Global strong solution for the Korteweg system

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Let us recall the Korteweg system:

- Mass equation :

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0,$$

- Momentum equation :

$$\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu(\rho)D(u)) - \nabla(\lambda(\rho)\mathrm{div} u) + \nabla P(\rho) = \mathrm{div} K,$$

- Initial data :

$$(\rho, u)_{/t=0} = (\rho_0, u_0).$$

Here $u = u(t,x) \in \mathbb{R}^N$ stands for the velocity field, $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density and $D(u) = \frac{1}{2}(\nabla u +^t \nabla u)$ the strain tensor.

We denote by λ and μ the two viscosity coefficients of the fluid, which are assumed to satisfy $\mu > 0$ and $2\mu + N\lambda > 0$.

P is the pressure and we consider a pressure of the type $P(\rho) = a\rho^\gamma$ (with $a > 0$, $\gamma \geq 1$) and the general Korteweg tensor reads as follows:

$$\mathrm{div} K = \mathrm{div}\Big(\big(\rho\kappa(\rho)\Delta\rho + \frac{1}{2}(\kappa(\rho) + \rho\kappa'(\rho))|\nabla\rho|^2\big) Id - \kappa(\rho)\nabla\rho \otimes \nabla\rho\Big). \quad (1)$$

Here κ is the capillary coefficient and is regular far away from 0.

Remark

In the sequel we will consider the particular case of the quantum pressure $\kappa(\rho)=\frac{\kappa_1}{\rho}$ *with shallow water viscosity coefficients* $\mu(\rho)=\mu\rho\,,\;\lambda(\rho)=0\,.$

- **Notion of scaling**

  We can easily verify that, if (ρ,u) solves our system, so does (ρ_l,u_l), where

$$\begin{aligned}&(\rho_0(x),u_0(x))\longrightarrow(\rho_0(lx),lu_0(lx)),\quad\forall\,l\in\mathbb{R}.\\&(\rho(t,x),u(t,x),P(t,x))\longrightarrow(\rho(l^2t,lx),lu(l^2t,lx),l^2P(l^2t,lx)).\end{aligned}\tag{2}$$

  provided that the pressure term has been changed in l^2P.

  A functional space X is critical for the scaling of the equations if the norm $\|\cdot\|_X$ is invariant by the transformation (2) (Example: $B^{\frac{N}{p}}_{p,1}\times B^{\frac{N}{p}-1}_{p,1}$).

- **Energy estimates**

$$\begin{aligned}&\mathcal{E}(\rho,u)(t)=\int_{\mathbb{R}^N}\big(\frac{1}{2}\rho(t,x)|u(t,x)|^2+(\Pi(\rho)(t,x)-\Pi(\bar\rho))+\kappa_1|\nabla\sqrt{\rho}(t,x)|^2\big)dx\\&+\int_0^t\int_{\mathbb{R}^N}2\mu\,\rho(s,x)|Du|^2(s,x)dsdx\leq\int_{\mathbb{R}^N}\big(\rho_0(x)|u_0(x)|^2+\Pi(\rho_0)(x)\big)dx,\end{aligned}\tag{3}$$

  with $\Pi(\rho)=a\big(\rho\ln(\frac{\rho}{\bar\rho})+\bar\rho-\rho\big)$ and $\bar\rho>0$.

Some results of global weak solution

- R. Danchin and B. Desjardins [01], Existence of global strong with small initial data ($(\rho_0 - 1) \in B^{\frac{N}{p}}_{2,1}$, $\kappa(\rho) = \kappa$).
- D. Bresch, B. Desjardins and C-K. Lin [01], Stability of the global weak solution with test functions depending on the solution itself ($\kappa(\rho) = \kappa$).
- A. Jungel [10] Global weak solutions for the quantum pressure with test functions depending on the solution itself.
- BH [13], Global weak solution for particular choice on $\kappa(\rho)$ and $\mu(\rho)$.
- BH [14], Global strong solution with small initial data such that $\rho_0 - 1 \in H^{\frac{N}{2}}$. The initial density can be chosen discontinuous.
- A. Vasseur and C. Yu [15], Existence of global weak solution for quantum pressure with friction terms.

