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Lagrangian Transport Through Surfaces in Volume-Preserving Flows

Daniel Karrasch

Setting

- fluid – d-dimensional continuum (manifold) $\mathcal{M} \subseteq \mathbb{R}^d$, $d = 2, 3$
- fluid motion – differentiable, volume-preserving flow F

(CM) (DS)

$$\begin{cases} F_0 \colon [0,T] \times \mathcal{M} \to \mathbb{R}^d, \\ F_0^0(p) = p, \\ F_0^t(p) = F_s^t \circ F_0^s(p), \\ \det DF \equiv 1; \end{cases} \iff \begin{cases} v \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d, \\ v(t,x) = \frac{\mathrm{d}}{\mathrm{d}t} F_0^t(F_t^0(x)), \\ \operatorname{div}(v) \equiv 0. \end{cases}$$

- fluid velocity – differentiable, *nonautonomous*, divergence-free vector field v

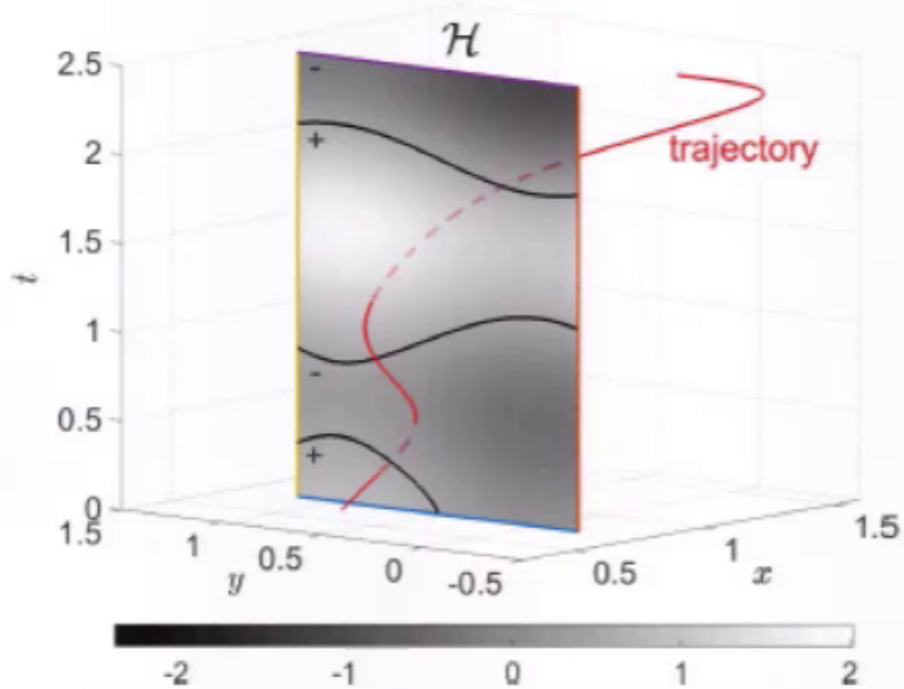
Eulerian transport through a surface

- $\mathcal{C}$ – codimension-one surface (section) in $\mathbb{R}^d$ with normal n
- $\mathcal{H} = [0, T] \times \mathcal{C}$ – extended section in extended state space

Transport through $\mathcal{H}$

$$\iint_{\mathcal{H}} \rho n \cdot v \, \mathrm{d}t \, \mathrm{d}x$$

ρ – material density



$n \cdot v$ – flux density

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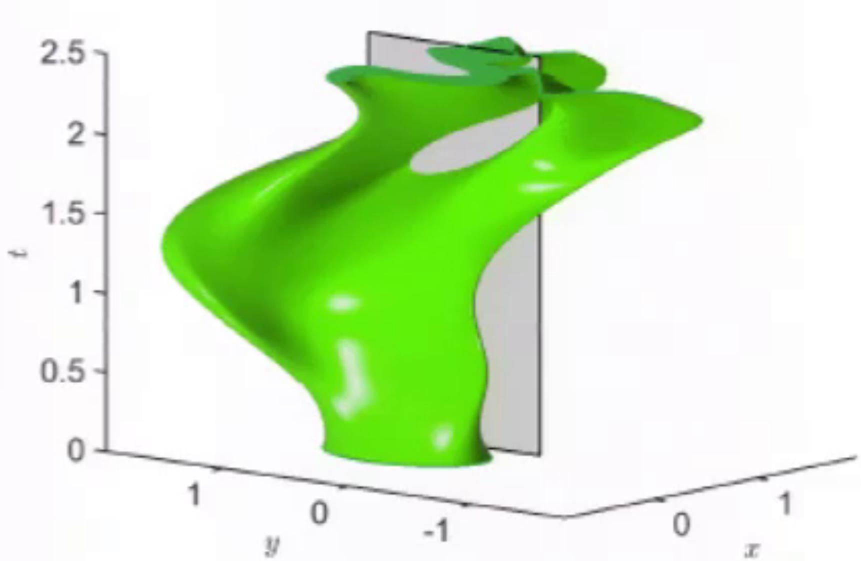
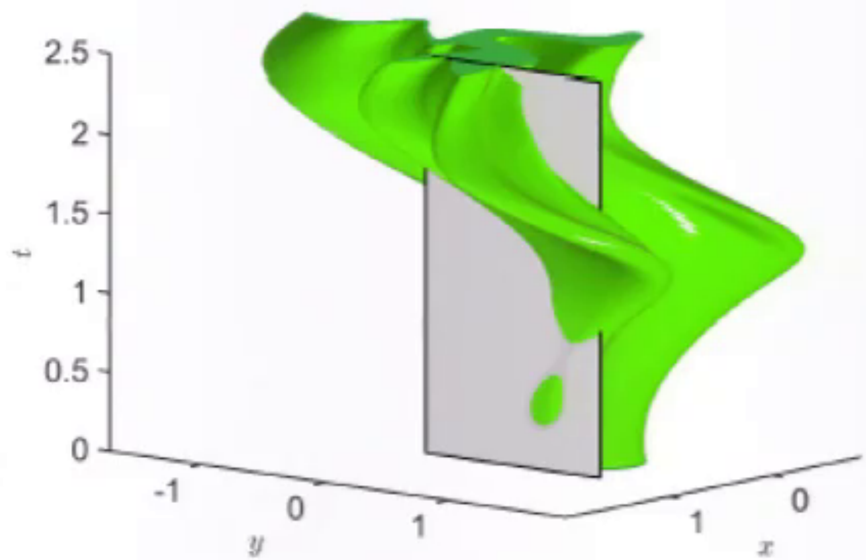
Lagrangian transport through a surface

