The Polynomial Atlas Method

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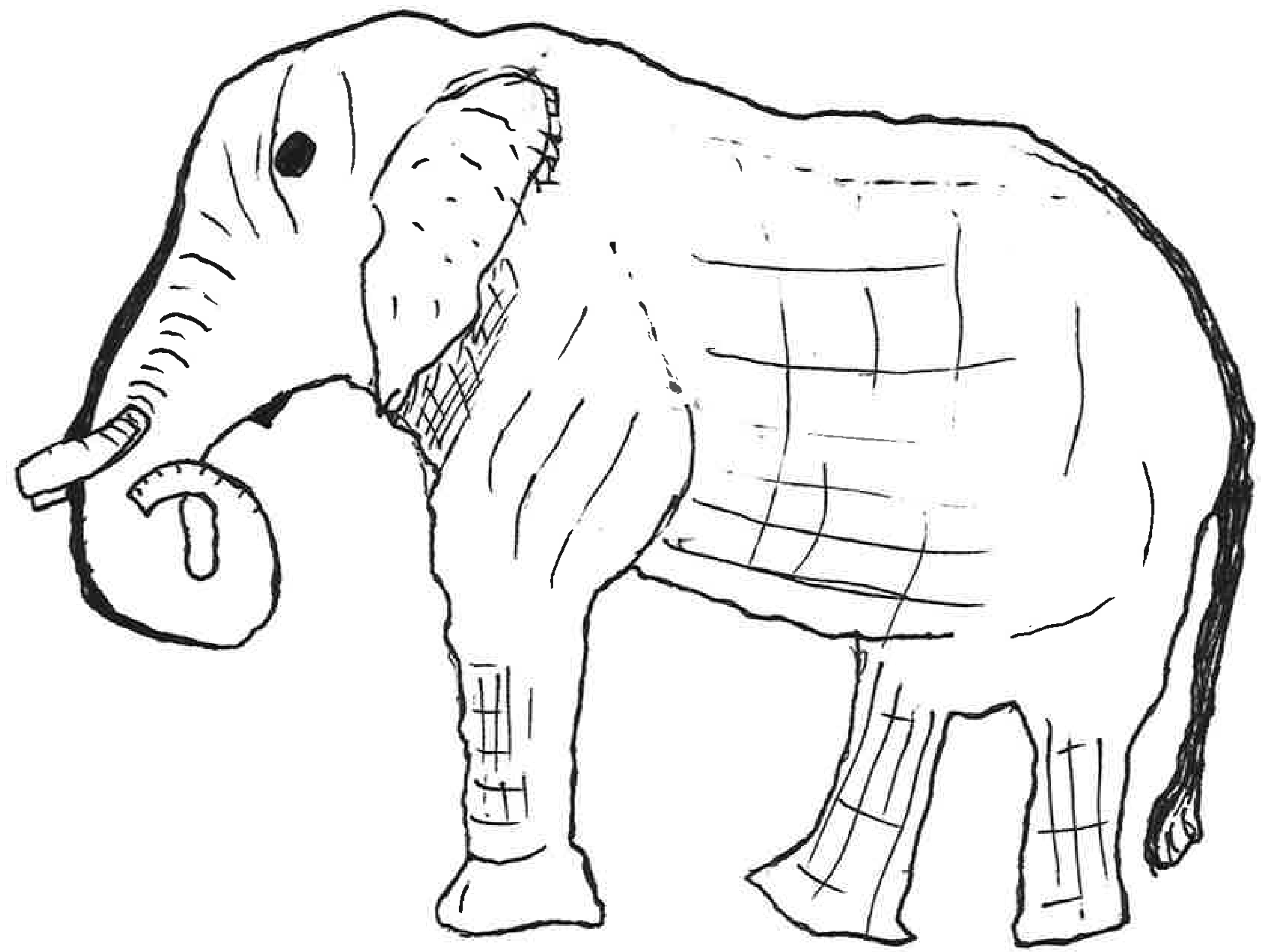
SIAM - LS
August 9, 2018



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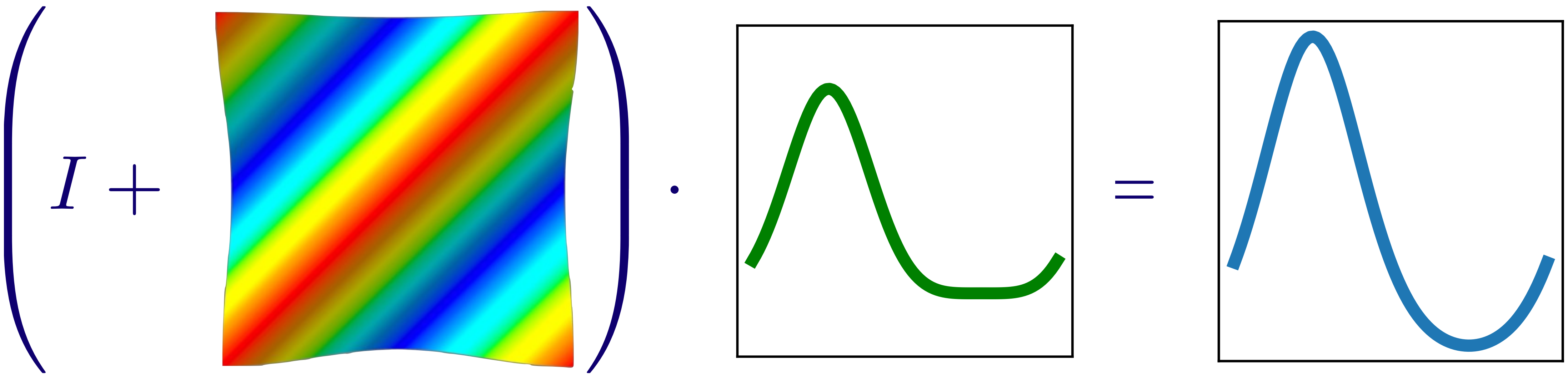
One Dimension: Periodic Model Problem



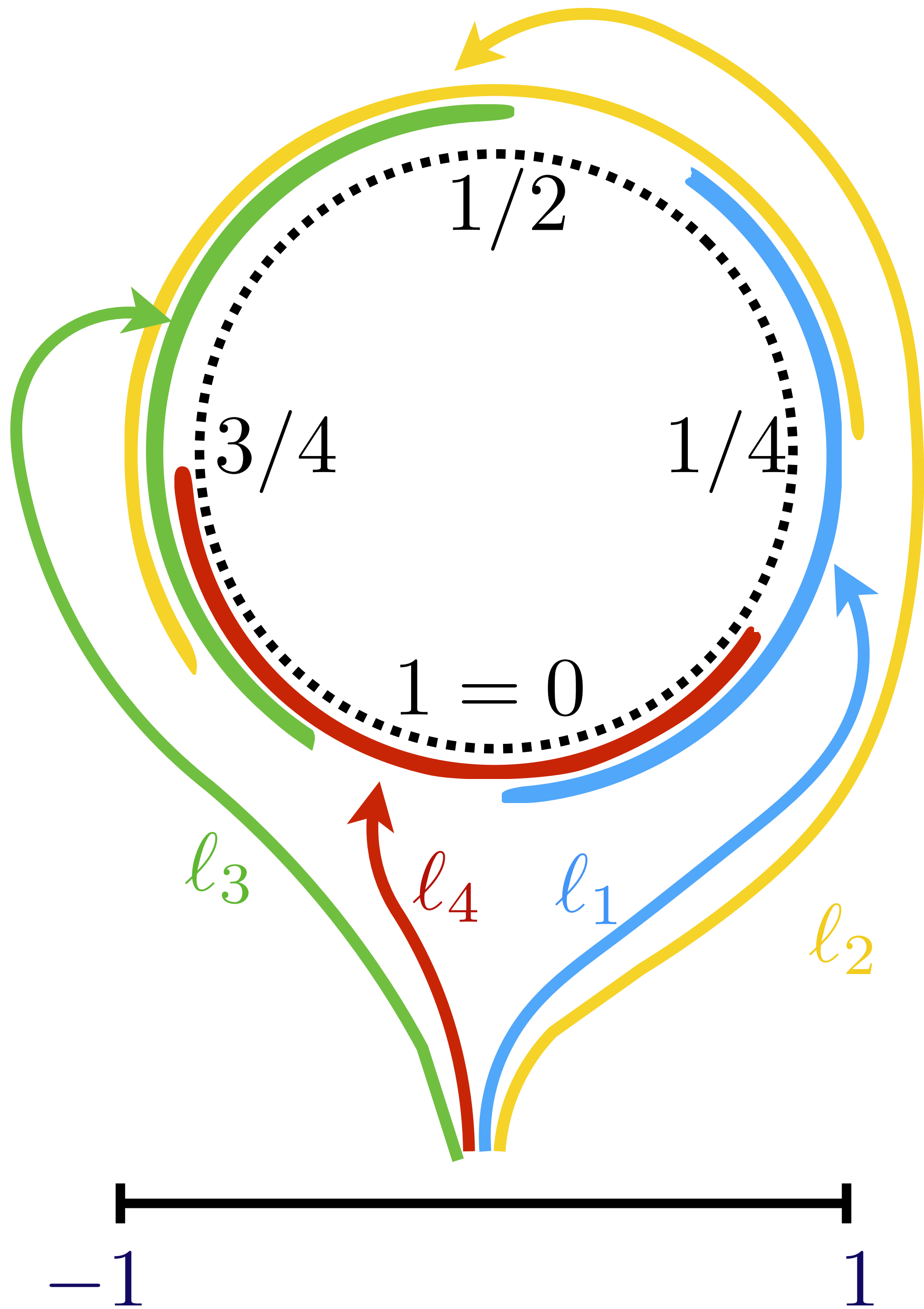
Find $f(x)$ on $[0,1]$ such that

$$f(x) + \int_0^1 \cos\left(2\pi(x-y)\right) f(y)\, dy = \exp\left(\sin(2\pi x)\right)$$

for each $x \in [0,1]$.