Existence of global strong solutions for Korteweg system with large initial data when $N \geq 2$

- In order to obtain such results, we have to provide additional energy estimates in order to have sufficiently regularity on our solutions.
- Particular Structure: When $\kappa_1 = \mu^2$, we can rewrite the Korteweg system as follows with $v = u + \mu\nabla \ln \rho$:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) - \mu \Delta \rho = 0, \\ \rho \partial_t v + \rho u \cdot \nabla v - \mu \mathrm{div}(\rho \nabla v) + \nabla P(\rho) = 0. \end{cases} \tag{4}$$

We have then an additional entropy, multiplying the momentum equation by v (see A. Jungel) we have:

$$\begin{aligned} &\mathcal{E}_1(\rho, v)(t) = \int_{\mathbb{R}^N} \big(\frac{1}{2}\rho(t,x)|v(t,x)|^2 + (\Pi(\rho)(t,x) - \Pi(\bar{\rho}))\big) dx \\ &+\mu \int_0^t \int_{\mathbb{R}^N} \rho(s,x)|\nabla v|^2(s,x) ds dx + a\mu \int_0^t \int_{\mathbb{R}^N} \frac{P'(\rho)|\nabla \rho|^2(s,x)}{\rho(s,x)} dx ds \\ &\leq \int_{\mathbb{R}^N} \big(\rho_0(x)|v_0(x)|^2 + (\Pi(\rho_0)(x) - \Pi(\bar{\rho})\big) dx. \end{aligned} \tag{5}$$

It implies the following estimates.

Proposition

- $(\sqrt{\rho} - \sqrt{\bar{\rho}}) \in L^\infty(H^1(\mathbb{R}^N))$, $\sqrt{\rho} u \in L^\infty(L^2(\mathbb{R}^N))$
- $\sqrt{\rho} \nabla u \in L^2(L^2(\mathbb{R}^N))$, $\sqrt{\rho} \Delta \ln \rho \in L^2(L^2(\mathbb{R}^N))$.

Definition

We denote $q_1 = \rho - \bar{\rho}$ and $q = \ln(\frac{\rho}{\bar{\rho}})$.

Theorem

Let $N \geq 2$ *with* $\kappa_1 = \mu^2$. *Let* $(q_0, v_0) \in B^{\frac{N}{p}}_{p,1} \times B^{\frac{N}{p}-1}_{p,1}$ *with* $1 \leq p < 2N$ *and* $0 < c \leq \rho_0$, *then there exists a time* T *such that system (4) has a unique solution on* $[0, T]$ *with:*

$$q \in \widetilde{C}_T(B^{\frac{N}{p}}_{p,1}) \cap L^1_T(B^{\frac{N}{p}+2}_{p,1}),\ \frac{1}{\rho}, \rho \in L^\infty_T(L^\infty(\mathbb{R}^N))\ v \in \widetilde{C}_T(B^{\frac{N}{p}-1}_{p,1}) \cap L^1_T(B^{\frac{N}{p}+1}_{p,1}). \tag{6}$$

Now assume that $(q_0, v_0) \in (B^{\frac{N}{p}}_{p,1} \cap B^{\frac{N}{p}+\varepsilon}_{p,1}) \times (B^{\frac{N}{p}-1}_{p,1} \cap B^{\frac{N}{p}-1+\varepsilon}_{p,1})$ *with* $\varepsilon > 0$ *then there exists* $C, C_1, c > 0$ *such that:*

$$T \geq \inf\Big(\frac{2(c\mu)^{\frac{2}{\varepsilon'}-1}\varepsilon^{\frac{2}{\varepsilon'}}}{(8C)^{\frac{2}{\varepsilon'}}\|q_0\|^{\frac{2}{\varepsilon'}}_{B^{\frac{N}{p}+\varepsilon'}_{p,1}}}, \frac{2(c\mu)^{\frac{2}{\varepsilon'}-1}\varepsilon^{\frac{2}{\varepsilon'}}}{(8C)^{\frac{2}{\varepsilon'}}\|v_0\|^{\frac{2}{\varepsilon'}}_{B^{\frac{N}{p}-1+\varepsilon'}_{p,1}}}, \frac{C_1}{4}, \frac{1}{16C_1^2(\|q_0\|_{B^{\frac{N}{p}}_{p,1}} + \|v_0\|_{B^{\frac{N}{p}-1}_{p,1}})(1 + \sqrt{\|q_0\|_{B^{\frac{N}{p}}_{p,1}} + \|v_0\|_{B^{\frac{N}{p}-1}_{p,1}}})^2}\Big). \tag{7}$$

Remark

The theorem of existence is not new (see R. Danchin and B. Desjardins). However we provide a new estimate (7) on the time of existence (for slightly subcritical initial data for the scaling of the equation). This estimate (7) will play an important role in the next theorem in order to extend the solution beyond T^* the maximal time of existence.