Task: Quantify transport due to a material/Lagrangian set of particles of interest!





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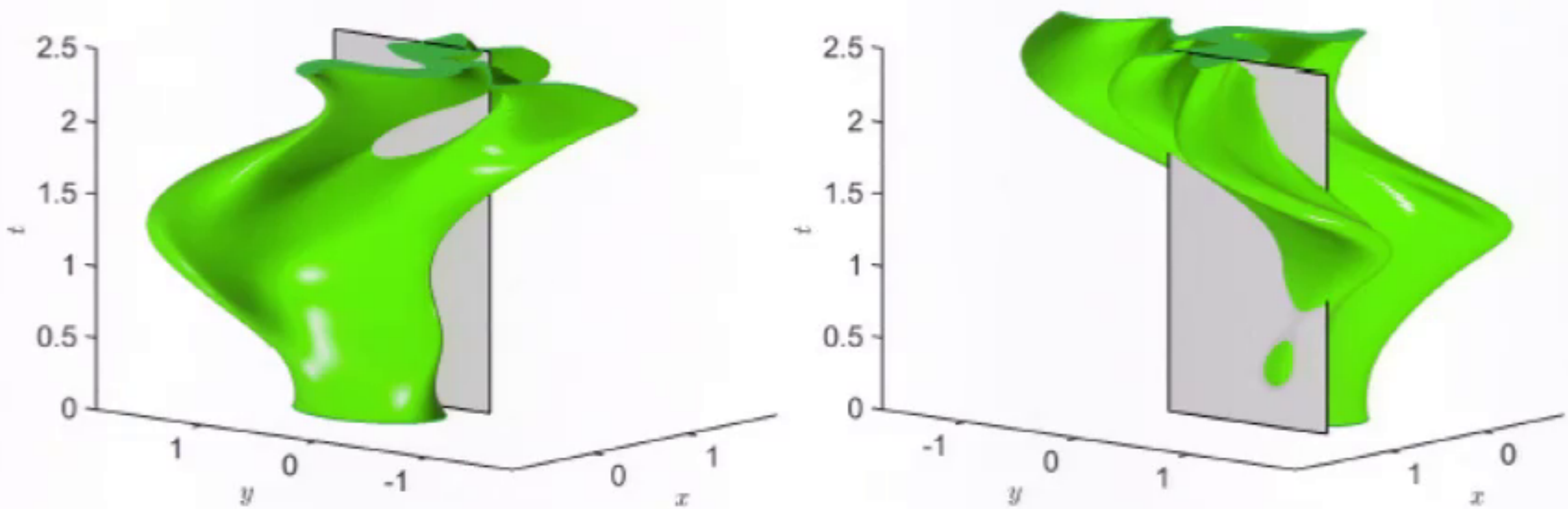
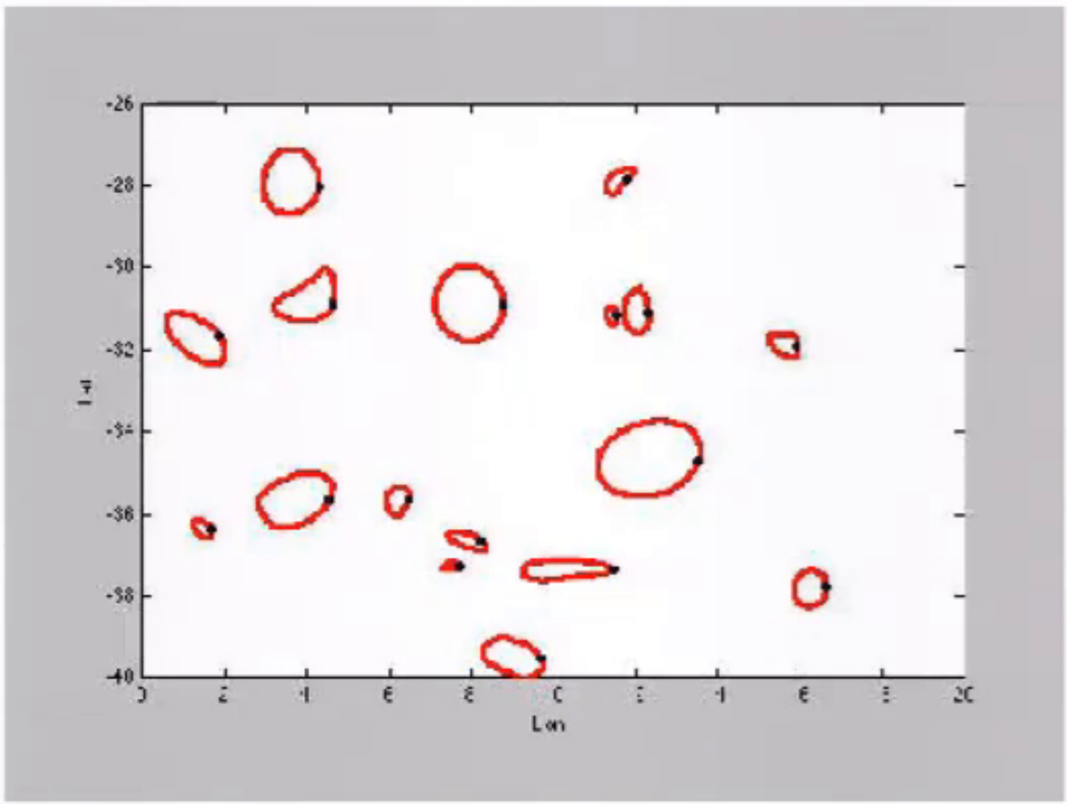


