PHYTOPLANKTON

The emergence of spatial patterns in a phytoplankton-nutrient model.

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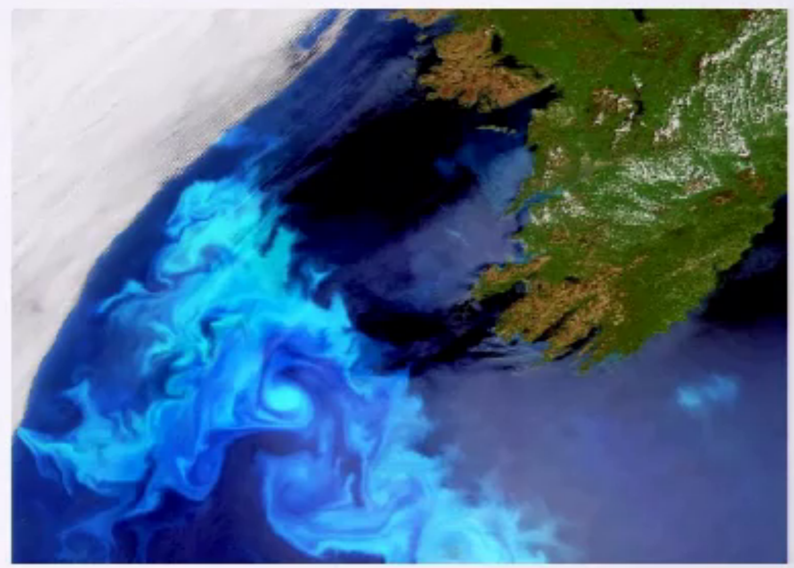
Arjen Doelman, Antonios Zagaris



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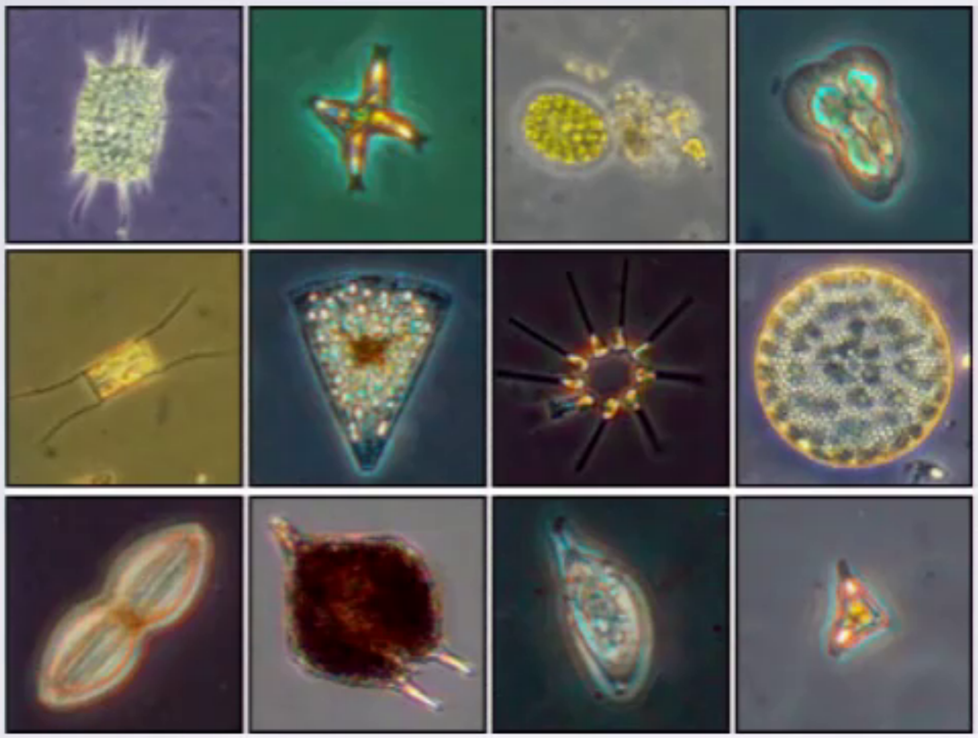
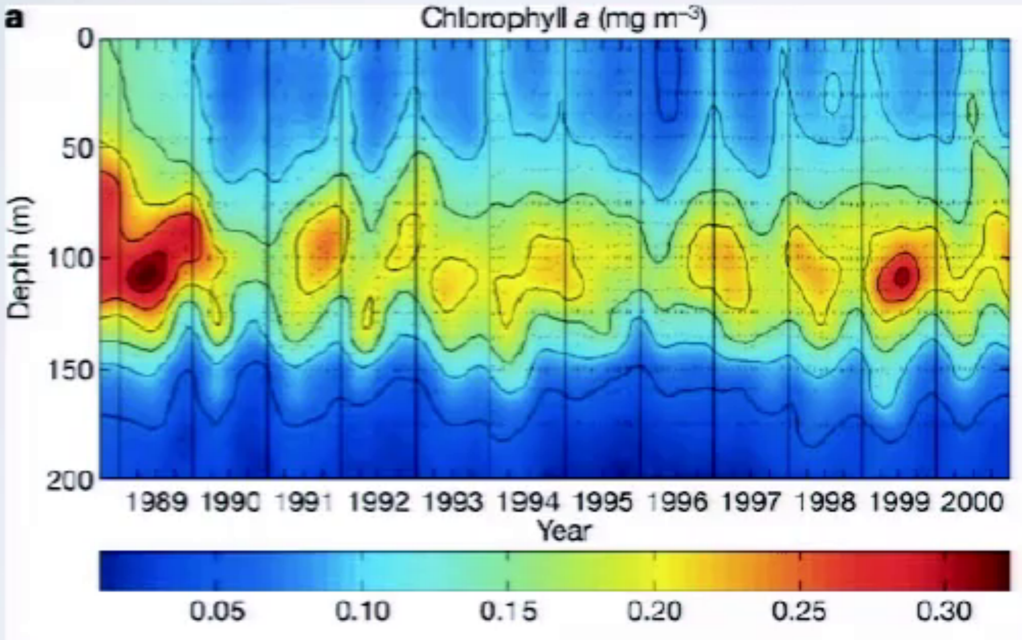


source: coralscience.org

- Photosynthesis.
- Light from the surface, nutrient from the bottom.
- Oxygen producer.
- Life cycle ~ 1 week.
- Size ~ 10 µm.