Exact solution: $f(x) = \exp\left(\sin(2\pi x)\right) - \frac{2}{3} I_1(1)\sin(2\pi x)$

Four-Chart Circle Atlas



$$\ell_1 : [-1, 1] \to [0, 0.4]$$
$$\ell_2 : [-1, 1] \to [0.25, 0.8]$$
$$\ell_3 : [-1, 1] \to [0.5, 0.9]$$
$$\ell_4 : [-1, 1] \to [0.75, 1] \cup [0, 0.15]$$

Communication: t_i^m: “up / back”

$$\ell_i(t_i^m(x)) = \ell_m(x)$$

$$t_i^m(x) = \ell_i^{-1}(\ell_m(x))$$

$$t_2^1(0.75) \approx$$

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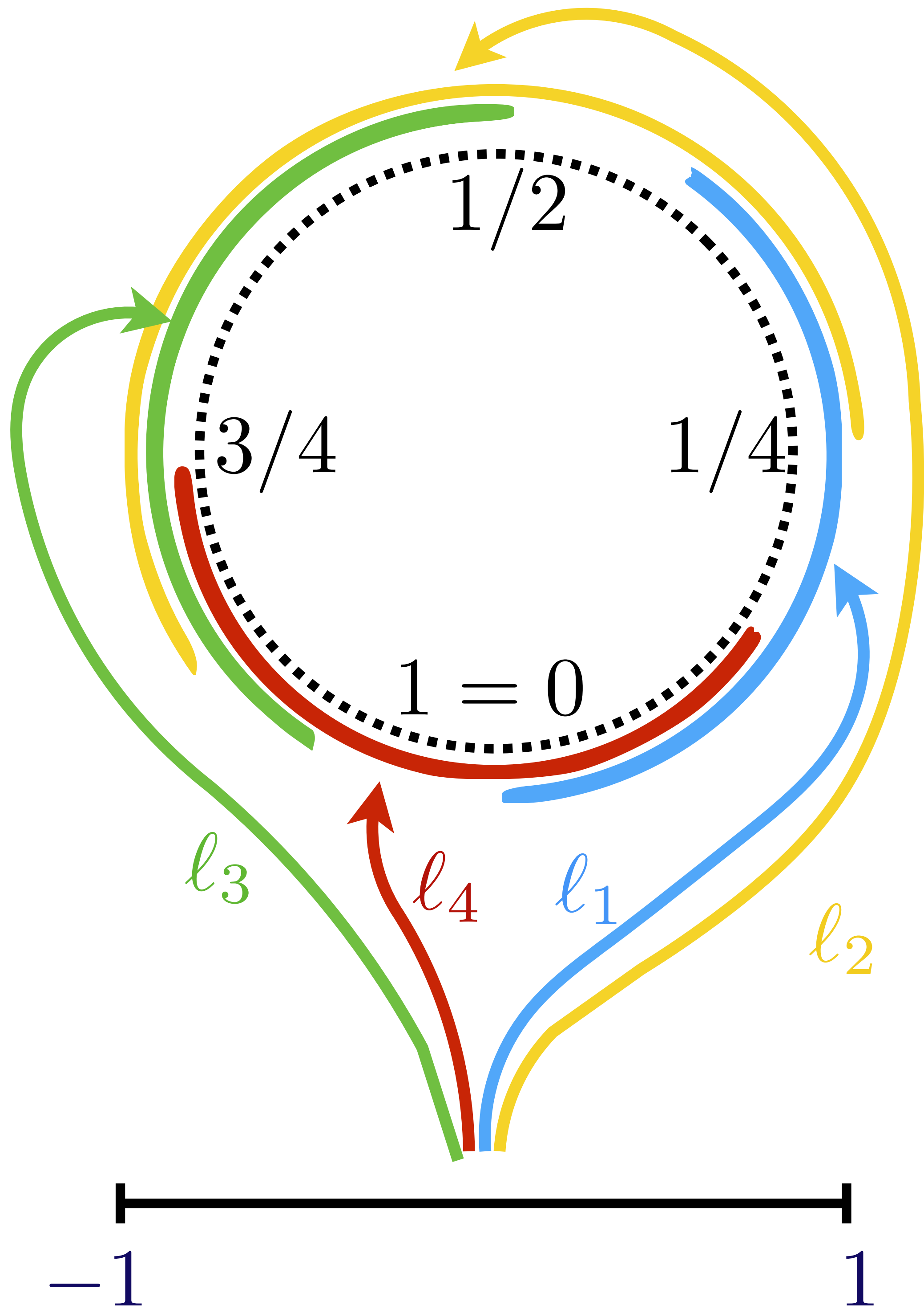

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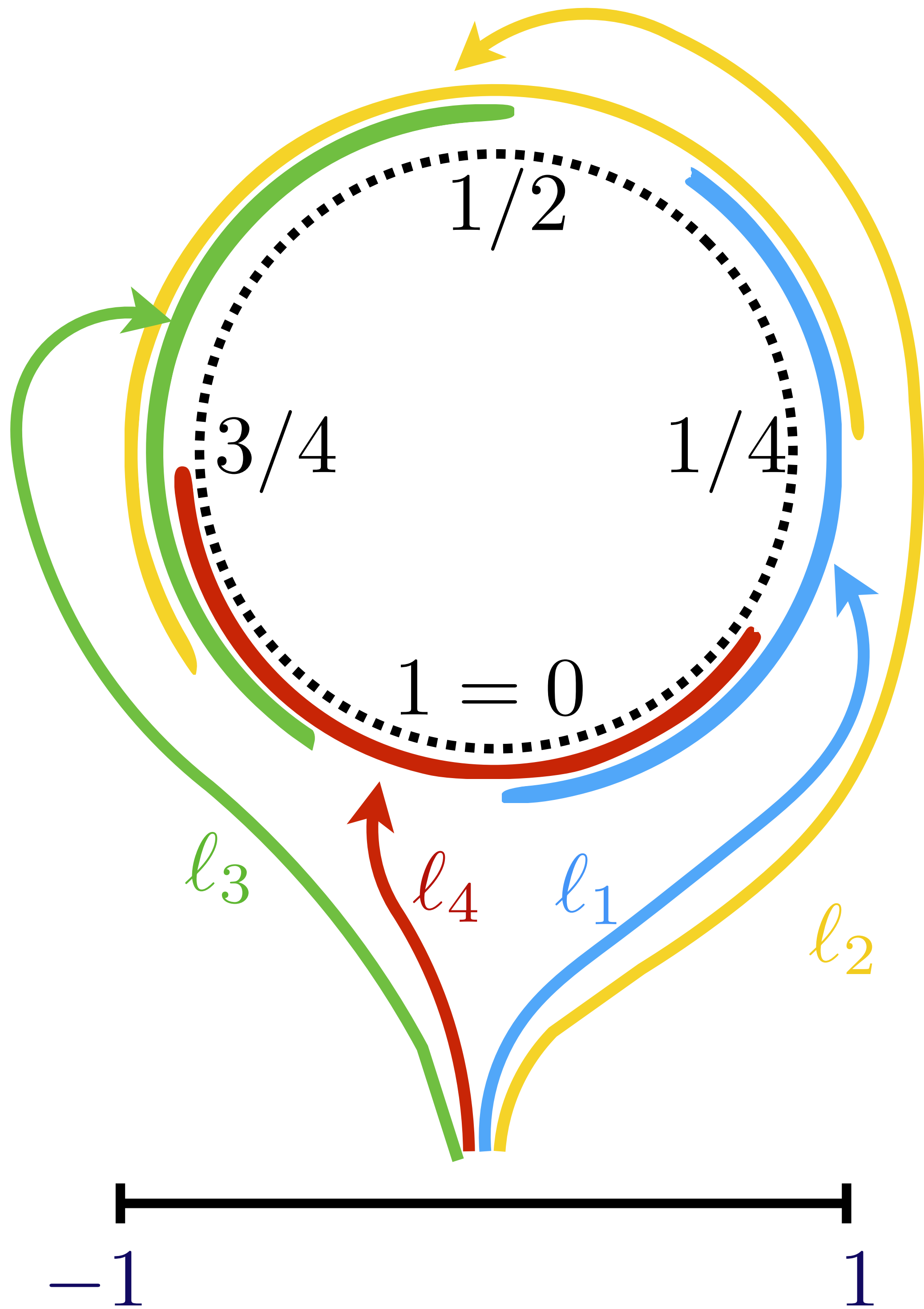

