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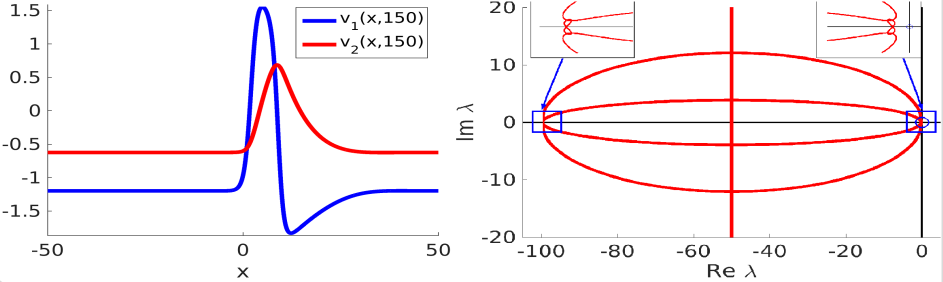
Computation and Stability of Waves in Second Order Evolution Equations

(with W.-J. Beyn and D. Otten)

Supported by DFG through CRC 1173 and CRC 701

SIAM DS 2017



Outline

Traveling Waves in 1st Order Systems

Extension to 2nd Order Systems

Transforming 2nd Order Systems to 1st Order Systems

Traveling Waves

Traveling waves are solutions of

(Evo) $\qquad u_t = \mathcal{F}(u), \quad x \in \mathbb{R}, t \geq 0, u(x,t) \in \mathbb{R}^m,$ ($\mathcal{F}$ nonlin PDO)

of the form

(TW) $\qquad u(x,t) = v_\star(x - \mu_\star t) =: v_\star(\xi), \quad v_\star$: profile, $\mu_\star$: velocity.

Examples:

- Reaction-Diffusion equations $\mathcal{F}(u) = Au_{xx} + g(u)$,
- Semilinear hyperbolic equations $\mathcal{F}(u) = Cu_x + g(u)$,
- Coupled hyperbolic-parabolic systems, including Reaction-Diffusion equations with non-invertible A
- ...

Traveling Wave - Boundary Value Problem

Typically: $v_\star$ and $\mu_\star$ unknown.
Inserting (TW) into (Evo) leads to BVP

Traveling wave equation

(BVP$_1$)
$$0 = \mathcal{F}(v_\star) + \mu_\star v_{\star x} \quad „= A v_{\star\xi\xi} + \mu_\star v_{\star\xi} + g(v_\star)“$$
$$\lim_{\xi\to\pm} v_\star(\xi) = v_{\star\pm}$$

Problem: How to find a good initial guess for the BVP-solver?
Solution: Direct forward simulation by **Freezing method** introduced by Beyn&Thümmler 2004 yields both:
Approximations of $v_\star$ and $\mu_\star$.

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Freezing in a Nutshell

(Evo) $u_t = \mathcal{F}(u), \quad x \in \mathbb{R}, t \geq 0, u(x,t) \in \mathbb{R}^m,$

Idea: Separate evolution of the profile from evolution of its position:

Freezing ansatz

$$u(x,t) = v\big(x - \gamma(t), t\big) =: v(\xi, t),$$

$v(\cdot, t)$: „profile at time t"

$\gamma(t)$: „position at time t"

Plug into (Evo): $u_t = \mathcal{F}(u)$

Freezing system

(FR$_1$)
$$\begin{cases} v_t = F(v) + \gamma_t v_\xi \\ \text{new degree of freedom } \gamma_t \end{cases}$$

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Plug into (Evo): $u_t = \mathcal{F}(u)$

Freezing system

(FR$_1$) $$\begin{cases} v_t = F(v) + \mu\, v_\xi \\ \gamma_t = \mu \\ 0 = \Phi(v, \mu), \quad \text{phase condition.} \end{cases}$$

Plan for Today

- Extend to 2nd order equations:

(DW) $$M u_{tt} = A u_{xx} + f(u, u_x, u_t), \quad x \in \mathbb{R},\ t \geq 0,\ u(x,t) \in \mathbb{R}^m$$

- Theorem on stability of traveling waves for (DW)
- The freezing System
- Indicate some key ideas in the proof
- Not today: For generalizations to multi-d and more general relative equilibria see Wolf-Jürgen Beyn's talk from Tuesday