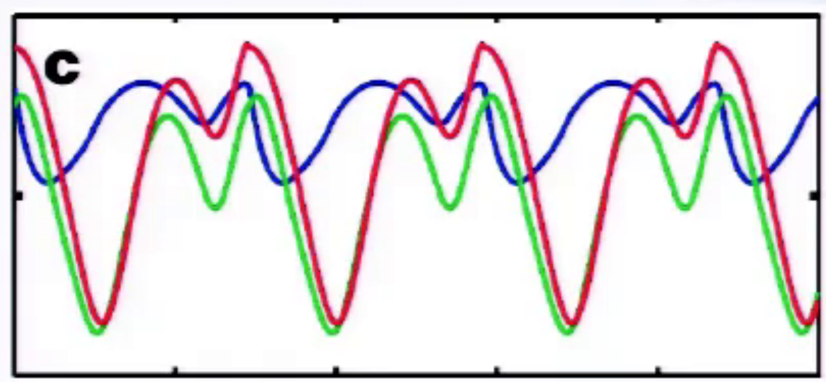
MOTIVATION



Field measurements of chlorophyll in the ocean near Hawaii.

Huisman *et al*, Nature 2006.

- Oscillating patterns.
- Field measurements indicate even chaotic behavior.



Phytoplankton density over a period of time, numerically derived for three species.

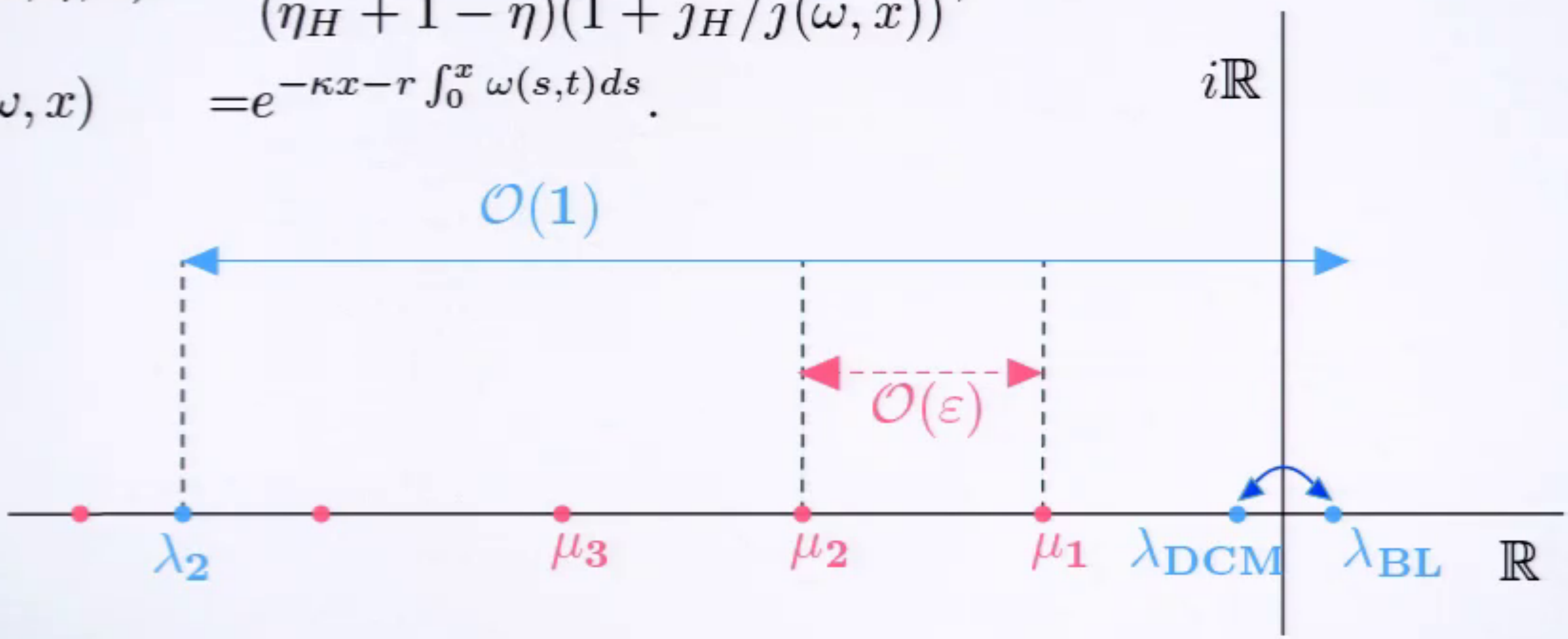
MATHEMATICAL MODEL

Plankton $\omega_t = \varepsilon \omega_{xx} - 2\sqrt{\varepsilon} v \omega_x + (p(\omega, \eta, x) - \ell)\omega,$