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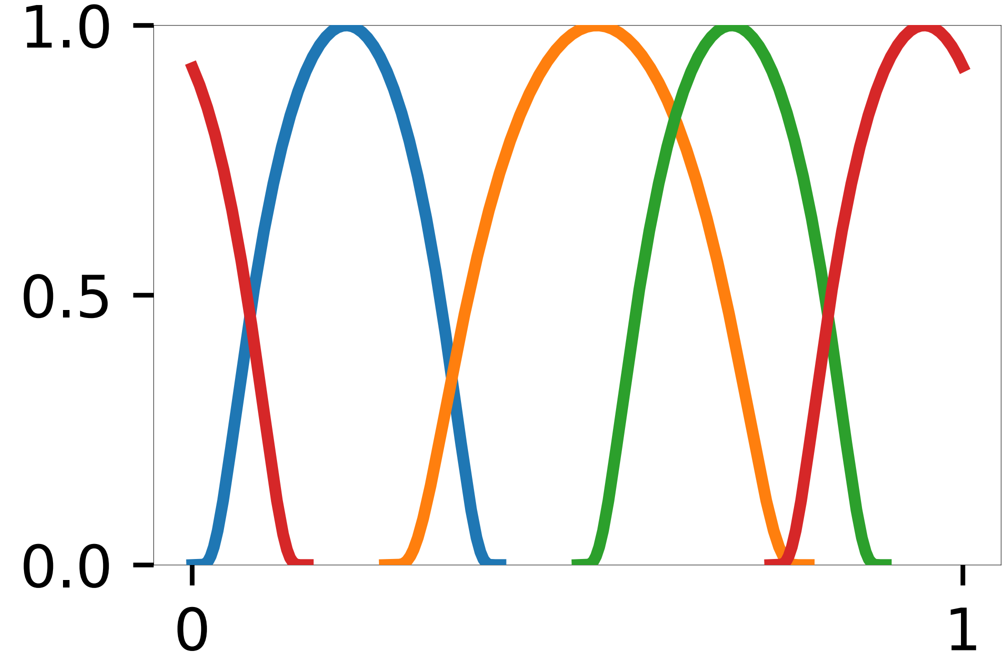
$$t_2^1(0.75) \approx -0.64$$

Four-Chart Circle Atlas: *Poutine*

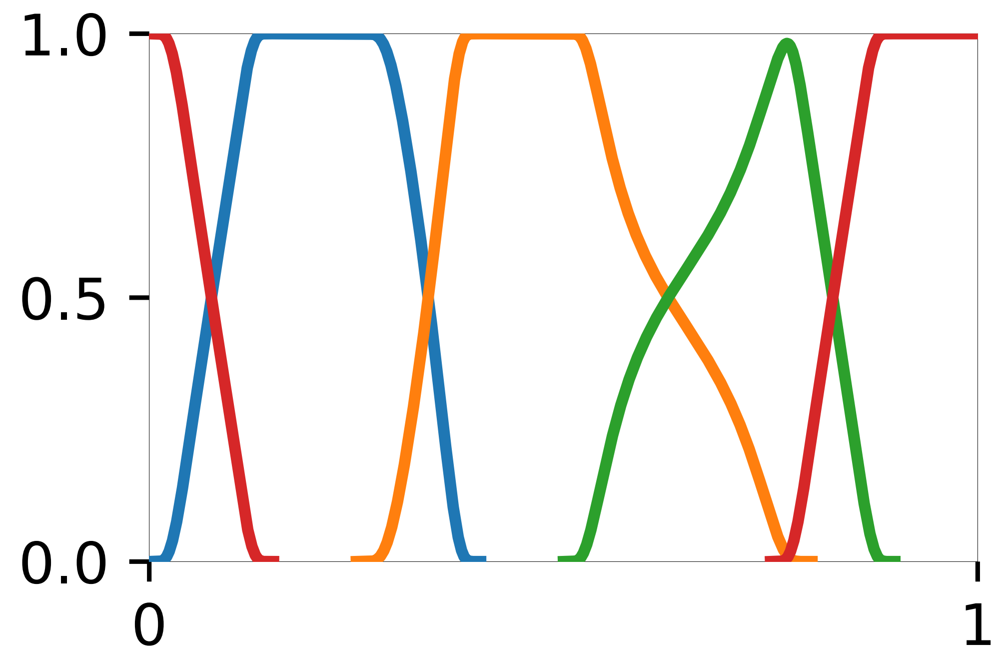
Smooth, but not analytic

$$\phi(x) = \begin{cases} \exp\left(1 - \frac{1}{(1+x)(1-x)}\right) & x \in (-1, 1) \\ 0 & |x| > 1 \end{cases}$$

$$\left(\ell_m(x), \phi(x)\right) \rightarrow$$



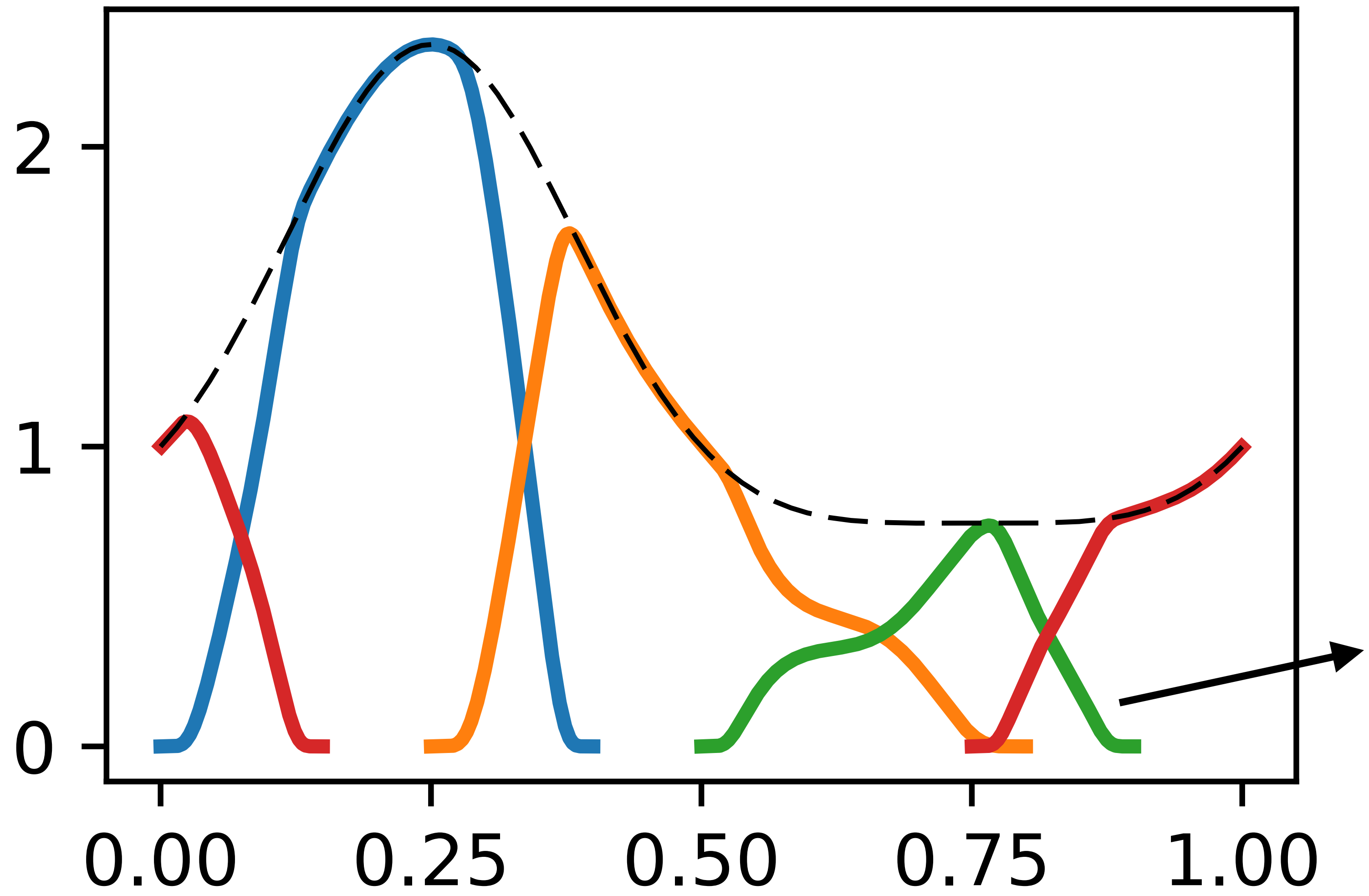
$$\psi_m(x) = \frac{\phi(x)}{\sum_i \phi(t_i^m(x))} \rightarrow$$



Degrees: $771, 1098, 781, 805$

Act Locally

Seek $\{f_i(u)\}$ and then build $f(x)$



Green is $\psi_3 \, f_3$

$$f(\ell_m(u)) = \sum_{i=1}^{N^c} \psi_i(t_i^m(u)) \, f_i(t_i^m(u))$$

Change of Domain

$$f(x) + \int_0^1 K(x,y) f(y)\, dy = b(x)$$

$$x = \ell_m(u)$$

$$f(\ell_m(u)) = \sum_{i=1}^{N^c} \psi_i(t_i^m(u))\, f_i(t_i^m(u))$$

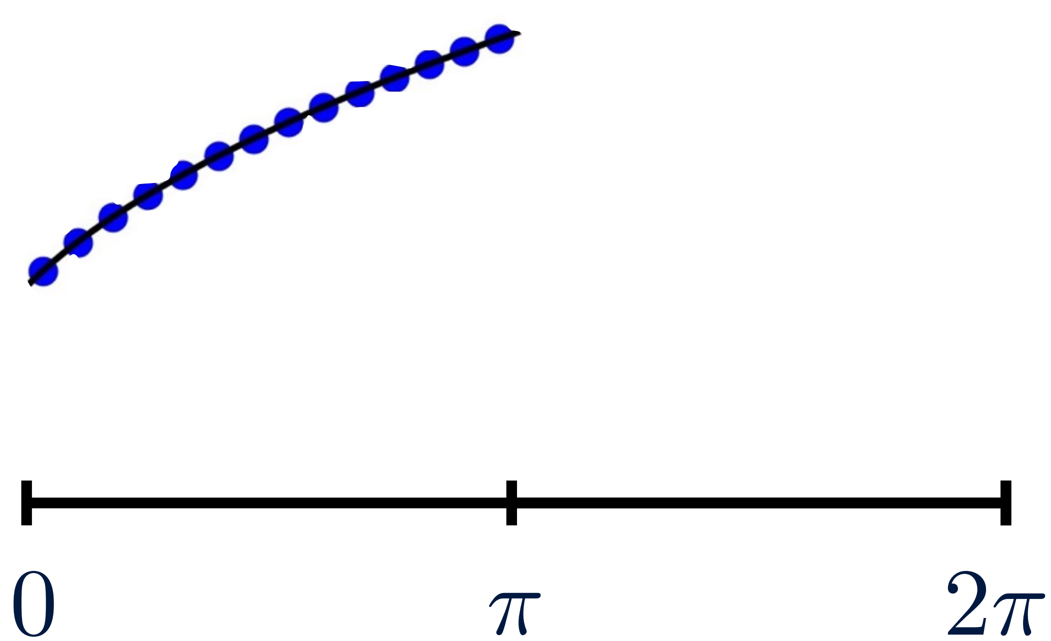
$$\sum_i \psi_i(t_i^m(u)) f_i(t_i^m(u)) + \int_0^1 K(\ell_m(u), y) \sum_i \psi_i(\ell_i^{-1}(y)) f_i(\ell_i^{-1}(y))\, dy = b(\ell_m(u))$$