Challenge: Complicated intersection of trajectories with section!
Hope: with Lagrangian analog of integral should be easy

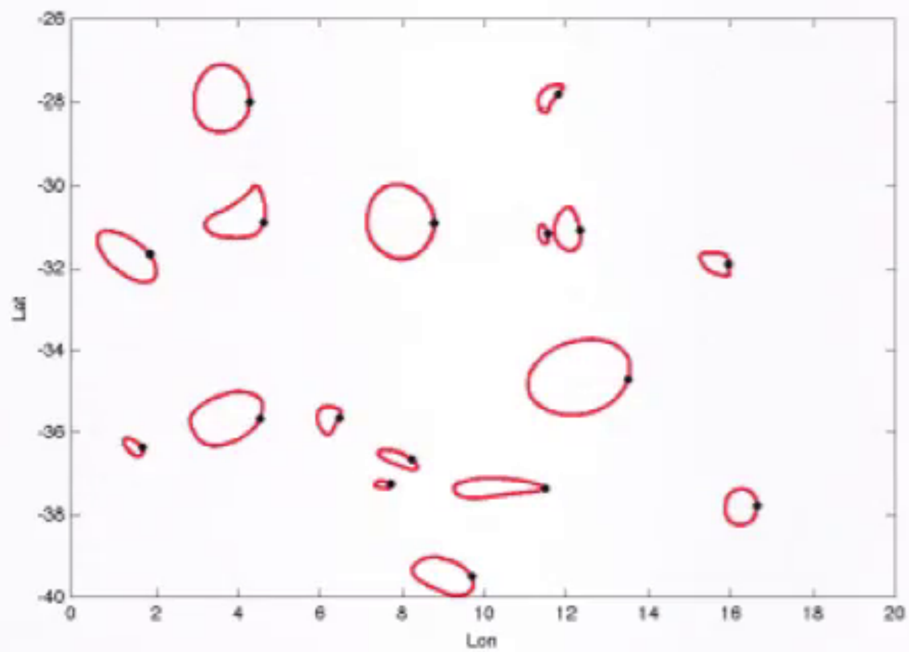
Is solving the task of any interest?



Video courtesy of Florian Huhn

Is solving the task of any interest? IT IS!

Transport by Lagrangian coherent structures / Lagrangian coherent vortices / coherent sets: K., Haller, Froyland, Padberg-Gehle, Bollt, Mezic, Rom-Kedar, Rowley, etc.