Let us define now the maximal time T^* of existence:

$$T^* = \sup\{T \in \mathbb{R}\,;\text{there exists a strong solution } (q,u) \text{ on } [0,T] \text{ verifying } (6)\}.$$

Theorem

Let $N \geq 2$, $\gamma = 1$, $\kappa_1 = \mu^2$. Let $(q_0, v_0) \in (B^{\frac{N}{p}}_{p,1} \cap B^{\frac{N}{p}+\varepsilon}_{p,1}) \times (B^{\frac{N}{p}-1}_{p,1} \cap B^{\frac{N}{p}-1+\varepsilon}_{p,1})$ with $\varepsilon > 0$ such that $\frac{N}{1-\varepsilon} < p < 2N$ and $0 < c \leq \rho_0$. In addition we assume that $v_0 \in L^\infty$ and:

$$\mathcal{E}(\rho_0, u_0) < +\infty, \;\; \mathcal{E}_1(\rho_0, v_0) < +\infty,$$

then we have $T^* = +\infty$. Furthermore the global solution so defined (ρ, v) is unique and verifies locally in time (6).

Remark

The important point to note is the form of the viscosity and capillary coefficients $\kappa_1 = \mu^2$. It corresponds to an intermediary regime if we consider a vanishing limit process for viscosity and capillary coefficients.
The fact that $P(\rho) = a\rho$ is also important in order to get a gain of integrability on v.

Idea of the Proof: We will denote by (q_L, v_L) the solution of linearized system:

$$\begin{cases} \partial_t q_L + \mathrm{div} v_L - \mu\Delta q_L = 0, \\ \partial_t v_L - \mu\Delta v_L = 0, \\ (q_L(0), v_L(0)) = (q_0, v_0). \end{cases} \tag{8}$$

We now define a sequence (q^n, v^n) as follows $q^n = q_L + \bar{q}^n$, $v^n = v_L + \bar{v}^n$, with $(\bar{q}_n, \bar{v}_n)$ solution of the following system:

$$\begin{cases} \partial_t \bar{q}^n + \mathrm{div}(\bar{v}^n) - \mu\Delta\bar{q}^n = F_{n-1} = -v_{n-1}\cdot\nabla q^{n-1} + \mu|\nabla q^{n-1}|^2, \\ \partial_t \bar{v}_n - \mu\Delta\bar{v}_n = G_{n-1} = -u^{n-1}\cdot\nabla v^{n-1} + \mu\nabla q^{n-1}\cdot\nabla v^{n-1} - a\nabla q^{n-1}, \\ (\bar{q}_n, \bar{u}_n)_{t=0} = (0,0), \end{cases}$$

We verify then that (q^n, v^n) converges to (q, v) a solution of the Korteweg system in $F_T = (\widetilde{C}([0,T], B^{\frac{N}{p}}_{p,1}) \cap \widetilde{L}^1_T(B^{\frac{N}{p}+2}_{p,1})) \times (\widetilde{C}([0,T], B^{\frac{N}{p}-1}_{p,1}) \cap \widetilde{L}^1_T(B^{\frac{N}{p}+1}_{p,1}))^N$.