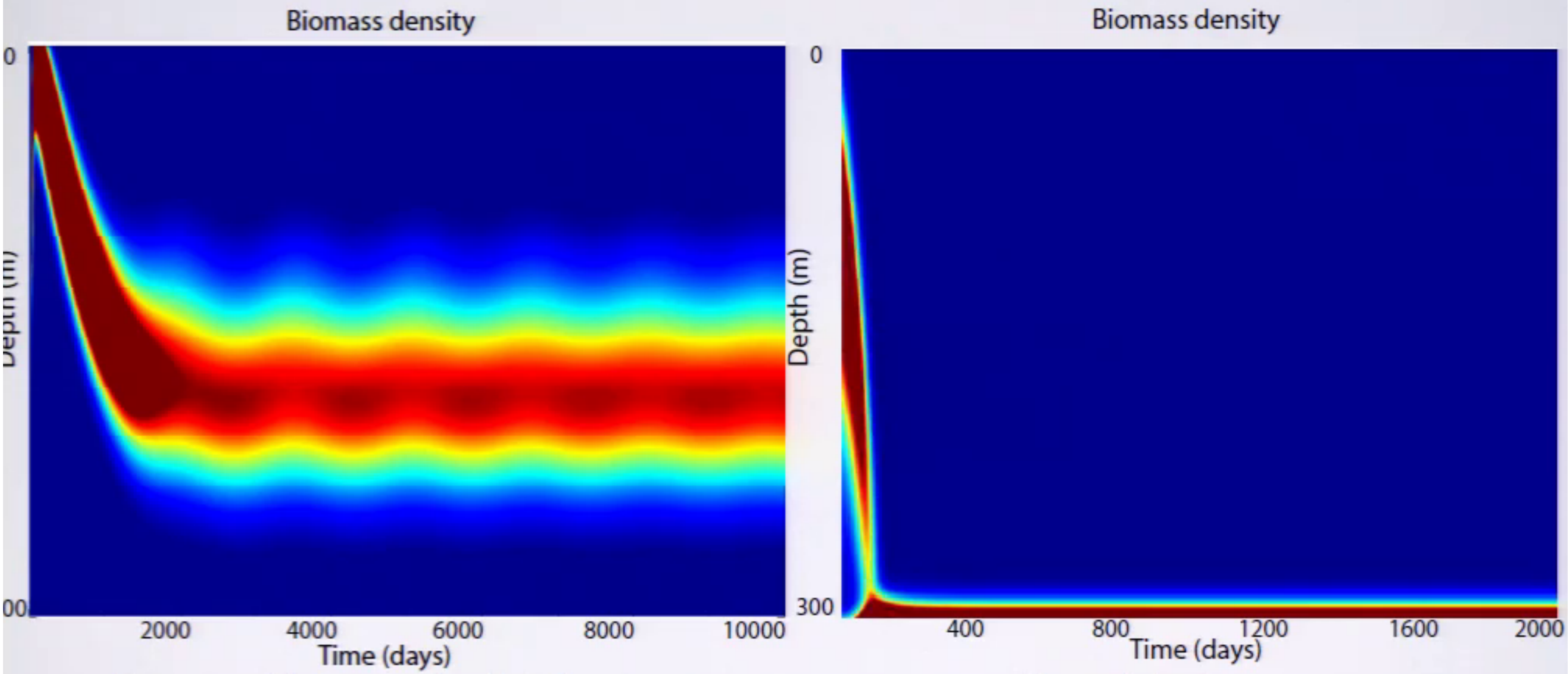
Nutrient $\eta_t = \varepsilon \left[\eta_{xx} + \ell^{-1} p(\omega, \eta, x)\omega\right];$ $\quad x \in \mathbb{R}$

$$p(\omega,\eta,x) = \frac{1-\eta}{(\eta_H + 1 - \eta)(1 + j_H / j(\omega, x))},$$