References

- K.P. Hadeler 1988, relating TWs of damped wave equations to those of parabolic equations
- T. Gallay and G. Raugel 1997, stability in the scalar case
- T. Gallay and R. Joly 2009, global stability in the scalar case
- M. Grillakis and J. Shatah and W. Strauss, 1987/1990 orbital stability in the undamped (Hamiltonian) case
- S. Dieckmann 2017, Freezing method in the Hamiltonian case

Outline

Traveling Waves in 1st Order Systems

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Traveling Waves in 2nd Order Equations

(DW) $$Mu_{tt} = Au_{xx} + f(u, u_x, u_t)$$

considered the co-moving frame

$$u(x,t) = v(x - \mu_\star t, t) =: v(\xi, t)$$

becomes

(DW_{co}) $$Mv_{tt} = (A - {\mu_\star}^2 M)v_{\xi\xi} + 2\mu_\star M v_{\xi t} + f(v, v_\xi, v_t - \mu_\star v_\xi).$$

A traveling wave (time-independent profile $v(\xi, t) = v_\star(\xi)$) satisfies

Traveling wave equation

$$0 = (A - {\mu_\star}^2 M)v_{\star\xi\xi} + f(v_\star, v_{\star\xi}, -\mu_\star v_\star).$$

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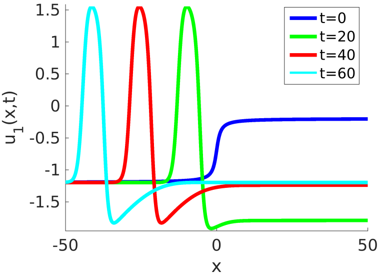
FitzHugh-Nagumo Wave Example I

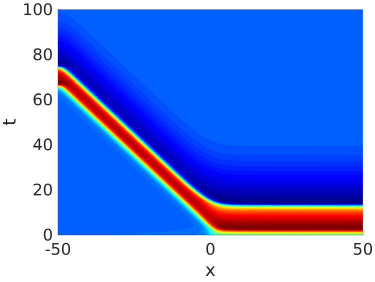
An observation of Hadeler 1988 can be used to show:

(FHN-W)

$$\varepsilon u_{1,tt} + u_{1,t} = u_{1,xx} + u_1 - \tfrac{1}{3}u_1^3 - u_2$$
$$\varepsilon u_{2,tt} + u_{2,t} = \nu u_{2,xx} + 0.08(u_1 + 0.7 - 0.8u_2)$$

with $\varepsilon = 0.01$ and $\nu \approx 0.1056$ has a traveling pulse solution, traveling with velocity $\mu_\star \approx -0.7868$.





Structural Assumptions

(DW$_{\text{co}}$) $$Mv_{tt} = (A - {\mu_\star}^2 M)v_{\xi\xi} + 2\mu_\star M v_{\xi t} + f(v, v_\xi, v_t - \mu_\star v_\xi)$$

- $f \in \mathcal{C}^3(\mathbb{R}^{3m}, \mathbb{R}^m)$
- M invertible, $M^{-1}A$ positive diagonalizable
- $\exists (v_\star, \mu_\star) \in \mathcal{C}_b^2 \times \mathbb{R}$, TW-solution with $v_{\star\xi} \in H^3(\mathbb{R}, \mathbb{R}^m) \setminus \{0\}$,

(BVP$_2$) $$0 = (A - {\mu_\star}^2 M)v_{\star\xi\xi} + f(v_\star, v_{\star\xi}, -\mu_\star v_\star),$$
$$v_\star(\xi) \to v_\pm,\ v_{\star\xi}(\xi) \to 0, \text{ as } \xi \to \pm\infty, \quad f(v_\pm, 0, 0) = 0,$$

- $(A - {\mu_\star}^2 M)$ is invertible

Spectral Assumptions

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(DW$_{\text{co}}$) $$Mv_{tt} = \left(A - {\mu_\star}^2 M\right)v_{\xi\xi} + 2\mu_\star M v_{\xi t} + f(v, v_\xi, v_t - \mu_\star v_\xi)$$

linearization about steady state $v_\star$, (abbreviate $f_\star = f(v_\star, v_{\star\xi}, -\mu_\star v_{\star\xi})$)

(DW$_{\text{lin}}$) $$\mathcal{P}(\partial_t, \partial_\xi)v = Mv_{tt} - (A - {\mu_\star}^2 M)v_{\xi\xi} - 2\mu_\star M v_{\xi t} + \left(\mu_\star D_3 f_\star - D_2 f_\star\right)v_\xi - D_3 f_\star v_t - D_1 f_\star v = 0,$$

ansatz $v(\xi, t) = e^{\lambda t} w(\xi)$ (Laplace transform) yields
Quadratic Eigenvalue Problem

$$\mathcal{P}(\lambda, \partial_\xi)w = \left(\lambda^2 P_2 + \lambda P_1(\partial_\xi) + P_0(\partial_\xi)\right)w = 0, \quad \mathcal{P}(\lambda, \partial_\xi) : H^2 \to L^2$$

Note: $\mathcal{P}(\lambda, \partial_\xi)$ depends on ξ!