In order to prove uniform estimates on (q^n, v^n) in F_T we need to choose T sufficiently small such that for $\varepsilon > 0$ sufficiently small:

$$\|q_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p}+2}_{p,1})} \leq C\big(\sum_{q\in\mathbb{Z}} 2^{q\frac{N}{p}} \|\Delta_q v_0\|_{L^p}\big(\frac{1-e^{-c\mu T 2^{2q}}}{c\mu}\big)\big) \leq \varepsilon. \tag{9}$$

Splitting q_0 in low and high frequencies, we can conclude:

$$\sum_{q\in\mathbb{Z}} 2^{q\frac{N}{p}} \|\Delta_q q_0\|_{L^p}\big(\frac{1-e^{-c\mu T 2^{2q}}}{c\mu}\big) \lesssim \sum_{q\leq l_0} 2^{q\frac{N}{p}} \|\Delta_q q_0\|_{L^p} T 2^{2q} + \frac{2}{c\mu}\sum_{q\geq l_0} 2^{q\frac{N}{p}} \|\Delta_q q_0\|_{L^p},$$

$$\leq (2^{(2-\varepsilon')l_0} T + \frac{2}{c\mu} 2^{-l_0\varepsilon'})\|q_0\|_{B^{\frac{N}{p}+\varepsilon'}_{p,1}}.$$

Additional energy estimates:

Lemma

Let (ρ, v) be our strong solution on $(0, T^*)$, then there exists C an increasing function depending only on the initial data such that for all $T \in (0, T^*)$ we have for all $p \in [2, +\infty)$:

$$\|\rho^{\frac{1}{p}} v(T, \cdot)\|_{L^p} \leq C(T). \tag{10}$$

- It suffices to multiply the momentum equation by $v|v|^{p-2}$ for any $p \geq 2$ and integrate over $(0,T) \times \mathbb{R}^N$. We conclude by using bootstrap argument and Gronwall inequality.
  The fact that $P(\rho) = a\rho$ is important here in order to end up the estimates.

Proposition

Under the assumption of theorem 3, the density ρ verifies for any $T \in (0,T^*)$:

$$\|\frac{1}{\rho}\|_{L^\infty_T(L^\infty(\mathbb{R}^N))} \leq C(T), \tag{11}$$

with C an increasing function in T.

We recall that ρ verifies the equation:

$$\partial_t \rho - \mu\Delta\rho + \mathrm{div}(\rho v) = 0.$$

- We can use De Giorgi technics (or a type of maximum principle) in order to prove that $\frac{1}{\rho}$ is bounded in $L^\infty_T(L^\infty)$.

Multiplying the previous equation by $(\rho^{-\alpha})^{(k)} = \max(\rho^{-\alpha}(t,x) - k, 0)$ and integrate over $(0,t_1) \times \mathbb{R}^N$, it gives with $\alpha > 0$:

$$\frac{1}{2}\int_{\mathbb{R}^N}[(\rho^{-\alpha})^{(k)}(t_1,x)]^2dx+\mu\int_0^{t_1}\int_{A_k(t)}|\nabla\rho^{-\alpha}(t,x)|^2dxdt$$
$$\lesssim\int_0^{t_1}\int_{A_k(t)}|v_i\partial_i(\rho^{-\alpha})(\rho^{-\alpha})^{(k)}(t,x)|dxdt+\int_0^{t_1}\int_{A_k(t)}|\rho^{-\alpha}v_i\partial_i(\rho^{-\alpha})|dxdt.$$

with:

$$A_k(t)=\{x\in\mathbb{R}^N;\ \frac{1}{\rho^{\alpha}(t,x)}\geq k\}=\{x\in\mathbb{R}^N;\ 0\leq\rho(t,x)\leq\frac{1}{k^{\frac{1}{\alpha}}}\}.$$

Setting:

$$|(\rho^{-\alpha})^{(k)}|^2_{Q_{t_1}(k)}=\sup_{0\leq t\leq t_1}\|(\rho^{-\alpha})^{(k)}(t)\|_{L^2}+\|\nabla(\rho^{-\alpha})^{(k)}\|_{L^2_{t_1}(L^2(\mathbb{R}^N))}.$$

we prove that for $0<t_1<T^*$ sufficiently small:

$$|(\rho^{-\alpha})^{(k)}|_{Q_{t_1}(k)}\leq\sqrt{C_{\alpha,\mu}}\|\frac{1}{\rho}\|^{\frac{1}{2q}}_{L^\infty_{t_1}(L^\infty)}\|\rho^{\frac{1}{2q}}v\|_{L^\infty_{t_1}(L^{2q}(\mathbb{R}^N))}t_1^{\frac{1}{r}}k\mu(k)^{\frac{1}{r_1}},\qquad(12)$$

with r, r_1 well chosen.