$$y = \ell_i(v)$$

$$\sum_i \psi_i(t_i^m(u)) f_i(t_i^m(u)) + \sum_i \frac{|\ell_i([-1,1])|}{2} \int_{-1}^1 K(\ell_m(u), \ell_i(v)) \psi_i(v) f_i(v)\, dv = b(\ell_m(u))$$

A case where polynomials really are bad

Discretizing f_m: don't oversample near endpoints
Instead, make FFT work with nonperiodic data:



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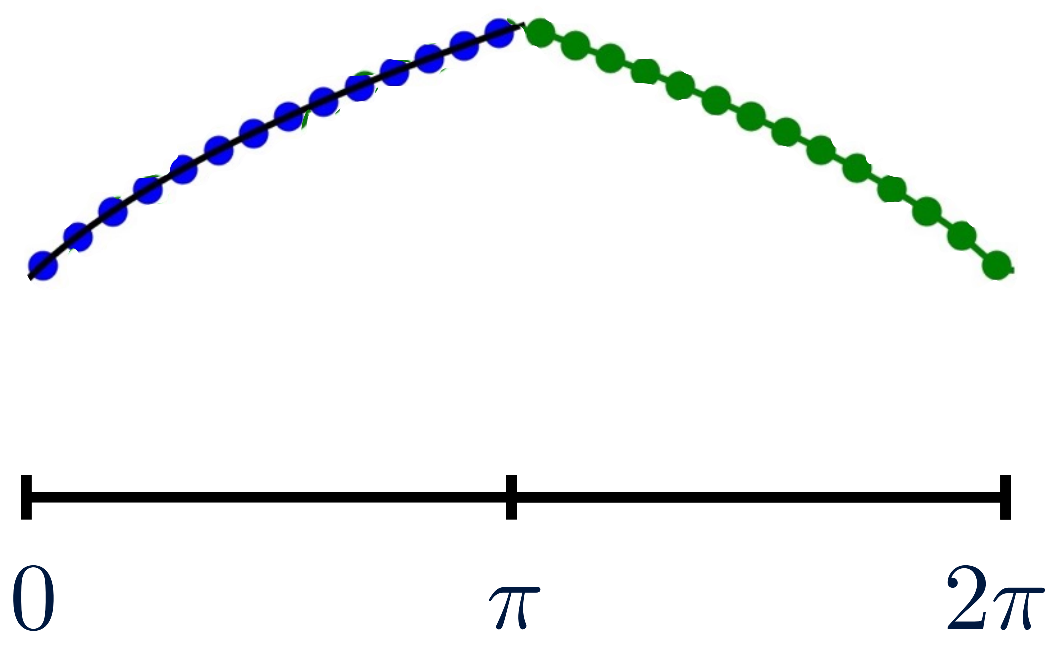

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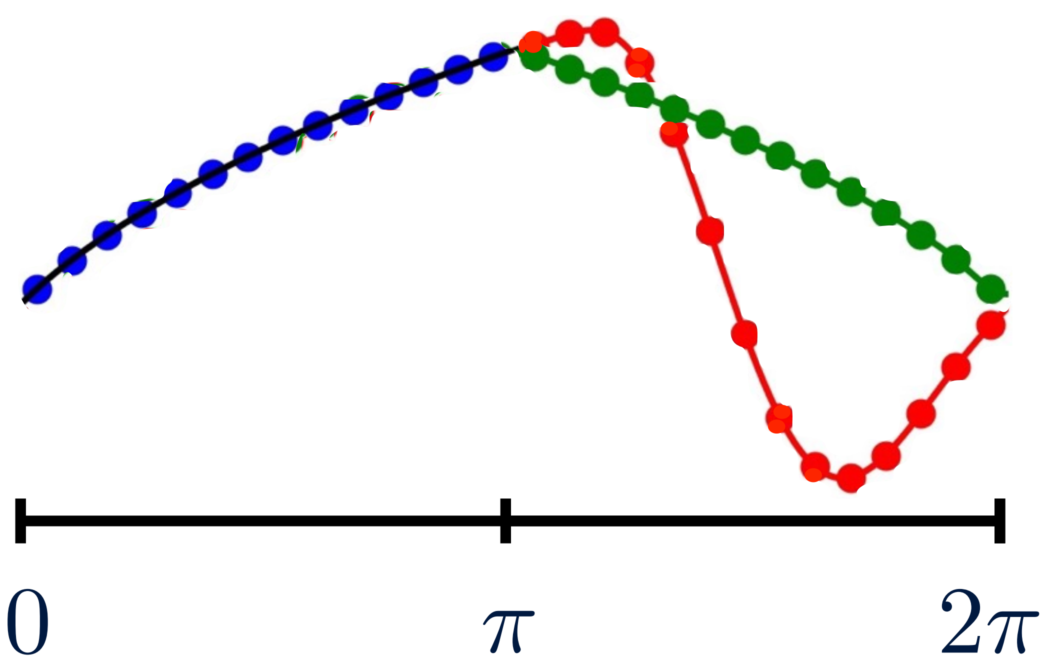
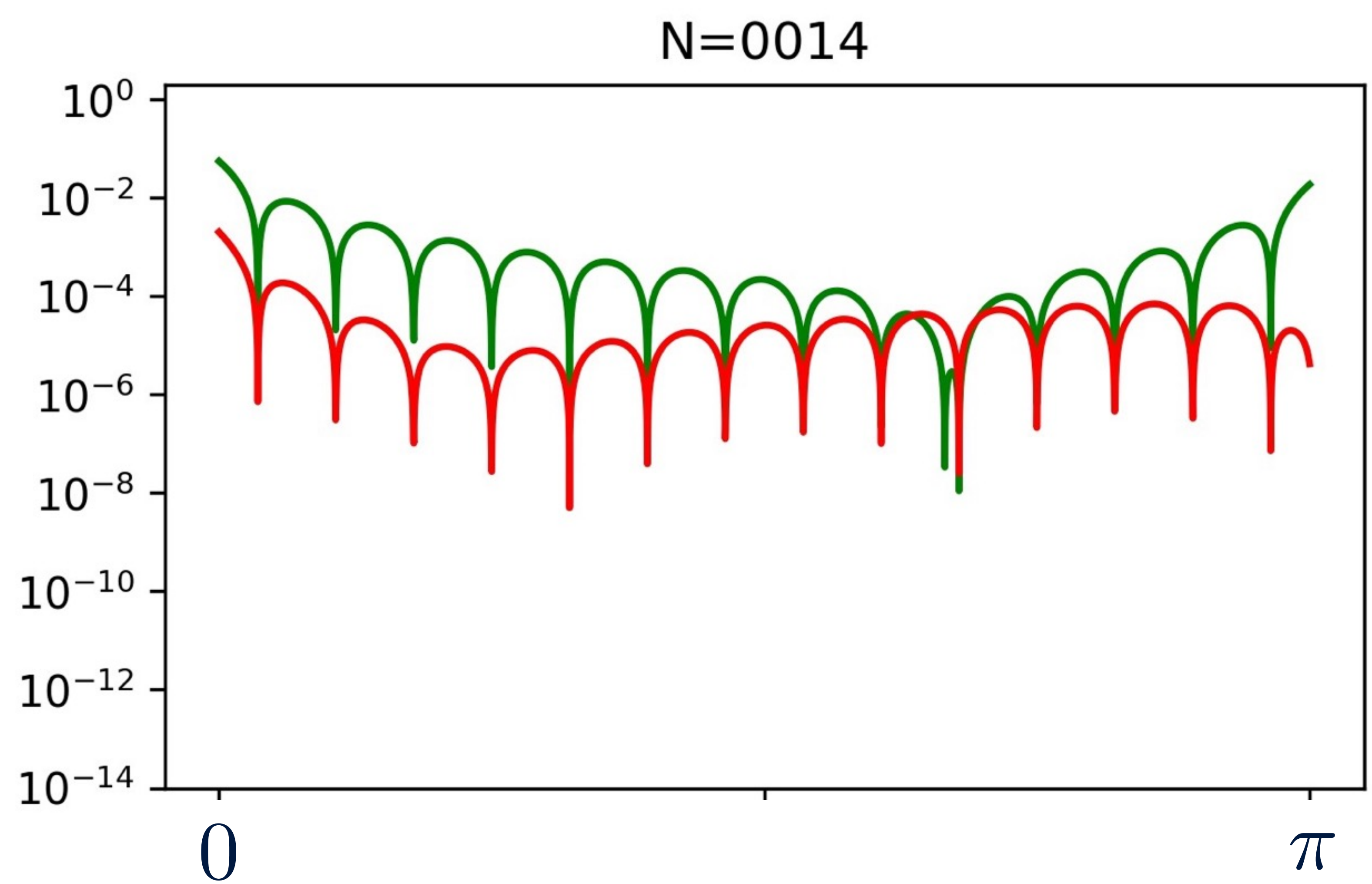
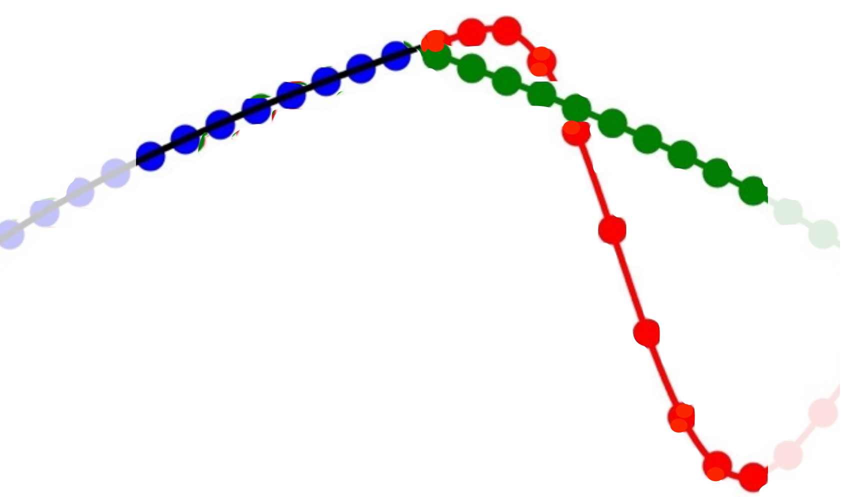
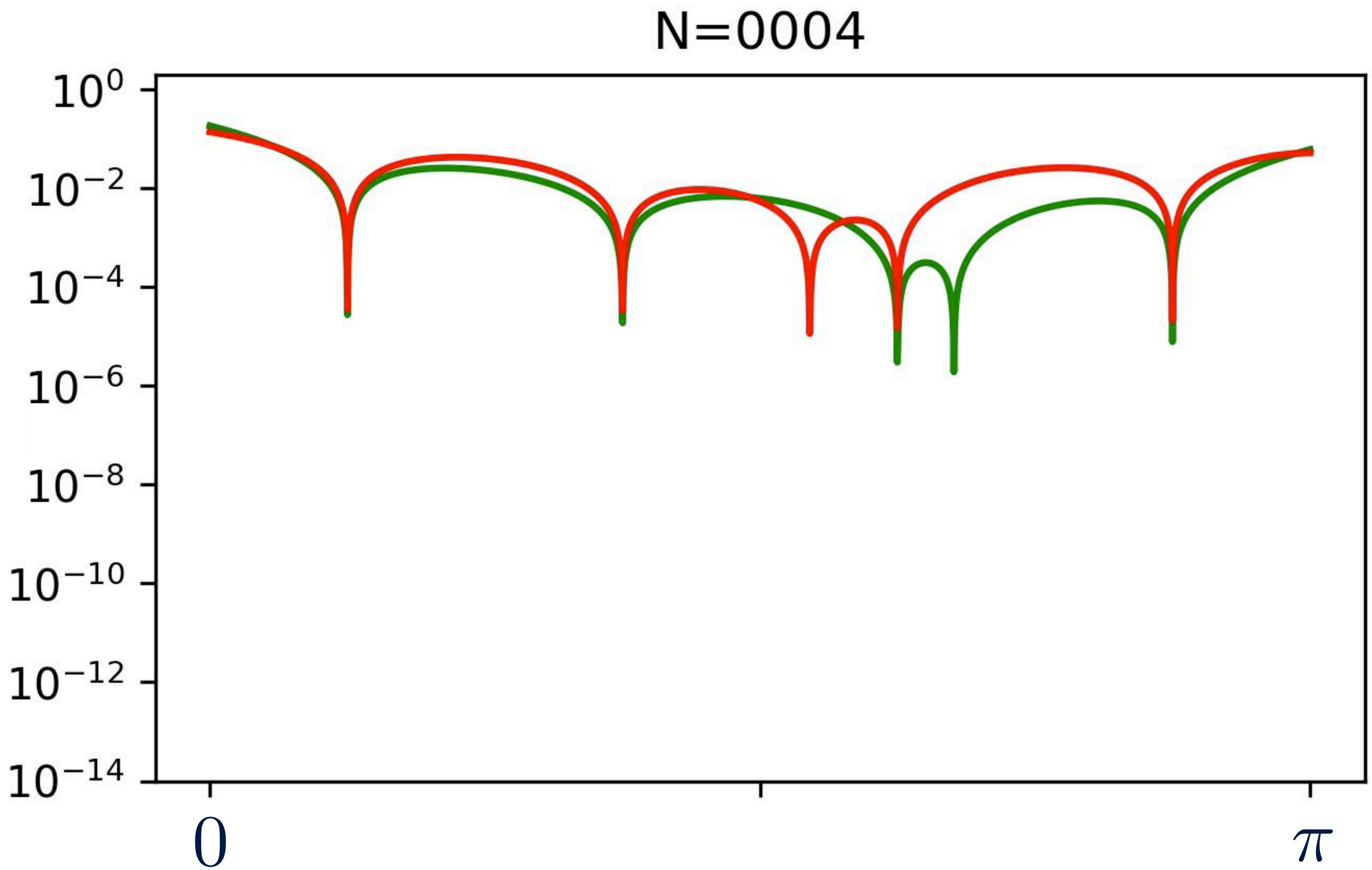