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There is $\delta > 0$, s.t.

- $\sigma_{\mathsf{pt}}(\mathcal{P}(\cdot,\partial_\xi)) \bigcap \{\operatorname{Re}\lambda > -\delta\} = \{0\}$ and 0 is simple eigenvalue

  $\mathcal{N}\big(\mathcal{P}(0,\partial_\xi)\big) = \operatorname{span}\left\{v_{\star\xi}\right\}$ and $P_1 v_{\star\xi} \notin \mathcal{R}\big(\mathcal{P}(0,\partial_\xi)\big)$, furthermore
  $\mathcal{N}\big(\mathcal{P}(\lambda,\partial_\xi)\big) \neq \{0\}$, $\operatorname{Re}\lambda > -\delta$ implies $\lambda = 0$
- Dispersion set of $\mathcal{P}$ to the left of $-\delta$

$$\operatorname{Re}\sigma_{\mathsf{disp}}(\mathcal{P}) = \operatorname{Re}\left\{\lambda \in \mathbb{C} : \det \mathcal{P}^{\pm}(\lambda, i\omega) = 0,\ \omega \in \mathbb{R}\right\} =$$

$$\operatorname{Re}\left\{\lambda \in \mathbb{C} : \det\left(\lambda^2 M + \lambda P_1^{\pm}(i\omega) + P_0^{\pm}(i\omega)\right) = 0, \omega \in \mathbb{R}\right\} \leq -\delta$$

Spectral Assumptions

Quadratic Eigenvalue Problem

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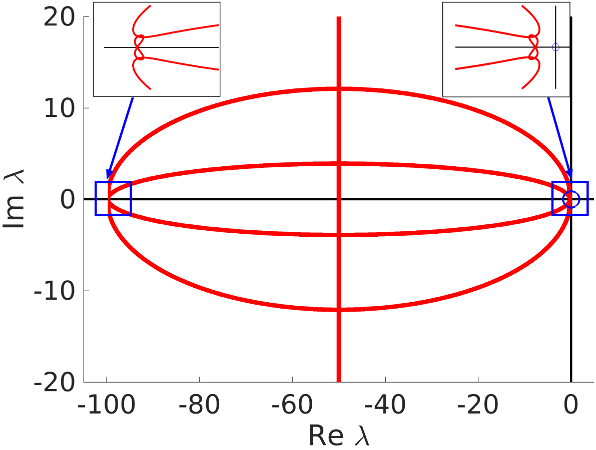
FitzHugh-Nagumo Wave Example II

Spectral set $\sigma_{\text{disp}}(\mathcal{P}_{FHN})$:



Stability with Asymptotic Phase

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Theorem (Beyn, Otten, RM 17)

Structural + Spectral Assumptions.
Then: $\forall 0 < \eta < \delta \;\; \exists \rho > 0$, *s.t. the Cauchy problem*

$$Mu_{tt} = Au_{xx} + f(u, u_x, u_t), \quad u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = v_0$$

with data $u_0 \in v_\star + H^3$, $v_0 \in H^2$, *satisfying*

$$\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star\xi}\|_{H^2} \le \rho,$$

has a unique global solution $u \in v_\star + \bigcap_{j=0}^{2} \mathcal{C}^{2-j}([0,\infty); H^j) =: v_\star + \mathcal{CH}^2$.
Moreover, there are $\varphi_\infty = \varphi_\infty(u_0, v_0)$ *and* $C = C(\eta, \rho)$, *s.t.*

$$|\varphi_\infty| \le C\big(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star\xi}\|_{H^2}\big),$$

$$\|u(\cdot,t) - v_\star(\cdot - \mu_\star t - \varphi_\infty)\|_{H^2} + \|u_t(\cdot,t) + \mu_\star v_{\star\xi}(\cdot - \mu_\star t - \varphi_\infty)\|_{H^1} \le C\big(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star\xi}\|_{H^2}\big)e^{-\eta t}, \quad \forall t \ge 0.$$