- We can estimate t_1 in function of the initial data.

We conclude using a suitable bootstrap argument and De Giorgi, Ladyzenskaya arguments by repeating the procedure for a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n>T^*$ for n large enough.

How to estimate $\|q_1(T,\cdot)\|_{B^{\frac{N}{p}}_{p,1}\cap B^{\frac{N}{p}+\varepsilon'}_{p,1}}$ for $0<T<T^*$ and $\varepsilon'>0$.

We recall that:

$$\partial_t q_1 - \mu\Delta q_1 = -\mathrm{div}(\rho v).$$

Classical estimates on heat equation provide:

$$\|q_1\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}+\varepsilon'}_{p,1})} \leq C(\|q_0^1\|_{B^{\frac{N}{p}+\varepsilon'}_{p,1}} + \|\rho v\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}-1+\varepsilon'}_{p,1})}).$$

- Since we have proved that for $p\geq 2$:

$$\|\rho^{\frac{1}{p}}v\|_{L^\infty_T(L^p)} \leq C(T) \ \text{ and } \ \|\frac{1}{\rho^{\frac{1}{p}}}\|_{L^\infty_T(L^\infty)} \leq C_1^{\frac{1}{p}}(T), \tag{13}$$

  by Besov embedding we observe that for $p\geq 2$ there exists $C>0$ such that:

$$\|v\|_{L^\infty_T(B^0_{p,\infty})} \leq CC(T)C_1^{\frac{1}{p}}(T) \ \text{ and } \ \|v\|_{L^\infty_T(B^{-N(\frac{1}{2}-\frac{1}{p})}_{p,\infty})} \leq CC(T)C_1^{\frac{1}{2}}(T). \tag{14}$$

- It implies via paraproduct law that:

$$\|\rho v\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}-1+\varepsilon'}_{p,1})} \leq \|\rho\|^{1-\frac{1-\varepsilon'}{N}}_{L^\infty_T(L^\infty)} M_1(T). \tag{15}$$

- Using the previous estimate we have:

$$\|q_1\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}+\varepsilon'}_{p,1})} \leq C(\|q_0^1\|_{B^{\frac{N}{p}+\varepsilon'}_{p,1}} + (\|q_1\|_{L^\infty_T(L^\infty)} + \bar{\rho})^{1-\frac{1-\varepsilon'}{N}} M_1(T)). \tag{16}$$

From Besov embedding and interpolation, we know that there exists $C, C' > 0$ such that:

$$\|q_1\|_{L^\infty_T(L^\infty)} \leq C\|q_1\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})} \leq C'\|q_1\|^{\theta}_{\widetilde{L}^\infty_T(B^{-N(\frac{1}{2}-\frac{1}{p})}_{p,\infty})} \|q_1\|^{1-\theta}_{\widetilde{L}^\infty_T(B^{\frac{N}{p}+\varepsilon'}_{p,\infty})}, \tag{17}$$

for $0 < \theta < 1$ and we conclude by bootstrap.

Assume that $T^* < +\infty$, we have proved that for C, C_1, C_2 increasing continuous functions, we have using composition theorem that for any $0 < T < T^*$:

$$\|q_1(T,\cdot)\|_{B^{\frac{N}{p}}_{p,1} \cap B^{\frac{N}{p}+\varepsilon'}_{p,1}} \leq C(T)$$

$$\|v(T,\cdot)\|_{B^{\frac{N}{p}-1}_{p,1} \cap B^{\frac{N}{p}-1+\varepsilon'}_{p,1}} \leq C_1(T)$$

$$\|\frac{1}{\rho}\|_{L^\infty_T(L^\infty)} \leq C_2(T).$$

We can now extend our solution (q, v) beyond T^*.

- Taking α small enough we can consider the existence of strong solution in finite time for initial data $(q(T^* - \alpha, \cdot), v(T^* - \alpha, \cdot))$. Using (7), we can show that the strong solution exists on a time interval $(0, T_1)$ with $T_1 > \alpha$. By uniqueness of the solution we can then extend our initial solution beyond T^*. It concludes the proof of our theorem.

Perspectives:

- What happens for general capillarity and viscosity?

THANK YOU FOR YOUR ATTENTION!!!