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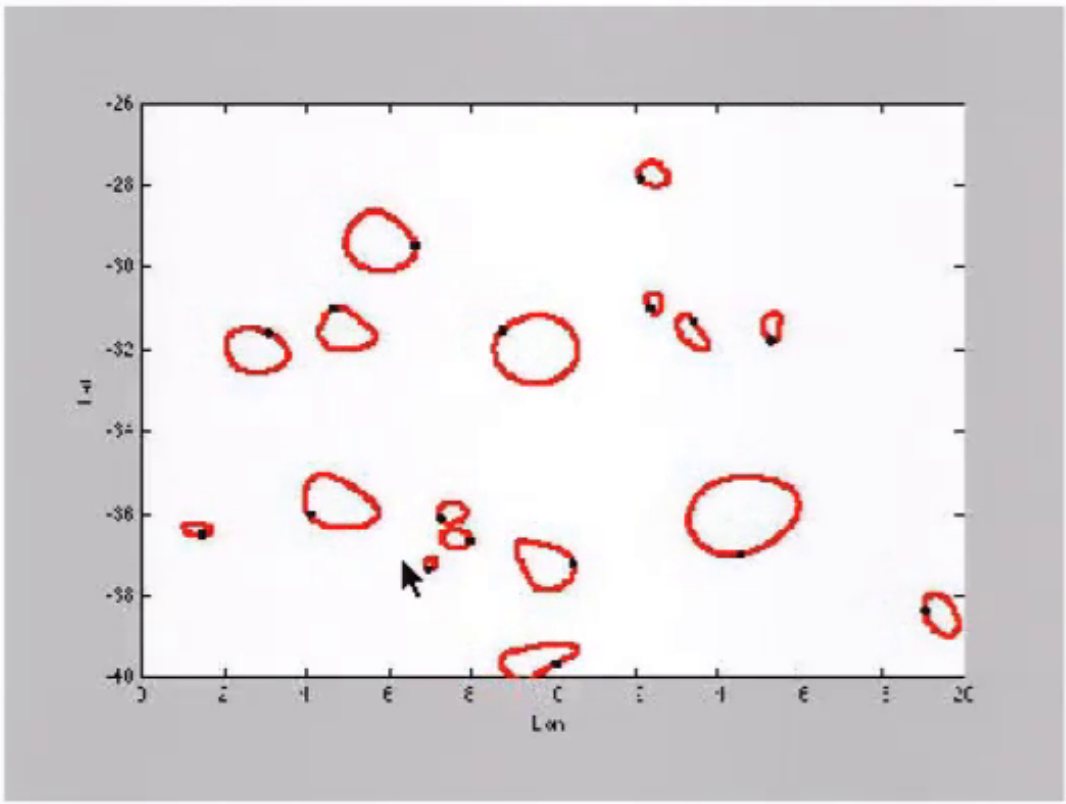

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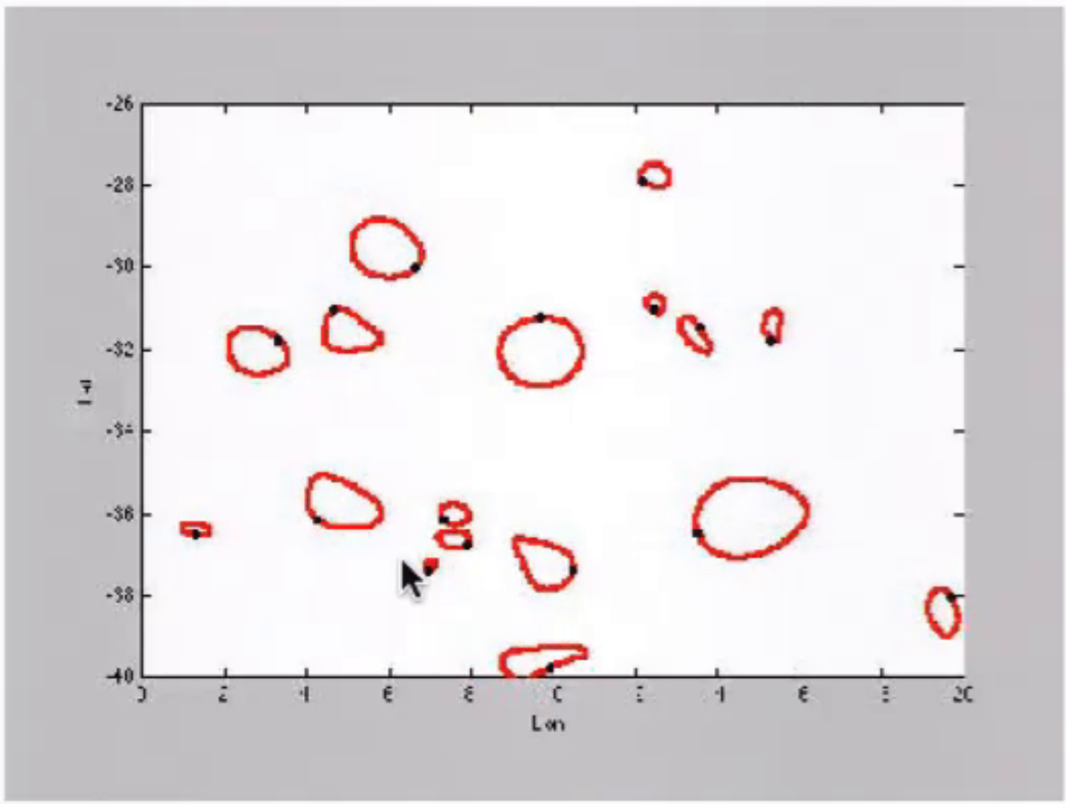