$$j(\omega, x) = e^{-\kappa x - r\int_0^x \omega(s,t)ds}.$$



PATTERN FORMATION



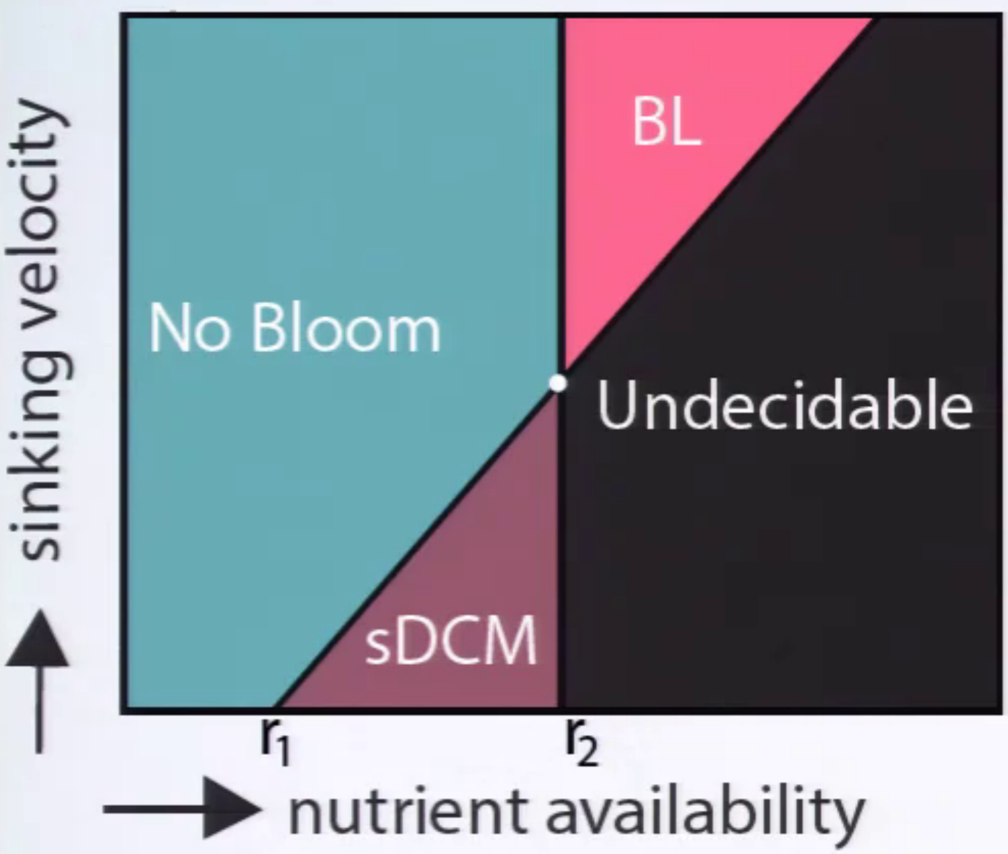
Deep Chlorophyll Maximum

Benthic Layer

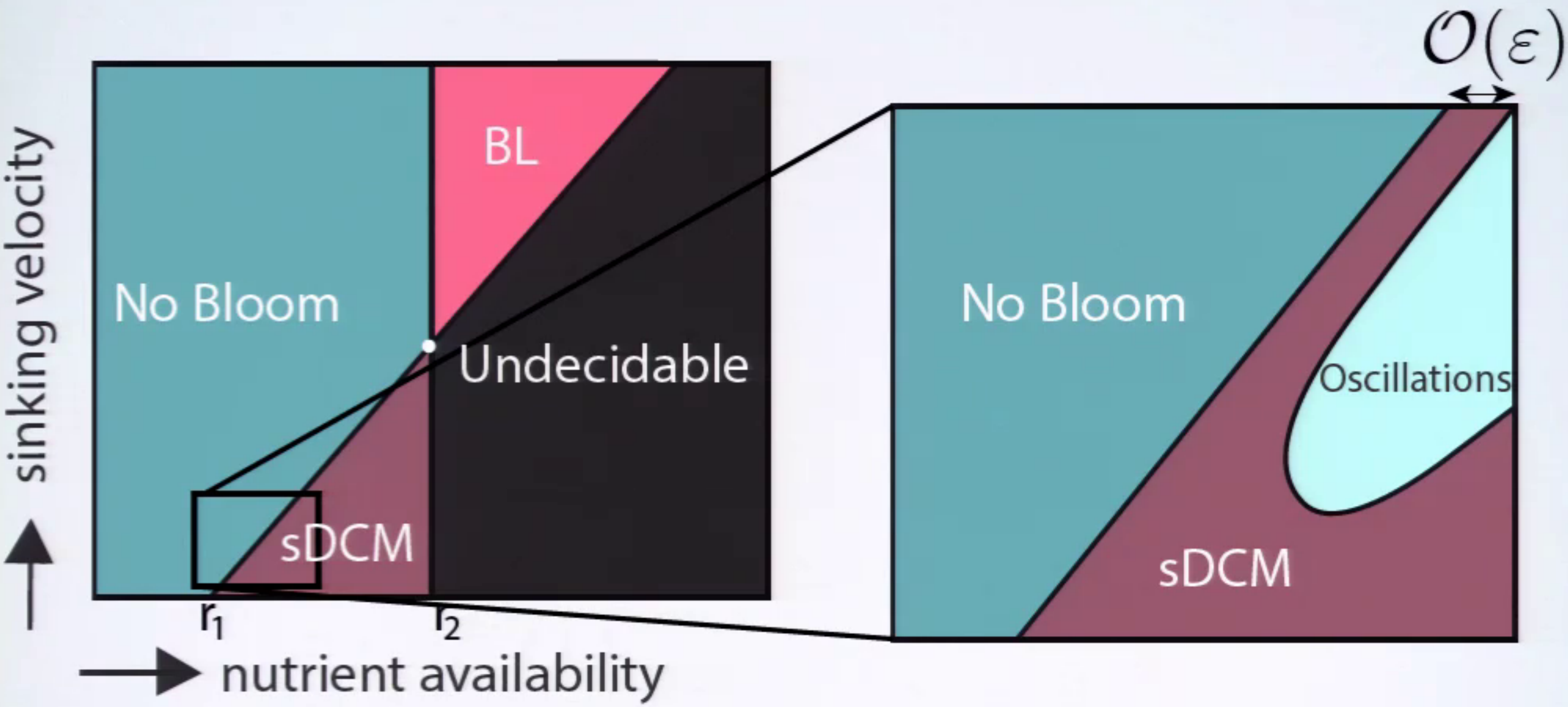
Low density

High density

LINEAR STABILITY & NUMERICS

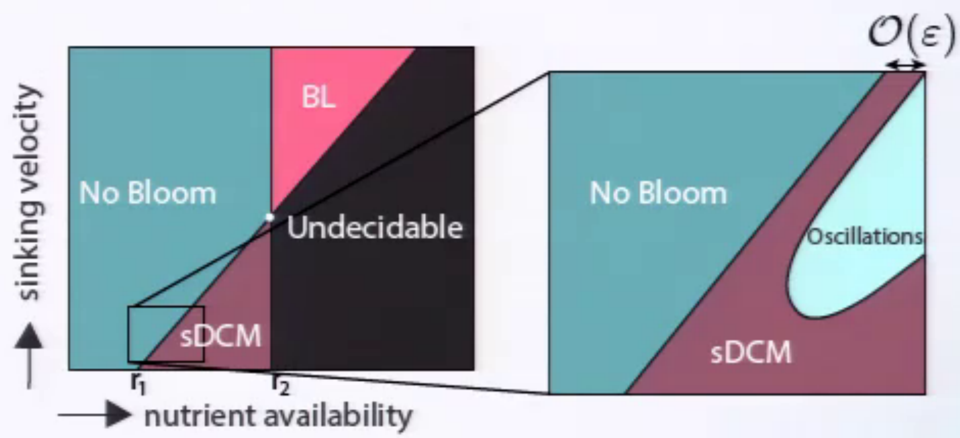


LINEAR STABILITY & NUMERICS