Freezing Method for 2nd Order Equations

Consider (DW)

$$Mu_{tt} = Au_{xx} + f(u, u_x, u_t)$$
$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x).$$

Ansatz as before: $u(x,t) = v\big(x - \gamma(t), t\big) =: v(\xi, t)$

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Freezing system for 2nd order evolution equations

(FR$_2$)

$$Mv_{tt} = (A - \gamma_t{}^2 M)v_{\xi\xi} + 2\gamma_t Mv_{\xi t} + \gamma_{tt} Mv + f(v, v_\xi, v_t - \gamma_t v_\xi)$$
$$\gamma_t = \mu_1$$
$$\mu_{1,t} = \mu_2$$

again we lack 1 equation, need a phase condition!

Consistent initial data:

(Init)

$$\gamma(0) = 0,\ v(\cdot,0) = u_0,\ v_t(\cdot,0) = v_0 + \mu_1(0)u_{0,x},$$
$$\mu_1(0) = -\frac{\langle v_0, \widehat{v}\rangle}{\langle u_{0,x}, \widehat{v}_\xi\rangle},\ \mu_2(0) = \dots$$

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Freezing system for 2nd order evolution equations

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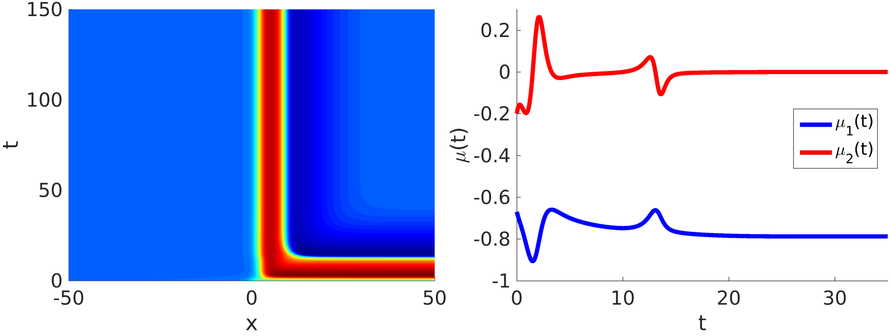
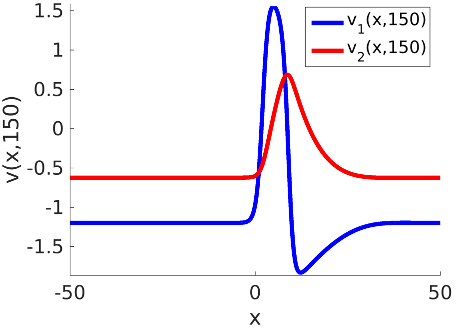
FitzHugh-Nagumo Wave Example III

Result of the freezing method, applied to the FHN-Wave system



FitzHugh-Nagumo Wave Example III

Result of the freezing method, applied to the FHN-Wave system



Lyapunov Stability for Freezing Method

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Theorem (Beyn, Otten, RM 17)

Structural + **Spectral Assumptions** + *the template* $\widehat{v} \in v_\star + H^1$ *satisfies,* $\langle v_\star - \widehat{v}, \widehat{v}_\xi \rangle = 0$, $\langle v_{\star\xi}, \widehat{v}_\xi \rangle \neq 0$.
Then: $\forall 0 < \eta < \delta\ \exists \rho > 0$, *s.t. for all* $u_0 \in v_\star + H^3$, $v_0 \in H^2$ *with*

$$\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star\xi}\|_{H^2} \leq \rho,$$

and consistent $\mu_1(0)$, $\mu_2(0)$, *s.t.* $\langle u_0 - \hat{v}, \hat{v}_\xi \rangle = 0$ *holds:*
The freezing system (FR$_2$) *has a unique global solution* $(v, \mu_1, \mu_2, \gamma)$,
$v \in v_\star + \mathcal{CH}^2$, $\mu_1 \in \mathcal{C}^1$, $\mu_2 \in \mathcal{C}$, $\gamma \in \mathcal{C}^2$.
Moreover, $\exists C = C(\rho, \eta) > 0$, *s.t.*

$$\|v(\cdot,t) - v_\star\|_{H^2} + \|v_t(\cdot,t)\|_{H^1} + |\mu_1(t) - \mu_\star| \\ \leq C\big(\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star\xi}\|_{H^2}\big)e^{-\eta t}, \quad \forall t \geq 0.$$