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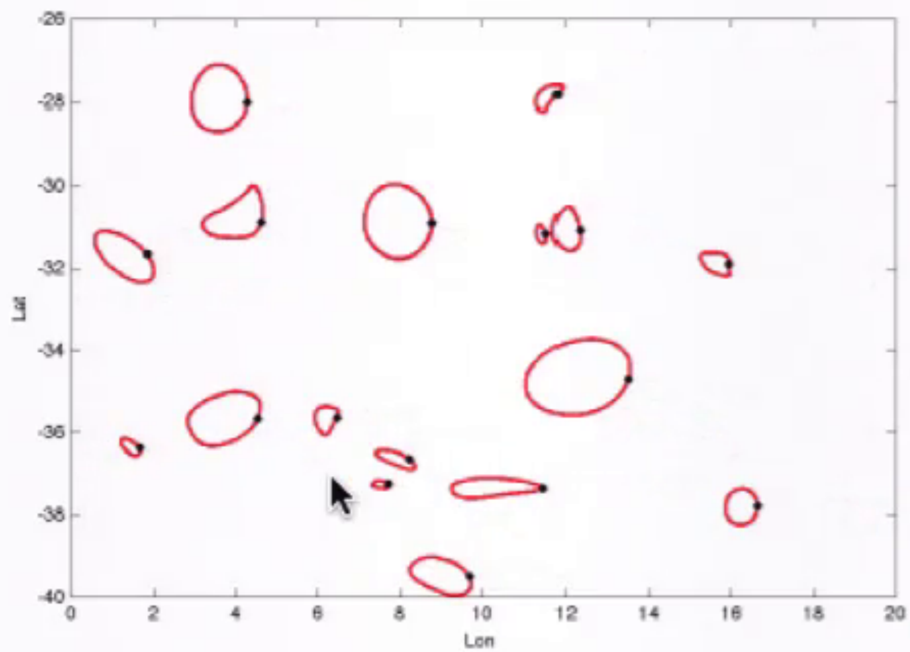

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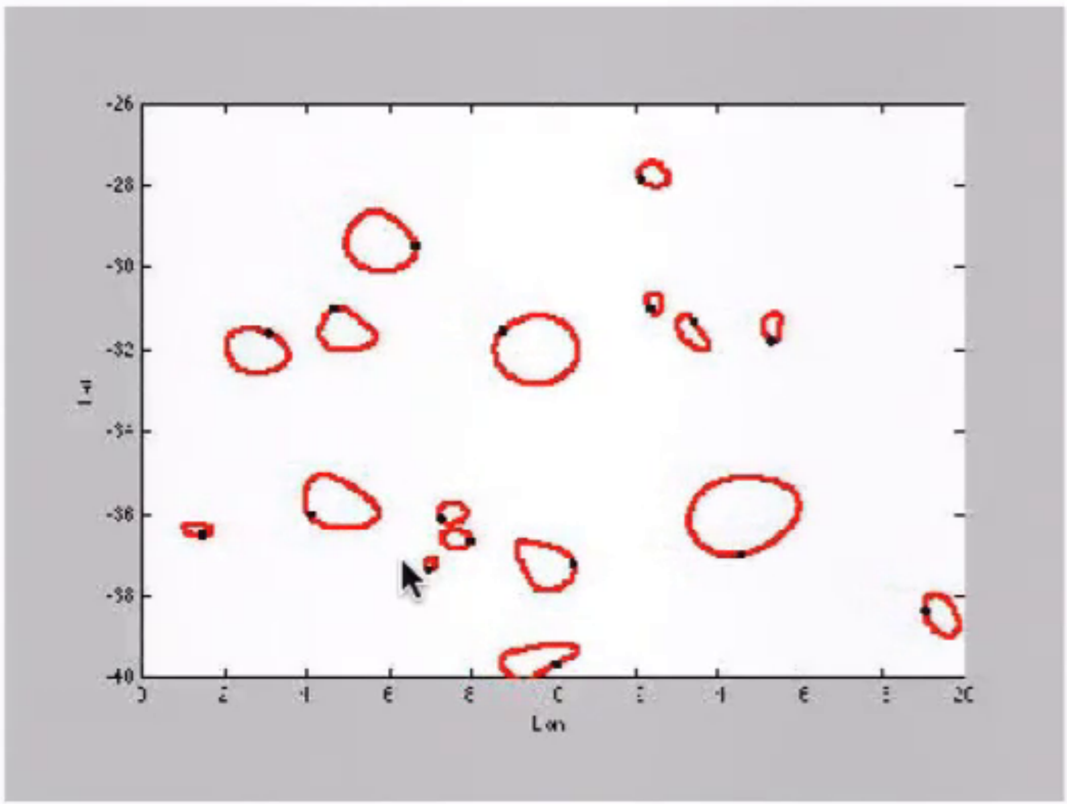

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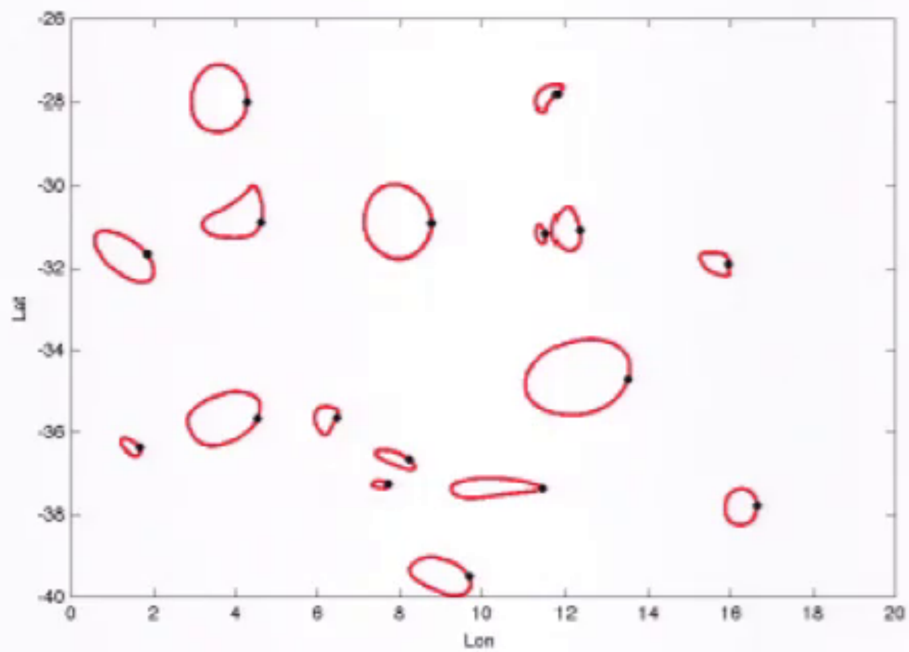