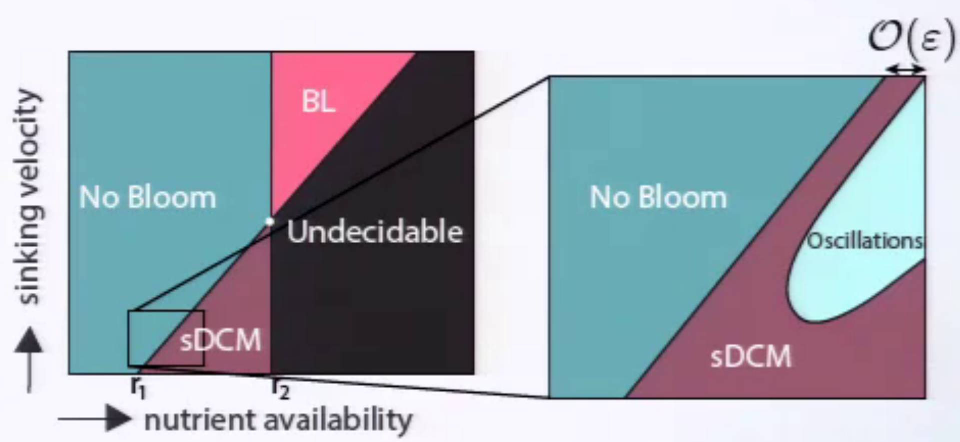
DEEP CHLOROPHYLL MAXIMUM

- Onset of patterns via a transcritical bifurcation.
- When bifurcation parameter is $\mathcal{O}(\varepsilon)$, a Hopf bifurcation occurs, indicating oscillations.
- (Numerically) That Hopf induces chaos through a cascade of period doubling bifurcations.