Outline

Traveling Waves in 1st Order Systems

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Result for First Order Hyperbolic Balance Laws

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Theorem (RM 12)

If the first order Cauchy problem

(HYP)
$$U_t = EU_x + F(U), \quad x \in \mathbb{R},\ t \geq 0,\ U(x,t) \in \mathbb{R}^l,$$
$$U(\cdot,0) = U_0$$

satisfies the **structural** *and* **spectral Assumptions**.
Then: $\forall 0 < \eta < \delta\ \exists \rho_0 > 0$, *s.t.* (HYP) *has for all* $U_0 \in V_\star + H^2$ *with*

$$\|U_0 - V_\star\|_{H^2} \leq \rho_0$$

a unique global solution $U \in V_\star + \bigcap_{j=0}^{1} \mathcal{C}^{1-j}([0,\infty);H^j) =: V_\star + \mathcal{CH}^1$.
Moreover, there are $\varphi_\infty = \varphi_\infty(U_0)$ *and* $C = C(\eta, \rho_0)$, *s.t.*

$$|\varphi_\infty| \leq C\|U_0 - V_\star\|_{H^2}$$
$$\|U(\cdot,t) - V_\star(\cdot - \mu_\star t - \varphi_\infty)\|_{H^1} \leq C\|U_0 - V_\star\|_{H^2} e^{-\eta t},\ \forall t \geq 0.$$

The Assumptions

(HYP) $$U_t = EU_x + F(U), \quad x \in \mathbb{R},\ t \geq 0,\ U(x,t) \in \mathbb{R}^l$$

Structural assumptions

1. E is real diagonalizable
2. $F \in \mathcal{C}^3(\mathbb{R}^l, \mathbb{R}^l)$
3. exists $(V_\star, \mu_\star)$ TW solution $U(x,t) = V_\star(x - \mu_\star t)$, $V_\star \in \mathcal{C}^1_b$, $V_{\star\xi} \in H^2$
4. $Z(\xi) := DF\big(V_\star(\xi)\big) \to Z_\pm$, $Z'(\xi) \to 0$, as $\xi \to \pm\infty$
5. the matrix $(E + \mu_\star I)$ is invertible

Spectral assumptions: There is $\delta > 0$ s.t.

6. $\det \mathcal{P}^\pm_{1\text{st}}(\lambda, i\omega) = \det\big(\lambda I - i\omega(E + \mu_\star I) - Z_\pm\big) = 0$ for some $\omega \in \mathbb{R}$ $\Rightarrow \operatorname{Re}\lambda \leq -\delta < 0$
7. $\mathcal{P}_{1\text{st}}(\lambda, \partial_\xi) = \lambda I - (E + \mu_\star I)\partial_\xi - Z(\xi) : H^1(\mathbb{R}, \mathbb{C}^l) \to L^2(\mathbb{R}, \mathbb{C}^l)$ satisfies $\mathcal{N}\big(\mathcal{P}_{1\text{st}}(\lambda, \partial_\xi)\big) \neq \{0\} \Rightarrow \lambda = 0$ or $\operatorname{Re}\lambda \leq -\delta$ and 0 is alg. simple

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Transformation to 1st Order

(DW) $$u_{tt} = N^2 u_{xx} + M^{-1} f(u, u_x, u_t)$$

where $N^2 = M^{-1}A$, N is positive square root.
How to transform?

- Need a semilinear first order system
- Nonlinearity f depends on u, u_x and u_t

$\Rightarrow$ Need a $3m$-dimensional system!

Transformation to 1st Order

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where $N^2 = M^{-1}A$, N is a positive square root.
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$U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = \begin{pmatrix} u \\ u_t + Nu_x \\ u_t - Nu_x \end{pmatrix}$, so that

$$U_t = \begin{pmatrix} NU_{1x} \\ NU_{2x} \\ -NU_{3x} \end{pmatrix} + \begin{pmatrix} U_3 \\ \widetilde{f}(U) \\ \widetilde{f}(U) \end{pmatrix} = \begin{pmatrix} N & & \\ & N & \\ & & -N \end{pmatrix} U_x + F(U),$$

$$\widetilde{f}(U) = M^{-1} f\left(U_1, \tfrac{1}{2}N^{-1}(U_2 - U_3), \tfrac{1}{2}(U_2 + U_3)\right)$$