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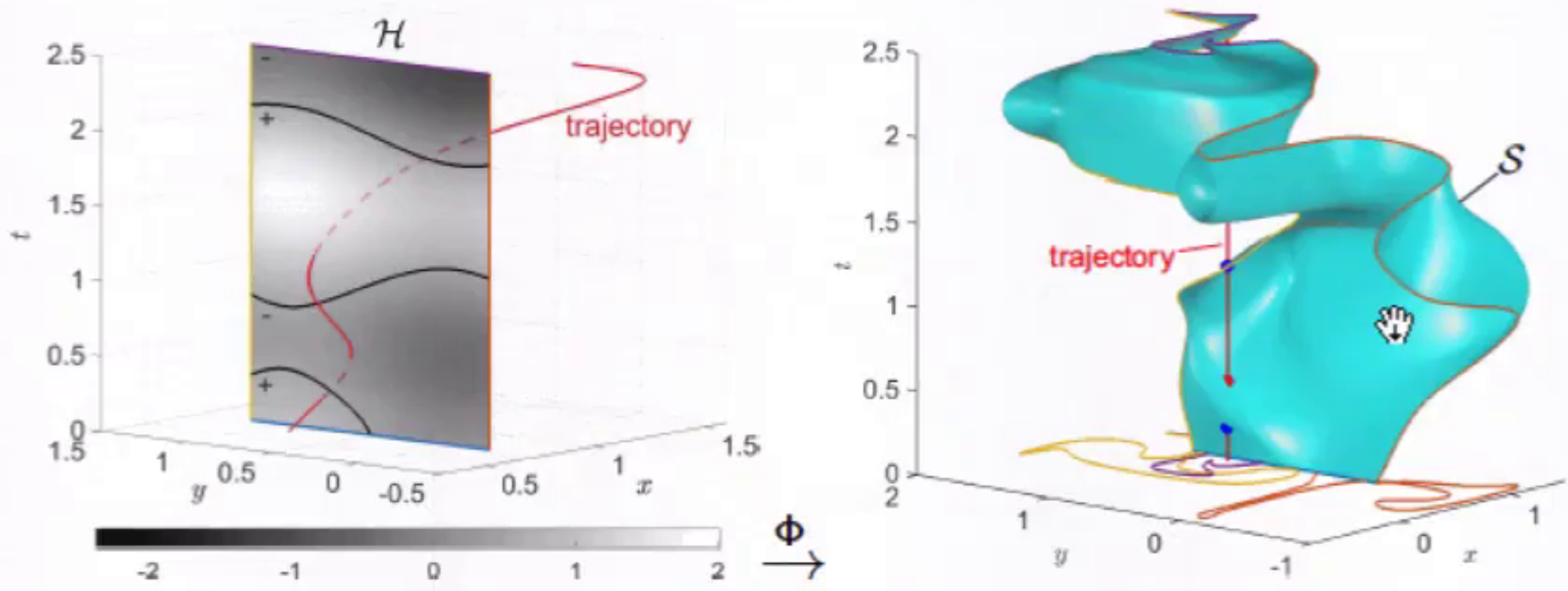
Proof I

Consider (extended) Eulerian↔Lagrangian coordinate change Φ

Input: (t, x) – a space-time point from $\mathcal{H}$

Output: (t, p) – the Lagrangian particle $p = F_t^0(x)$ occupying x at t



$\Phi(\mathcal{H}) = \mathcal{S}$ – streak surface

Lagrangian transport through a surface

Theorem (K., 2015, submitted))

Define $\mathcal{D}_k$ as the set of Lagrangian particles p which have only transversal crossings with $\mathcal{H}$ and exactly k net crossings.