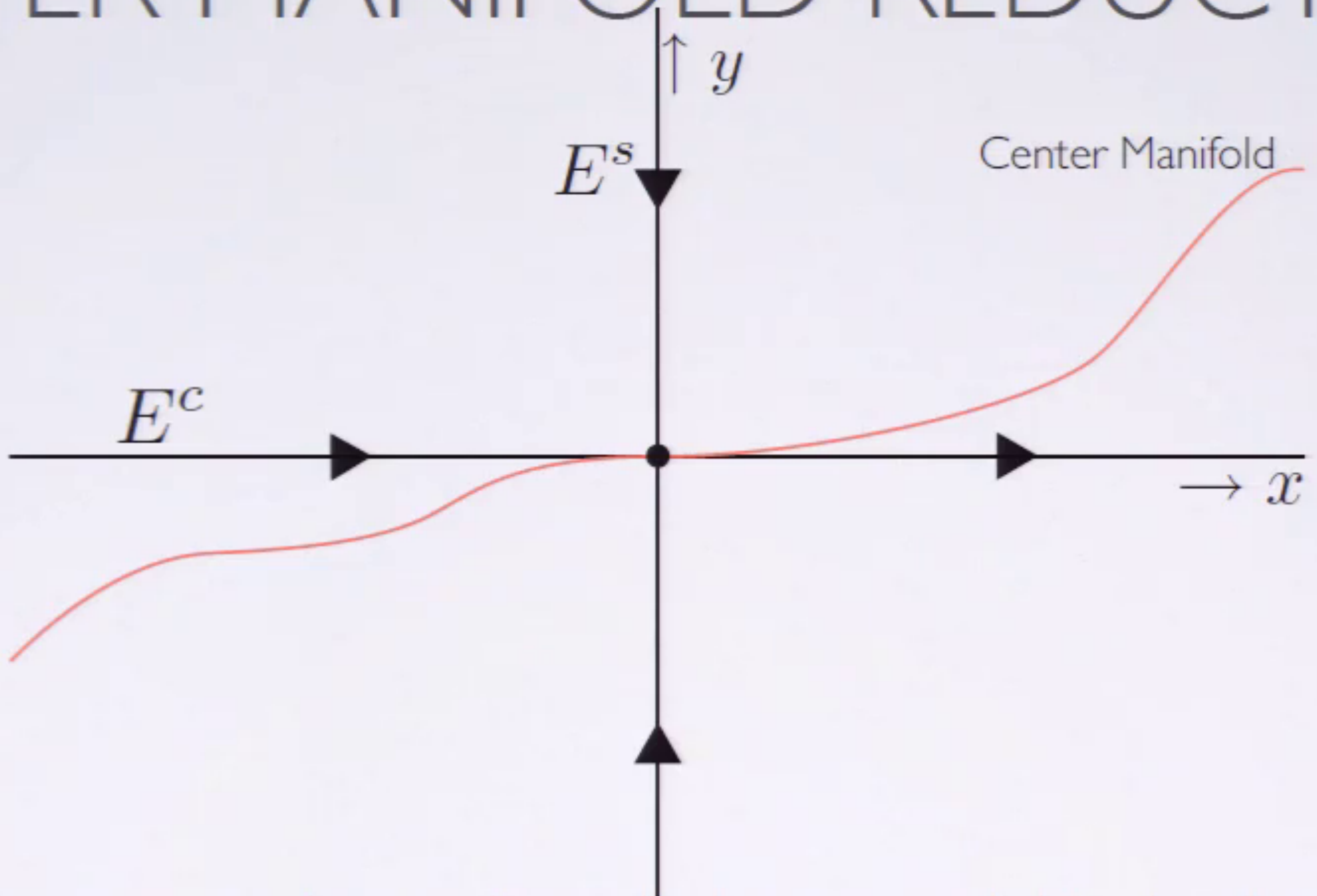


BENTHIC LAYER

- Onset of pattern via a transcritical bifurcation.
- Stationary Benthic Layer remains stable even for bifurcation parameter $\mathcal{O}(\varepsilon)$.
- Destabilization may occur closer to the codimension 2 point.



CENTER MANIFOLD REDUCTION



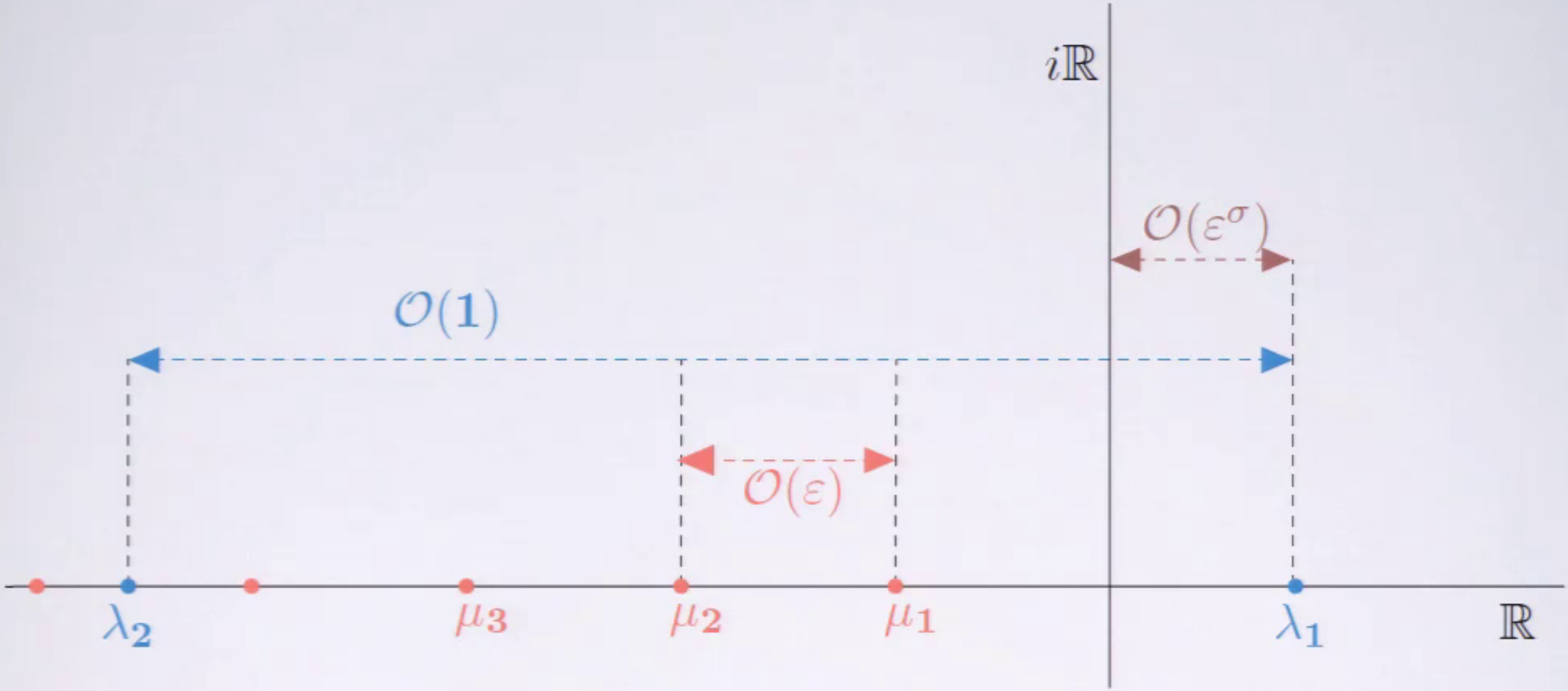
In case of a center eigenvalue, all the essential behavior is captured in the center manifold, whose dimension is equal to the number of center eigenvalues.

PARAMETER DEPENDENCE

$$\dot{x} = A(\mu)x + f(x, y, \mu), \qquad x \in \mathbb{R}^n$$
$$\dot{y} = B(\mu)x + g(x, y, \mu), \qquad y \in \mathbb{R}^m$$
$$\dot{\mu} = 0. \qquad \mu \in \mathbb{R}^k$$

'Theorem'

Suppose that for $\mu = 0$ the system has only stable and center eigenvalues. Then, there is a family of center manifolds parametrized by μ which captures all essential behavior for μ small enough.



- $\sigma > 1 \quad \Rightarrow \quad \varepsilon^\sigma \ll \varepsilon \quad \Rightarrow \quad \lambda_1 \ll \mu_k.$ OK!
- $\sigma = 1 \quad \Rightarrow \quad \varepsilon^\sigma = \varepsilon \quad \Rightarrow \quad \mathcal{O}(\lambda_1) = \mathcal{O}(\mu_k).$

  Center manifold reduction is infinite dimensional.