Transformation to 1st Order – Structural

TW solution $u(x,t) = v_\star(x - \mu_\star t)$ of $u_{tt} = N^2 u_{xx} + M^{-1} f(u, u_x, u_t)$

leads to TW solution $(V_\star, \mu_\star)$, $V_\star = \begin{pmatrix} v_\star \\ (N - \mu_\star I) v_{\star\xi} \\ -(N + \mu_\star I) v_{\star\xi} \end{pmatrix}$, of

$$U_t = \begin{pmatrix} N & & \\ & N & \\ & & -N \end{pmatrix} U_x + \begin{pmatrix} U_3 \\ \tilde{f}(U) \\ \tilde{f}(U) \end{pmatrix} = E\, U_x + F(U).$$

2nd order Structural assumptions $\Rightarrow$ **1st order** Structural assumptions:

1. E is real diagonalizable (since $N^2 = M^{-1} A$ pos. diagonalizable)
2. $F \in \mathcal{C}^3(\mathbb{R}^{3m}, \mathbb{R}^{3m})$ (since $f \in \mathcal{C}^3$)
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5. $(E + \mu_\star I)$ is invertible (since $A - {\mu_\star}^2 M = M(N - \mu_\star I)(N + \mu_\star I)$ is)

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Linearize $V_t = (E + \mu_\star I)V_\xi + \begin{pmatrix} V_3 \\ \widetilde{f}(V) \\ \widetilde{f}(V) \end{pmatrix}$ about $V_\star$:

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6. $\det\big(\lambda I - i\omega(E + \mu_\star I) - Z_\pm\big) = 0$ for some $\omega \in \mathbb{R}$ implies $\operatorname{Re}\lambda \le -\delta < 0$ **But impossible:** $\omega = 0 \Rightarrow \det(-Z_\pm) = 0$ **!**
7. $\mathcal{P}_{\text{1st}}(\lambda, \partial_\xi) : H^1(\mathbb{R}, \mathbb{C}^{3m}) \to L^2(\mathbb{R}, \mathbb{C}^{3m})$ satisfies $\mathcal{N}\big(\mathcal{P}_{\text{1st}}(\lambda, \partial_\xi)\big) \neq \{0\} \Rightarrow \lambda = 0$ or $\operatorname{Re}\lambda \le -\delta$ and 0 is alg. simple

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Relate 2nd and 1st Order Problems

Relate spectral properties of $\mathcal{P}(\lambda, \partial_\xi) : H^2 \to L^2$,

$$\mathcal{P}(\lambda, \partial_\xi) = \left(\lambda^2 P_2 + \lambda P_1(\partial_\xi) + P_0(\partial_\xi)\right)$$

to spectral properties of $\mathcal{P}_{\text{1st}}(\lambda, \partial_\xi) : (H^1)^3 \to (L^2)^3$,

$$\mathcal{P}_{\text{1st}}(\lambda, \partial_\xi) = \begin{pmatrix} \mathcal{P}_{-N}(\lambda, \partial_\xi) + cI & 0 & -I \\ -\phi_1 & \mathcal{P}_{-N}(\lambda, \partial_\xi) - \phi_2 & -\phi_3 \\ -\phi_1 & -\phi_2 - cI & \mathcal{P}_N(\lambda) - \phi_3 \end{pmatrix},$$

where $\quad \mathcal{P}_{\pm N}(\lambda, \partial_\xi) = \lambda I + \left(\pm N - \mu_\star I\right)\partial_\xi.$

The key is the following „magical factorization“:

Factorization and its Consequences

$$\begin{pmatrix} 0 & 0 & I \\ 0 & I & -I \\ I & 0 & 0 \end{pmatrix} \mathcal{P}_{\text{1st}}(\lambda, \partial_\xi) =$$

$$\begin{pmatrix} M^{-1}\mathcal{P}(\lambda, \partial_\xi) & -\phi_2 - cI & \mathcal{P}_N(\lambda, \partial_\xi) - \phi_3 \\ 0 & \mathcal{P}_{-N}(\lambda, \partial_\xi) + cI & -\mathcal{P}_N(\lambda, \partial_\xi) \\ 0 & 0 & -I \end{pmatrix}$$

$$\begin{pmatrix} I & 0 & 0 \\ -\mathcal{P}_N(\lambda, \partial_\xi) & I & 0 \\ -\mathcal{P}_{-N}(\lambda, \partial_\xi) - cI & 0 & I \end{pmatrix}$$