A case where polynomials really are bad

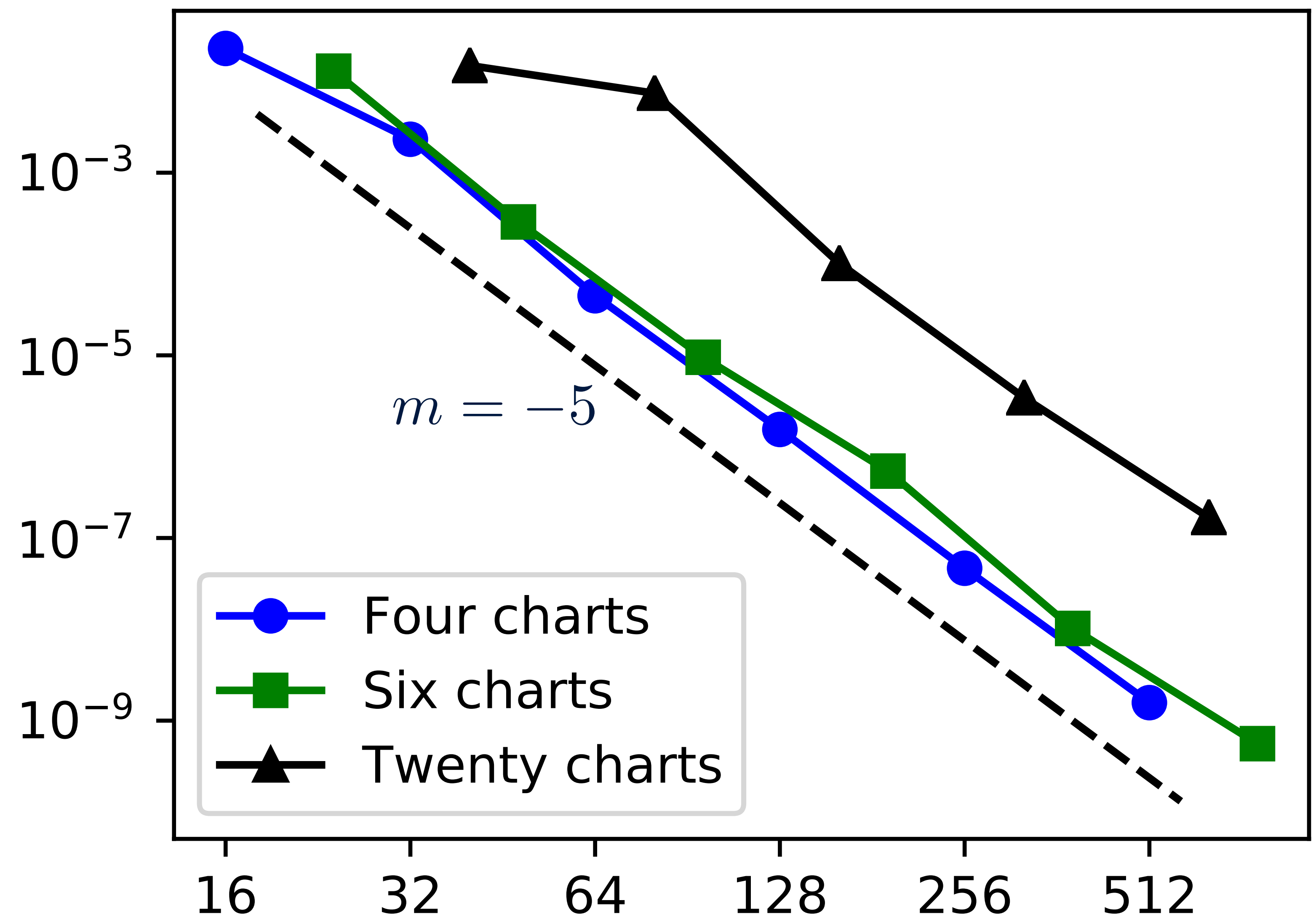
The fourth derivative is discontinuous at the endpoints.
This is still not a spectral interpolation scheme.
Suggestions welcome. Idea: discretize ψf instead of f ?