OUR APPROACH

- Fourier analysis and amplitude equations.

$$\omega(x,t) = \sum_{n\geq 1} A_n(t)\phi_n(x),$$

$$\eta(x,t) = \sum_{n\geq 1} B_n(t)\psi_n(x).$$

- What do we expect? A small pattern growing in the 'shape' of the bifurcating eigenfunction.

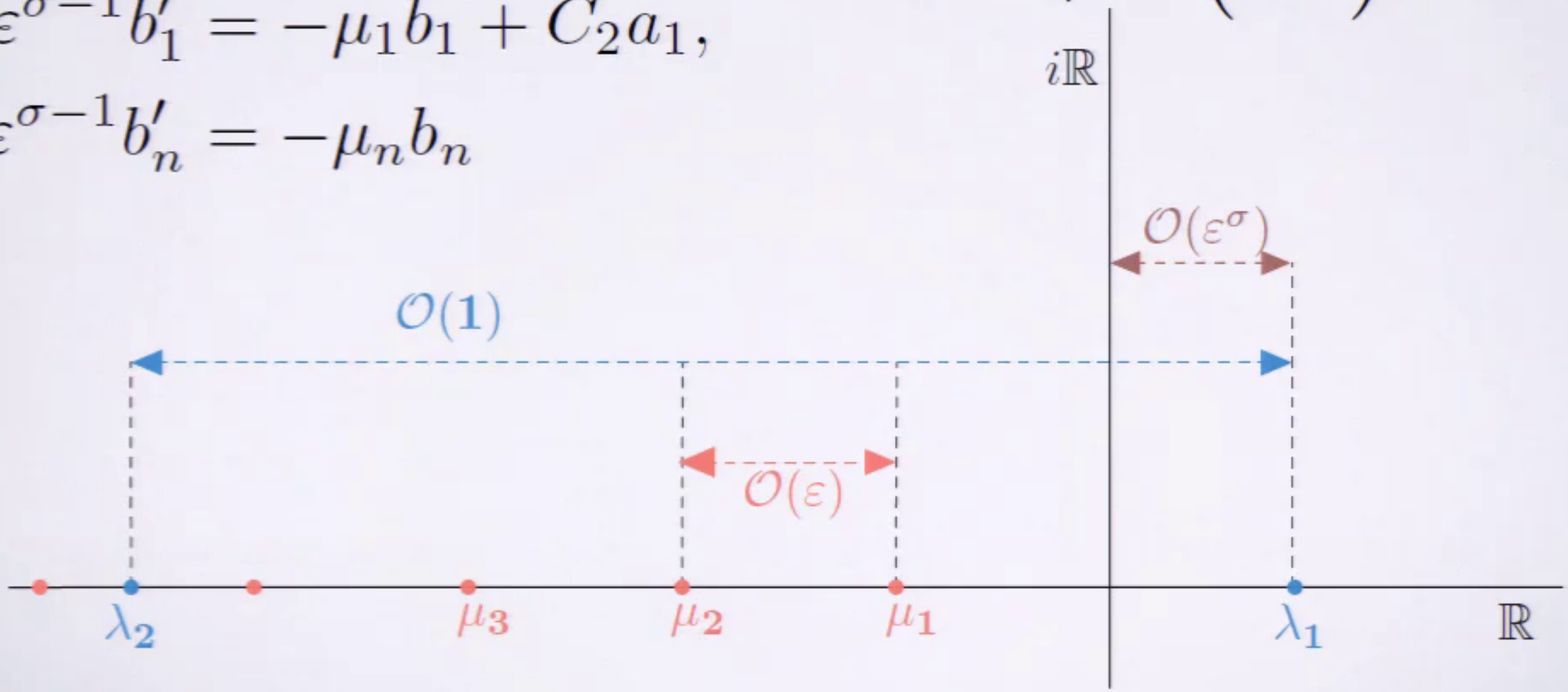
- Rescale amplitudes accordingly.

$$A_1 = \varepsilon a_1 \quad A_n = \varepsilon^2 a_n \quad B_n = \varepsilon b_n.$$

AMPLITUDE EQUATIONS

$$a_1' = \lambda_1 a_1 + C_1 a_1^2,$$
$$\varepsilon a_n' = \lambda_n a_n + \text{nonl. terms}$$
$$\varepsilon^{\sigma-1} b_1' = -\mu_1 b_1 + C_2 a_1,$$
$$\varepsilon^{\sigma-1} b_n' = -\mu_n b_n$$

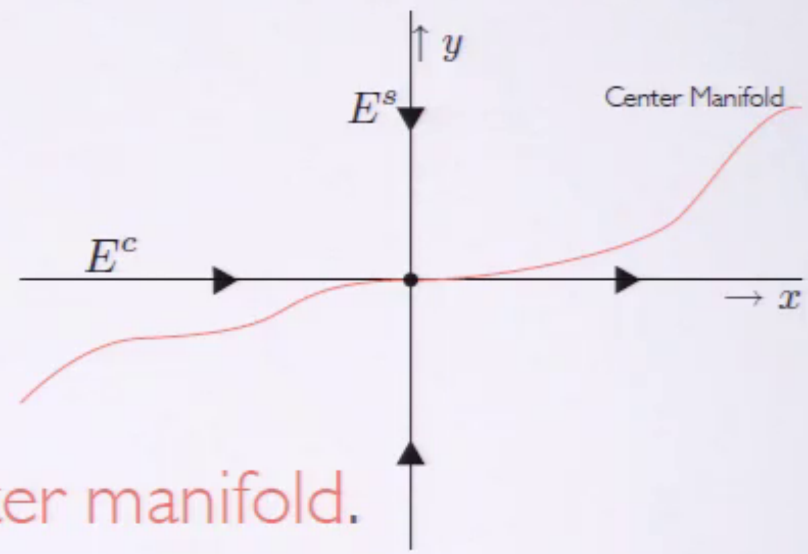
$$+\mathcal{O}(\varepsilon^\sigma)$$



CLASSICAL CMR $\sigma > 1$

$$
\begin{aligned}
a_1' &= \lambda_1 a_1 + C_1 a_1^2, \\
0 &= \lambda_n a_n + \text{nonl. terms} \\
0 &= -\mu_1 b_1 + C_2 a_1, \\
0 &= -\mu_n b_n
\end{aligned}
$$