Factorization and its Consequences

$$\begin{pmatrix} 0 & 0 & I \\ 0 & I & -I \\ I & 0 & 0 \end{pmatrix} \mathcal{P}^{\pm}_{\text{1st}}(\lambda, i\omega) =$$

$$\begin{pmatrix} M^{-1}\mathcal{P}^{\pm}(\lambda, i\omega) & -\phi_2^{\pm} - cI & \mathcal{P}_N(\lambda, i\omega) - \phi_3^{\pm} \\ 0 & \mathcal{P}_{-N}(\lambda, i\omega) + cI & -\mathcal{P}_N(\lambda, i\omega) \\ 0 & 0 & -I \end{pmatrix}$$

$$\begin{pmatrix} I & 0 & 0 \\ -\mathcal{P}_N(\lambda, i\omega) & I & 0 \\ -\mathcal{P}_{-N}(\lambda, i\omega) - cI & 0 & I \end{pmatrix}$$

Proposition: Dispersionrelation

$\det \mathcal{P}^{\pm}_{\text{1st}}(\lambda, i\omega) = 0 \Leftrightarrow \det \mathcal{P}^{\pm}(\lambda, i\omega) = 0$ or
$\det\big(\mathcal{P}_{-N}(\lambda, i\omega) + cI\big) = \det\big(\lambda I - i\omega(N + \mu_\star I) + cI\big) = 0,$

$$\Rightarrow \sigma_{\text{disp}}(\mathcal{P}_{\text{1st}}) = \sigma_{\text{disp}}(\mathcal{P}) \cup (-c + i\mathbb{R})$$

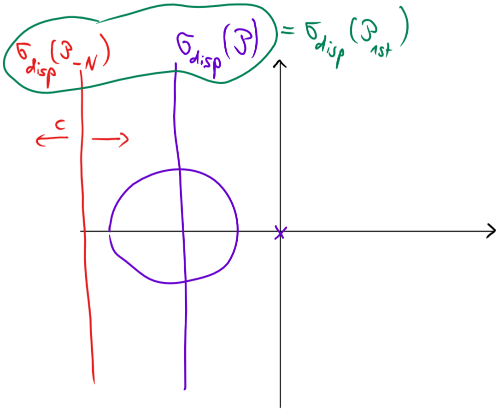
Factorization and its Consequences

$$\begin{pmatrix} 0 & 0 & I \\ 0 & I & -I \\ I & 0 & 0 \end{pmatrix} \mathcal{P}_{\text{1st}}(\lambda, \partial_\xi) = \begin{pmatrix} M^{-1}\mathcal{P}(\lambda,\partial_\xi) & -\phi_2 - cI & \mathcal{P}_N(\lambda,\partial_\xi) - \phi_3 \\ 0 & \mathcal{P}_{-N}(\lambda,\partial_\xi) + cI & -\mathcal{P}_N(\lambda,\partial_\xi) \\ 0 & 0 & -I \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ -\mathcal{P}_N(\lambda,\partial_\xi) & I & 0 \\ -\mathcal{P}_{-N}(\lambda,\partial_\xi) - cI & 0 & I \end{pmatrix}$$

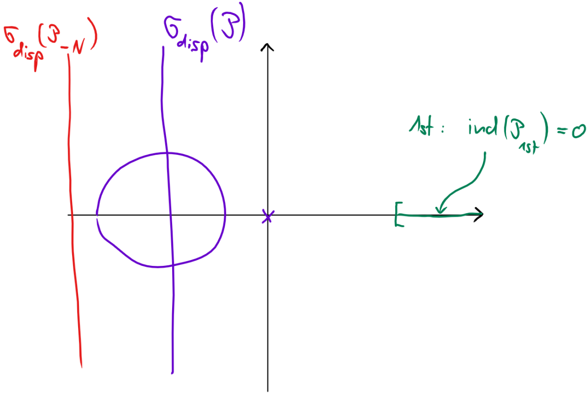
Proposition: Eigenvalues

- $\exists \lambda_\star > -c$, s.t. $\sigma_{\text{disp}}(\mathcal{P}_{\text{1st}}) \cap [\lambda_\star, \infty) = \emptyset$
- $\mathcal{P}_{\text{1st}}(\lambda, \partial_\xi) : H^1 \to L^2$ is Fredholm of index 0 for all $\lambda \in \rho_+$, connected component of $\{\operatorname{Re}\lambda > -c\} \setminus \sigma_{\text{disp}}(\mathcal{P}_{\text{1st}})$ containing $[\lambda_\star, \infty)$
- $\sigma_{\text{pt}}(\mathcal{P}) \cap \rho_+ = \sigma_{\text{pt}}(\mathcal{P}_{\text{1st}}) \cap \rho_+$ and simple eigenvalues in this set coincide