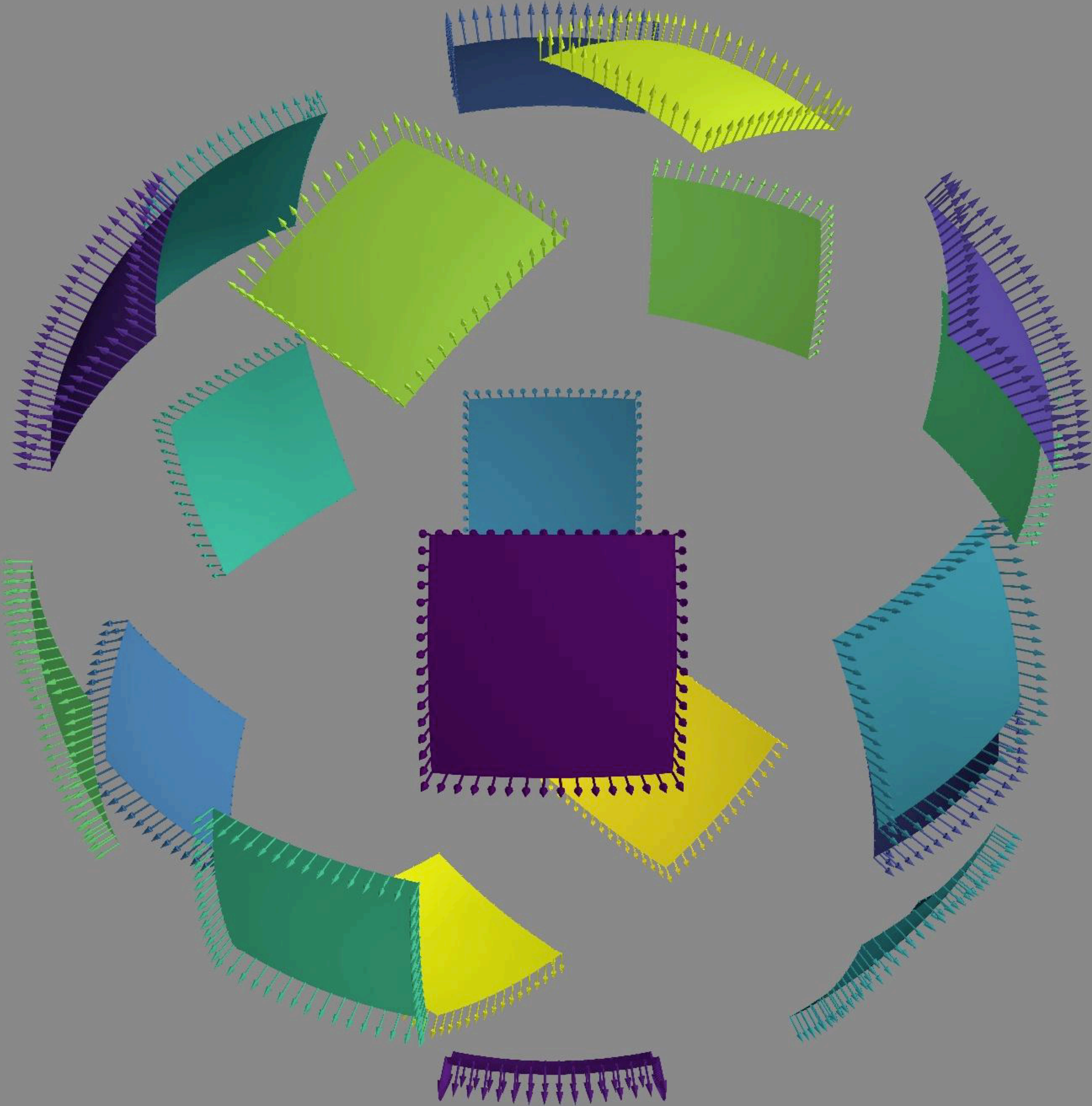
Results for the model problem on a circle

L^∞ error vs. exact solution



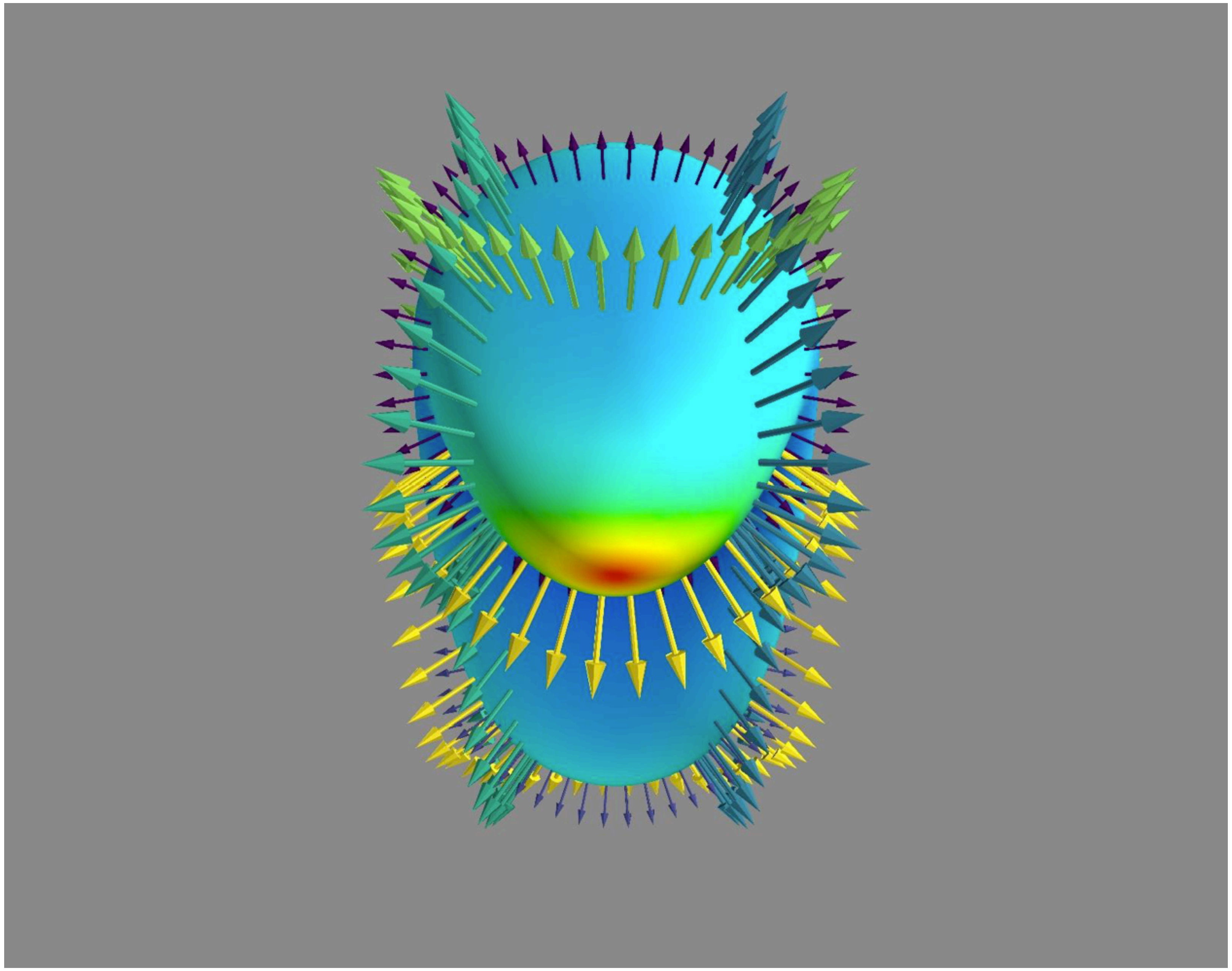
N : number of nodes for all f_i combined

