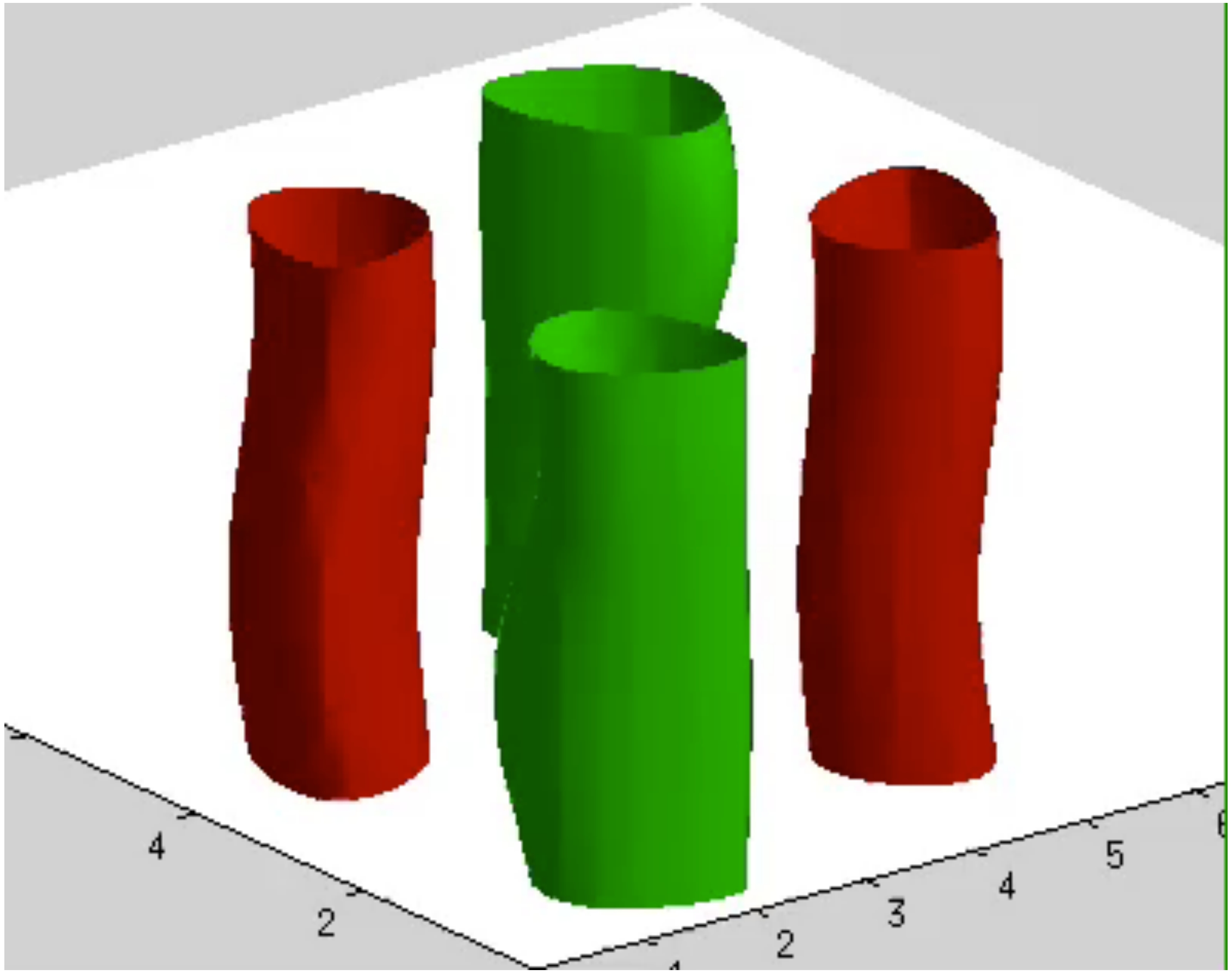




Isosurfaces of the vertical vorticity $\omega_3 = \partial_{x_1} u_2 - \partial_{x_2} u_1$ (the last component of $\omega = \nabla \times u$). For the forcing function g, the isosurfaces would be perfectly cylindrical, equal in size and stationary. Red and green indicate positive and negative values for the isosurfaces (at about 80% of the maximal value in each frame). The tubular structures represent vortex tubes, with anticlockwise (green) and clockwise (red) rotational motion around them.

Fully 3D periodic orbits in the Navier-Stokes equations?





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Thanks to my collaborators

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