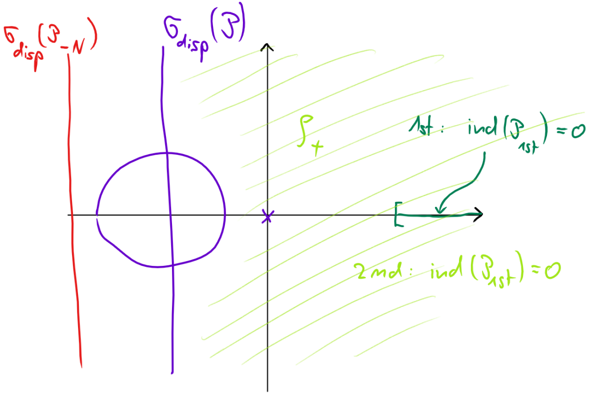
Sketches of Spectral Properties



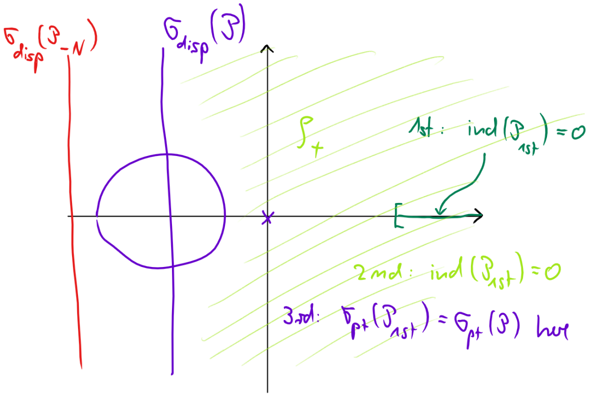
Sketches of Spectral Properties



Sketches of Spectral Properties



Sketches of Spectral Properties



Sketch of 2nd Order Stability Proof

1. Initial data for 2nd order problem small perturbation of traveling wave:

$$\|u_0 - v_\star\|_{H^3} + \|v_0 + \mu_\star v_{\star\xi}\|_{H^2} \le \rho$$

$$\Rightarrow \|U_0 - V_\star\|_{H^2} \le \text{const}\rho \quad \text{(1st order initial data)}$$

2. 1st order stability result: $U \in V_\star + \mathcal{CH}^1$ global solution, $U(\cdot,t) \to V_\star(\cdot - \mu_\star t - \varphi_\infty)$ in H^1

3. Remains:
   - Second and third component of U satisfy $U_2 = U_{1t} + NU_{1x}$ and $U_3 = U_{1t} - NU_{1x} + cU_1$
   - $U_{1t} = \frac{1}{2}\left(U_2 + U_3 - cU_1\right) \in \mathcal{CH}^1$
     $U_{1x} = \frac{N^{-1}}{2}\left(U_2 - U_3 + cU_1\right) \in \mathcal{CH}^1$
   - First component of U, U_1, really is a (the) solution to the 2nd order Cauchy problem, in particular: $U_1 \in v_\star + \mathcal{CH}^2$, not just $v_\star + \mathcal{CH}^1$!

Then H^1-convergence of U_1, U_2, U_3 implies H^2 convergence of $U_1 = u$

Summary / Conclusion / Take home

- Traveling waves for 2nd order semilinear wave equations
- Generalization of freezing method
- Stability results for traveling waves and freezing method
- Proof by transformation to $3m$-dimensional 1st order system
  - Advantage 1: Fully semilinear case possible
  - Advantage 2: Solutions of 1st and 2nd order problems directly related (for minimal dimension $2m$, we needed an auxiliary equation to show this)
  - Disadvantage: Introduction of additional spectrum
    Key idea: Shift this additional spectrum
  - Relating the spectral properties by a really beautiful factorization (personal opinion!)
- Generalization to higher spatial dimensions in W.-J. Beyn's talk from Tuesday

Thank You !