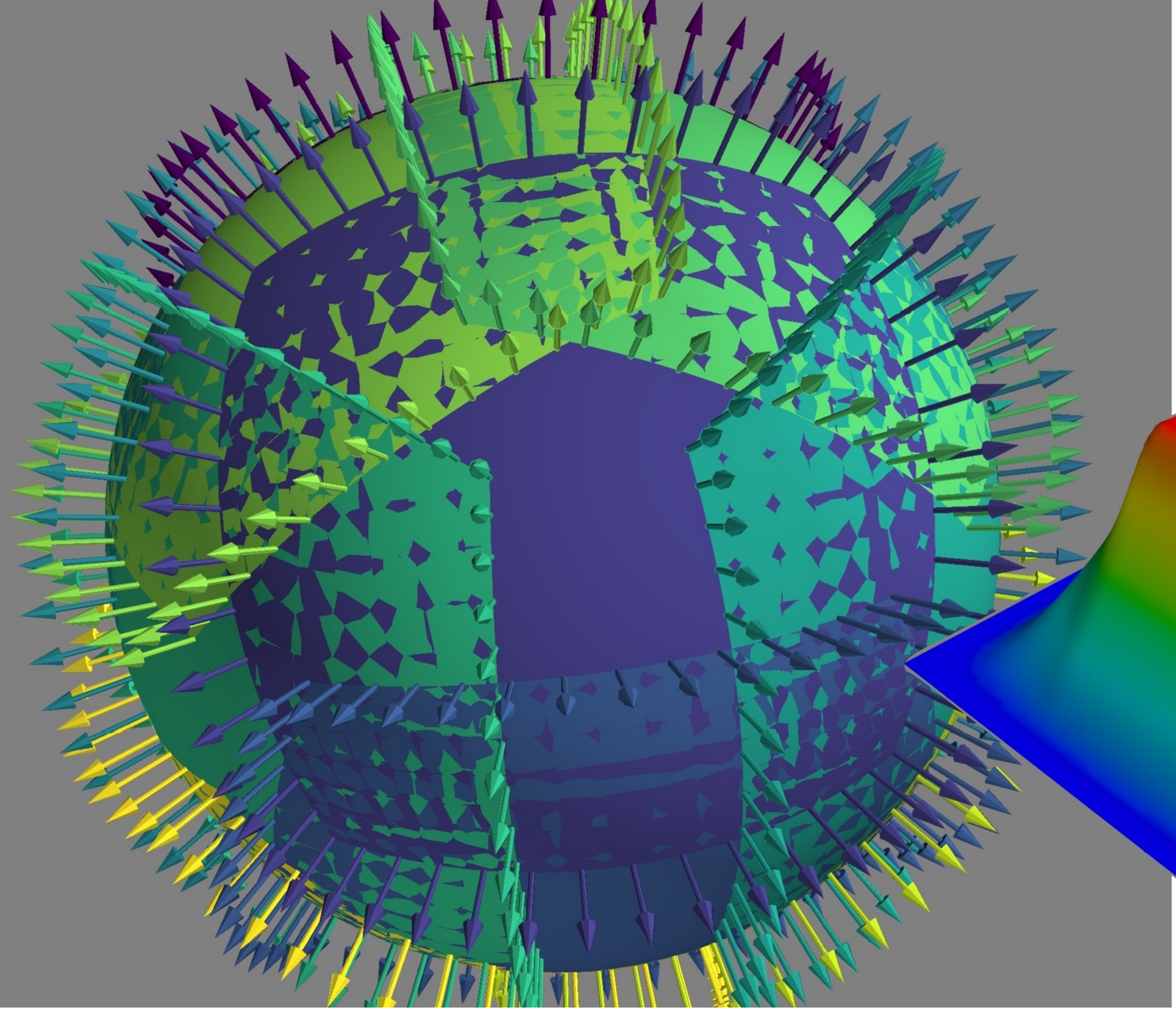
Three dimensional problems



Geometry is continuous (no mesh)

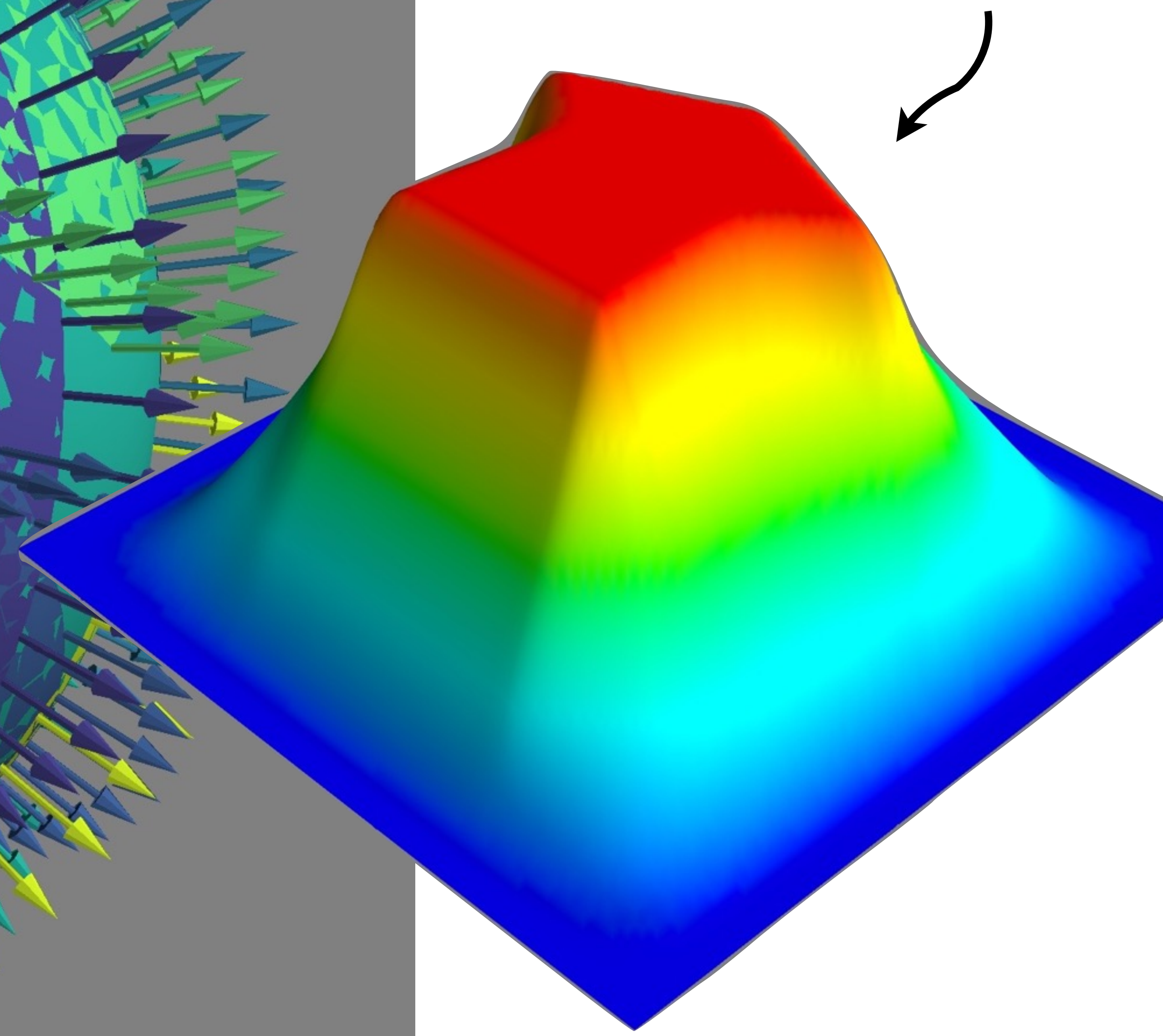
```
def mean_curvature(self,j,s,t):
    I,II = self.fundamental_forms(j,s,t)
    H = I[0]*II[1]-2*I[2]*II[2]+I[1]*II[0]
    return H / (I[0]*I[1] - I[2]**2)
```





Atlas haiku

This ψ is a bi-
variate polynomi-
al of rank thirty

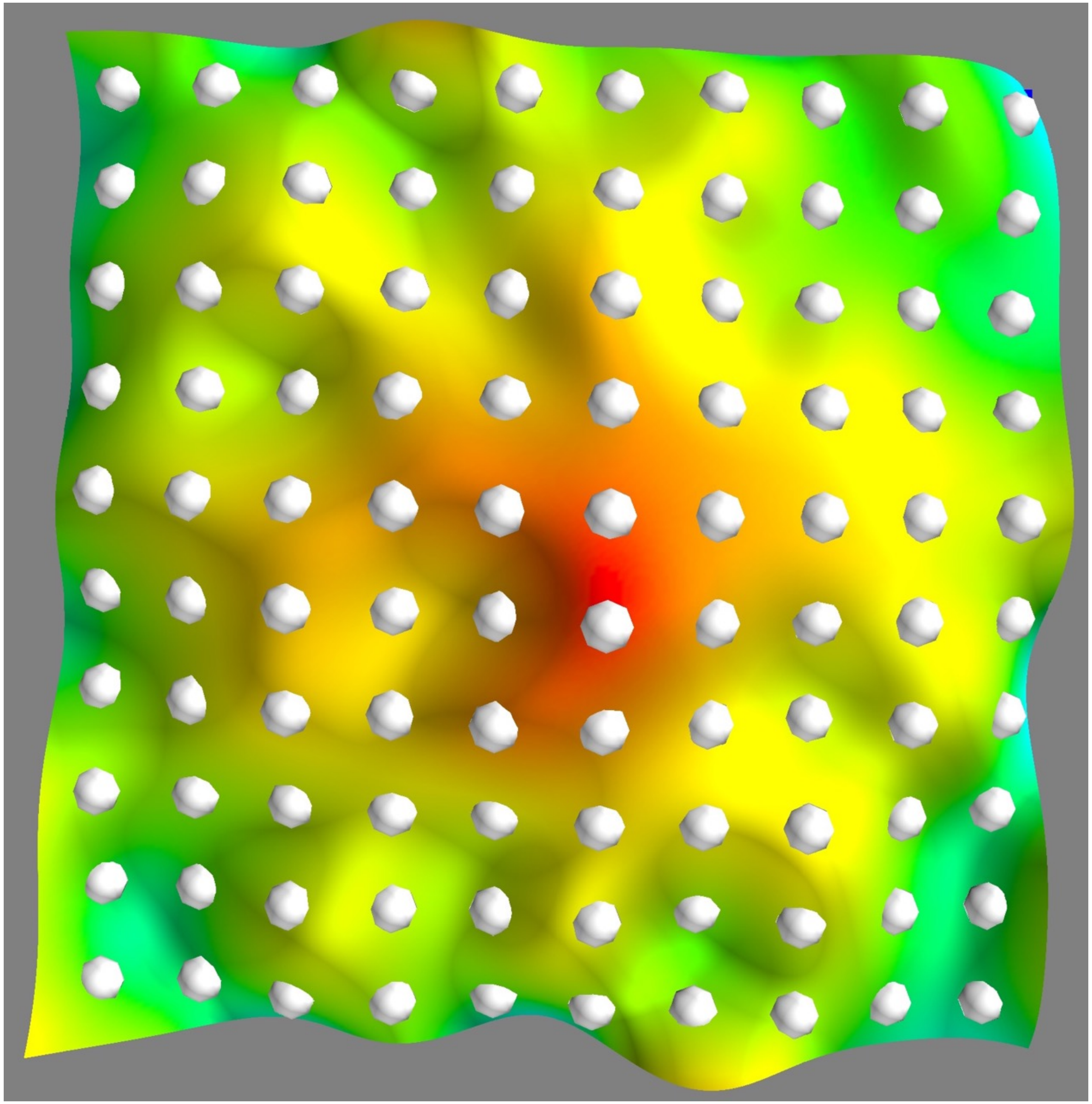


$$\psi(x,y) = \sum_{i=1}^{30} r_i(x) c_i(y)$$

as in Townsend & Trefethen, *An extension of Chebfun to two dimensions,* SIAM JSC 2013