Then:

$$\iint_{\mathcal{H}} \rho n \cdot v \, \mathrm{d}t \, \mathrm{d}x = \sum_{k \in \mathbb{Z}} k \int_{\mathcal{D}_k} \rho(0, p) \, \mathrm{d}p, \tag{1}$$

and for $\rho \equiv 1$:

$$\iint_{\mathcal{H}} n \cdot v \, \mathrm{d}t \, \mathrm{d}x = \sum_{k \in \mathbb{Z}} k \, \mathrm{vol}(\mathcal{D}_k) \,. \tag{2}$$

Proof I

Consider (extended) Eulerian↔Lagrangian coordinate change Φ

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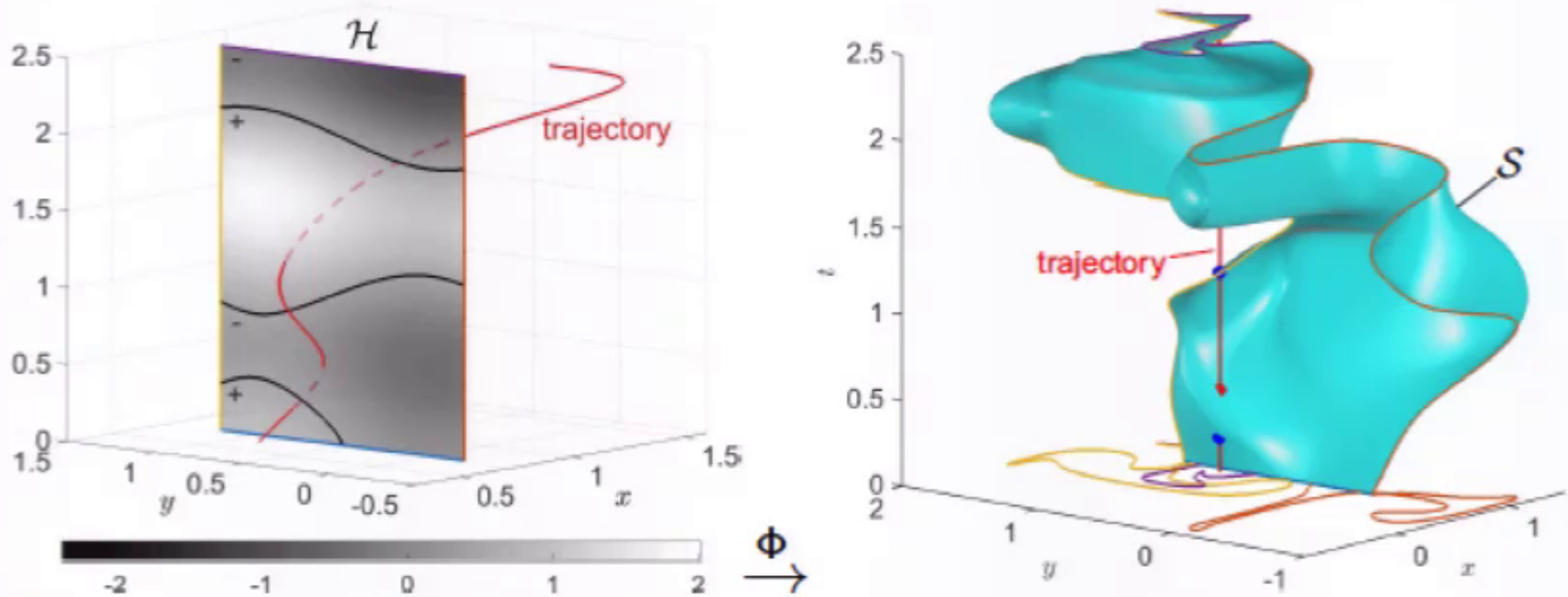
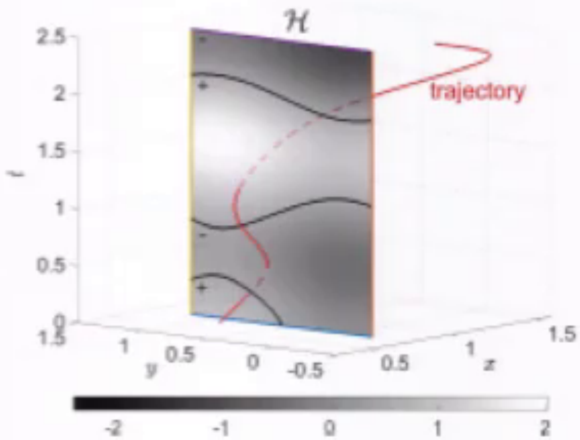


$\Phi(\mathcal{H}) = \mathcal{S}$ – streak surface

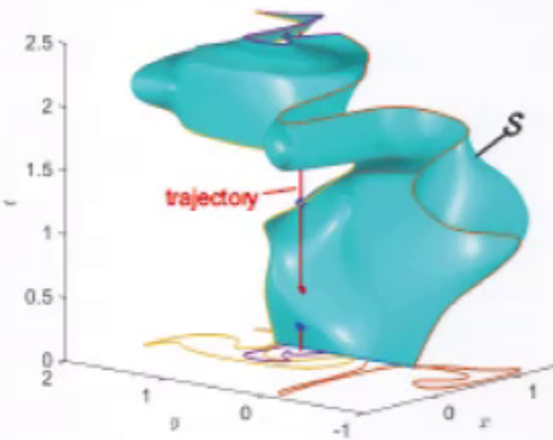
Proof II

Consider complete coordinate change:

Ψ: section $\xrightarrow{\Phi}$ streak surface $\xrightarrow{\Pi}$ material particles



$\xrightarrow{\Phi}$



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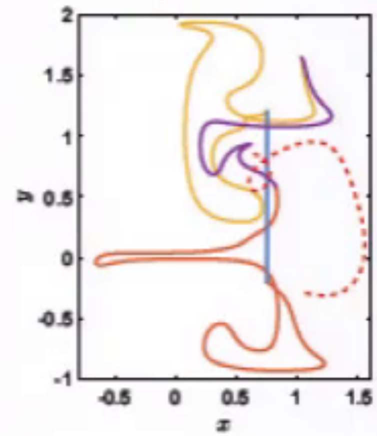