$$+\mathcal{O}(\varepsilon^{\sigma})$$



- Algebraic equations define the center manifold.
- Dynamic equation captures behavior on the center manifold.

EXTENDED CMR $\sigma = 1$

$$
\begin{aligned}
a_1' &= \lambda_1 a_1 + C_1 a_1^2, \\
0 &= \lambda_n a_n + \text{nonl. terms} \\
b_1' &= -\mu_1 b_1 + C_2 a_1, \\
b_n' &= -\mu_n b_n
\end{aligned}
\qquad + \mathcal{O}(\varepsilon^\sigma)
$$

- Even for $\sigma = 1$ we know what captures the essential behavior.
- Formal approach allows for reduction beyond the standard center manifold regime.

EXTENDED CMR $\sigma = 1$

$$a_1' = \lambda_1 a_1 + C_1 a_1^2,$$

$$+\mathcal{O}(\varepsilon^\sigma)$$

$$b_1' = -\mu_1 b_1 + C_2 a_1,$$

The center manifold is extended to two dimensions, which is easily studied for bifurcations.

SUMMARY

- Derive formally leading order approximations of both types of patterns in the phytoplankton model, and analyze their stability.

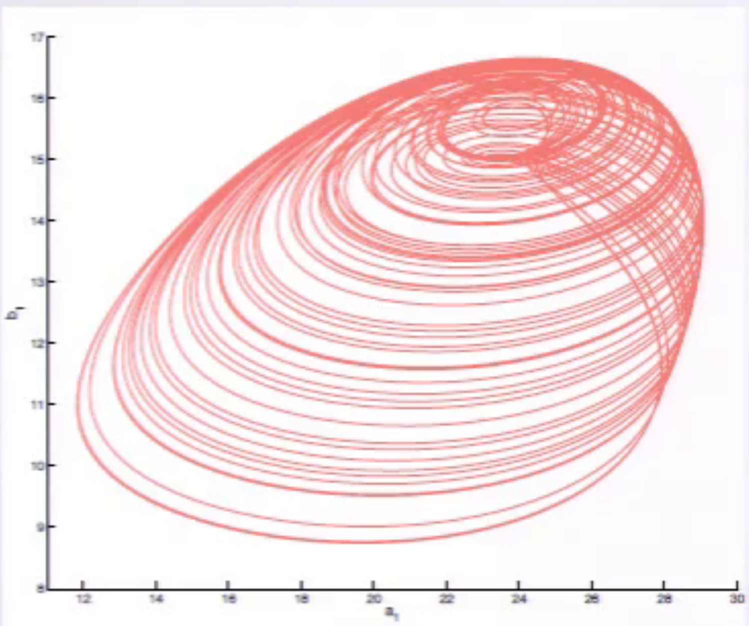
- The theory is applicable to a broad class of PDE systems.

$$u_t = \mathcal{L}u - f(x, u, v; \varepsilon)uv,$$
$$v_t = \varepsilon\mathcal{K}u - \varepsilon\mathcal{M}v - \varepsilon g(x, u, v; \varepsilon).$$

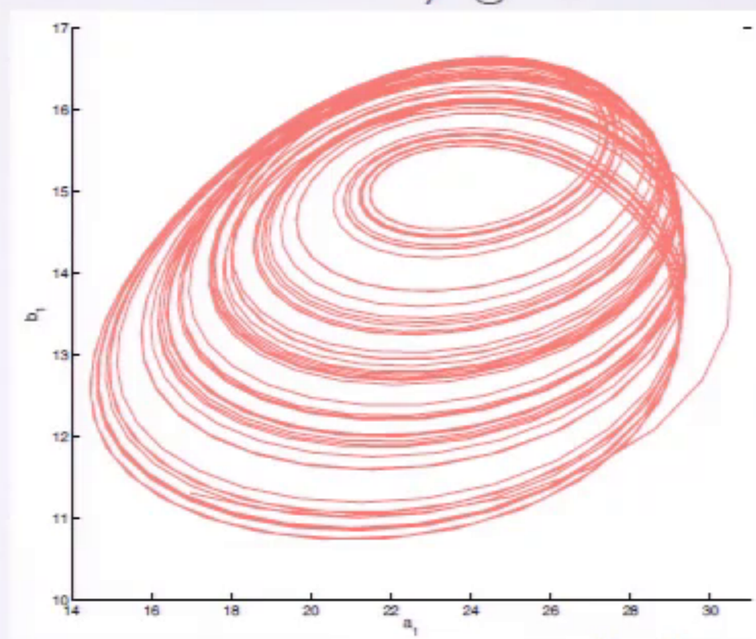
$\mathcal{L}, \mathcal{K}, \mathcal{M}$ differential operators

OPEN QUESTION: VALIDITY

- Our approach is formal, the persistence of the extended center manifold is not yet rigorously proved.
- Numerics indicate that the approximation is very good.



Reduction



Full PDE $\varepsilon = 0.1$