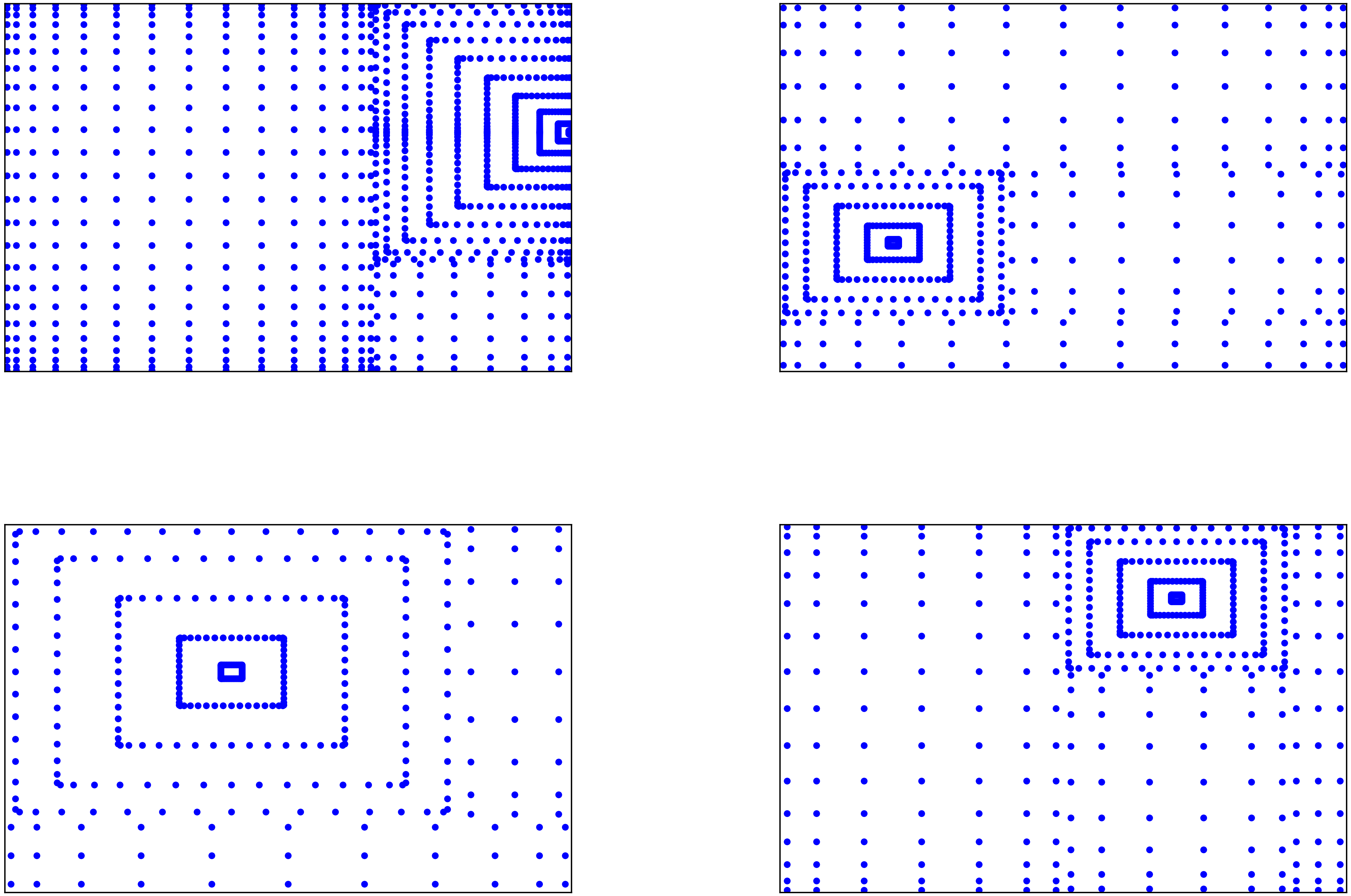
Interior regular grid interpolation

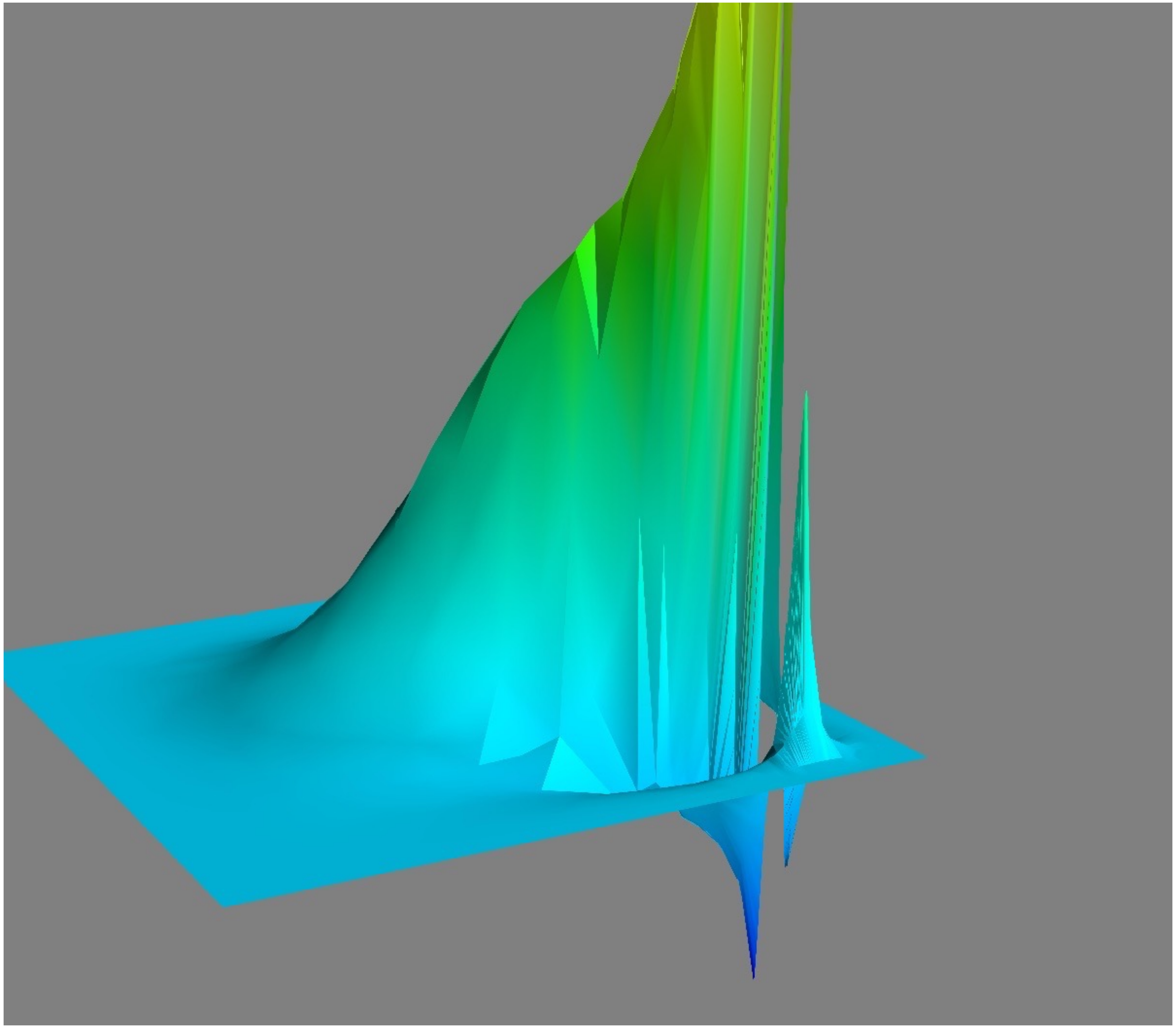
An f_i discretized by grid values



$$f_i = \sum_{i=1}^{10} R_i(x) C_i(y)$$

Specialized quadratures for $\frac{1}{r}$ singularities





Single layer integrand

Future work: boundary integral methods

Power & Miranda, SIAM JAM 1987

$$\boldsymbol{u}(\boldsymbol{x}) = -\int_S \boldsymbol{q}(\boldsymbol{y}) \cdot (\mathsf{T}(\boldsymbol{x}, \boldsymbol{y}) + \mathsf{T}^*(\boldsymbol{x}, \boldsymbol{y}^*)) \cdot \hat{\boldsymbol{n}}(\boldsymbol{y})\, \mathrm{d}S + \frac{1}{8\pi}(\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{x}_0) + \boldsymbol{G}^*(\boldsymbol{x}, \boldsymbol{x}^*)) \cdot \boldsymbol{F}$$

M. & Spagnolie, JFM 2017

$$\frac{1}{8\pi}\int_D T_{ijk}(y', y)\Big(f_i(y')n_k(y) + f_i(y)n_k(y')\Big)dS_{y'} + \frac{1}{8\pi}\int_D C_{ij}(y', y) f_i(y')\, dS_{y'} = (U_j)$$

Thank you!