- for all material points p with only transversal crossings: $\deg(\Psi, p)$ defined & counts the net number of crossings
- $\mathcal{D}_k \coloneqq \{p\colon\ \deg(\Psi, p) = k\}$ – *donating region*[1] of material particles with k net crossings

[1] Q. Zhang [2013,2015]

Proof III

- What about material particles p with tangential “crossing”? Characterized by:
  1. Eulerian: $u(t, x) \cdot n(x) = 0$ at crossing (t, x), i.e., no flux
  2. Lagrangian: *critical values* of Ψ, with zero measure (Sard’s Theorem)
- From a consequence of the Area Formula:

$$\int_{\mathcal{H}} \det \mathrm{d}\Psi(t, x) = \int_{\mathbb{R}^d} \deg(\Psi, p)\, \mathrm{d}p = \sum_{k \in \mathbb{Z}} k \operatorname{vol}(\mathcal{D}_k)$$

- From a calculation:

$$\det \mathrm{d}\Psi(t, x) = u(t, x) \cdot n(x)$$

$\Rightarrow$ Eq. (2)

Proof IV

- General conserved quantities ρ:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \partial_t \rho + u(t,\cdot) \cdot \nabla \rho = 0,$$

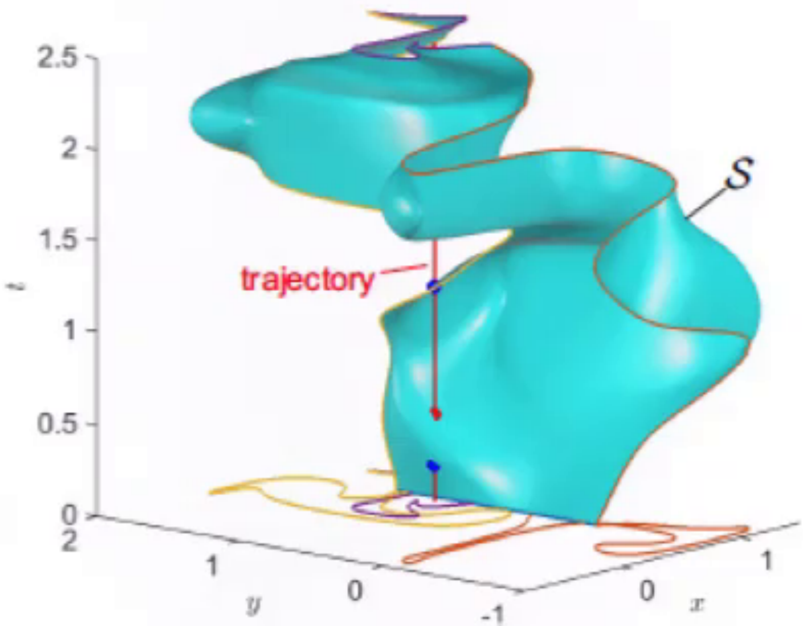
$$\rho(t,x) = \rho(0,\Psi(t,x)) = \rho(0,p).$$

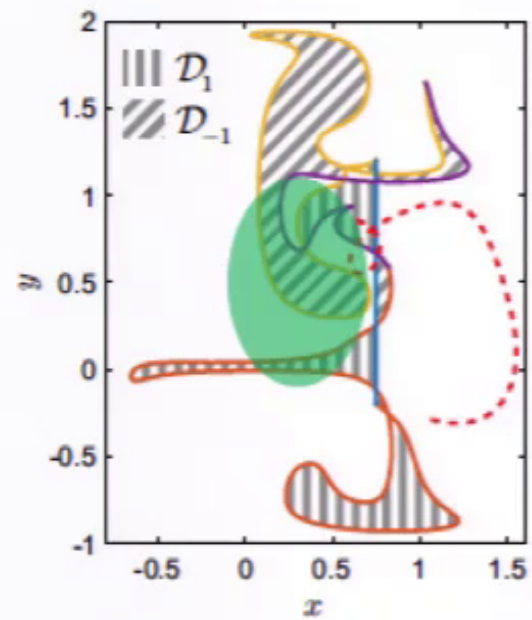
$$\Rightarrow \quad \begin{aligned} &\int_{\mathcal{H}} \rho(t,x) \det \mathrm{d}\Psi(t,x)\, \mathrm{d}t\, \mathrm{d}x = \\ &= \int_{\mathcal{H}} \rho(0,\Psi(t,x)) \det \mathrm{d}\Psi(t,x)\, \mathrm{d}t\, \mathrm{d}x = \\ &= \int_{\mathbb{R}^d} \deg(\Psi,p)\rho(0,p)\, \mathrm{d}p = \\ &= \sum_k k \int_{\mathcal{D}_k} \rho(0,p)\, \mathrm{d}p. \end{aligned}$$

■

Computation of $\mathcal{D}_k$'s in 2D

$\deg(\Psi, p)$ = winding number of closed curve $\Psi(\partial\mathcal{H})$ around p.





Transport contribution by green ellipse: restrict domain of integration in Lagrangian integral