Pacemakers in a Large Array of Non-locally Coupled Oscillators

SIAM Conference on Applications of Dynamical Systems
Snowbird, Utah

by

Gabriela Jaramillo and Arnd Scheel
University of Minnesota

18 May, 2015

Objectives

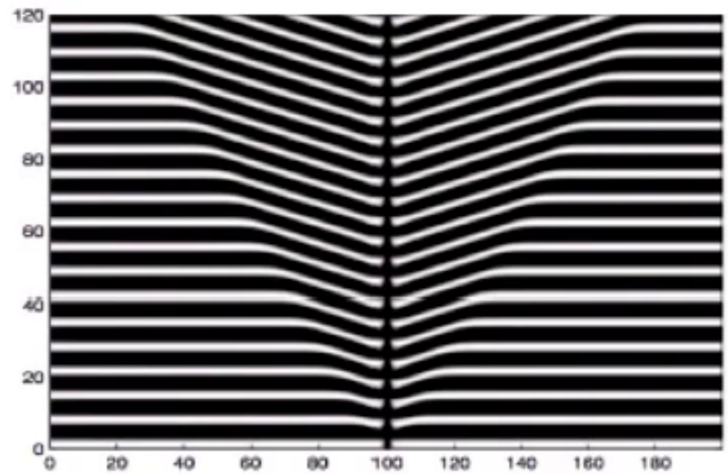
Objectives

- Motivate the following nonlocal equation

$$\phi_t = -\phi + G * \phi - (J * \phi_x)^2, \quad x \in \mathbb{R}$$

- Study the effects of a localized perturbation

$$\phi_t = -\phi + G * \phi - (J * \phi_x)^2 + \varepsilon g(x), \quad x \in \mathbb{R}$$



1-D Target Pattern/ Pacemaker

Motivation

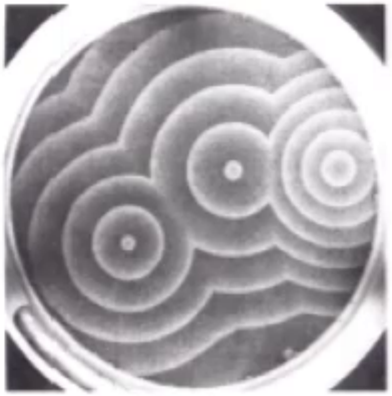
Examples from Biology:

- Neural networks
- Cardiac cells
- Insulin secreting cells in pancreas
- Crickets
- Fireflies

Synchronous fireflies in Gatlinburg, Tennessee: www.reservegatlinburg.com/travelguide

Interesting Phenomena

- Synchronization, frequency locking, incoherence, amplitude death
  Matthews, Mirollo, Strogatz (1991).
- 1-D traveling waves
  Bressloff, Coombes (1999).
- 2-D target patterns, rotating waves
  Goel, Ermentrout (2002).
- Chimera states, traveling chimera states
  Abrams, Strogatz (2004), *Kuramoto, Battogtokh (2002),* and *Xie, Knobloch, Kao (2014)*



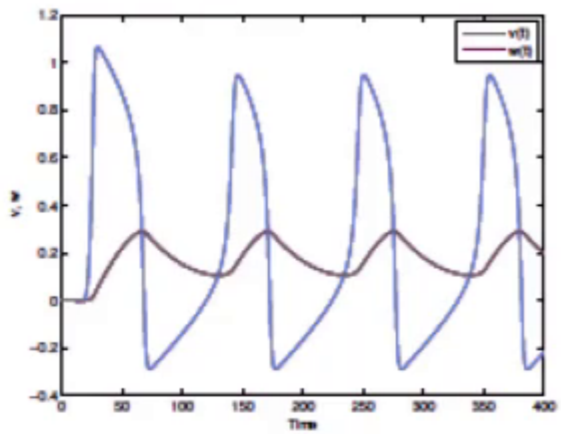
2-D target patterns in a chemical reaction. From Maselko, Reckley, Showalter, The J. Phys. Chem. 93.7 (1989)

Biological Oscillators

- System of ODEs

$$\dot{u} = f(u), \quad u \in \mathbb{R}^n$$

- Exponentially stable limit cycle



FitzHugh-Nagumo: variables vs. time

- Phase oscillators

$$\dot{\phi} = \omega$$

- Gauge-invariant phase amplitude oscillators

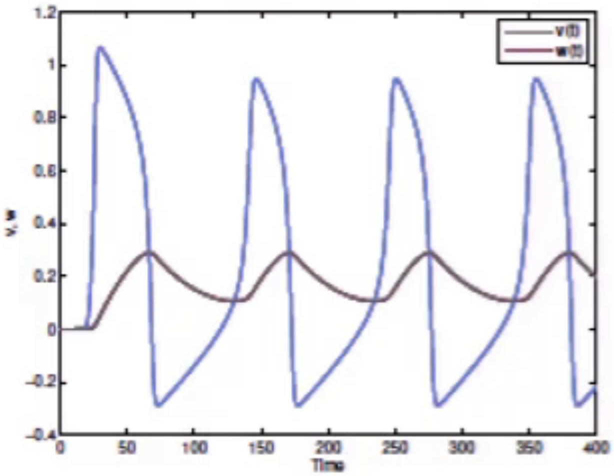
$$\dot{A} = (1+i\omega)A - (1+i\gamma)A|A|^2$$

Weak Coupling

- System of ODEs

$$\dot{u} = f(u), \quad u \in \mathbb{R}^n$$

- Exponentially stable limit cycle



FitzHugh-Nagumo: variables vs. time

- Assume weak coupling

$$\dot{u}_i = f_i(u_i) + \varepsilon h(u_i, u_j), \quad u \in \mathbb{R}^n$$

$$\varepsilon << 1, \quad i, j = 1, 2$$

- Leads to study phase dynamics

$$\dot{\phi}_i = \omega_i + \varepsilon H(u_i(\phi_i), u_j(\phi_j))$$

$$\phi_i \in S^1, \quad \varepsilon << 1, \quad i, j = 1, 2$$

Number of Oscillators

- N oscillators

$$\dot{\phi}_i = \omega_i + \frac{\varepsilon}{N}\sum_{j=1}^{N} G_{ij} H(\phi_i, \phi_j), \quad \varepsilon << 1, \quad i = 1, \cdots N$$

- Infinite number of oscillators
  - Continuum limit

$$\frac{\partial \phi}{\partial t}(x,t) = \omega(x) + \varepsilon \int G(|x-y|) H(\phi(y,t) - \phi(x,t))\, dy$$

Chemical Oscillators

$$\dot{U} = F(U) + D\Delta U, \quad U \in \mathbb{R}^n$$

For long temporal and spatial scales

$$\phi_t = \alpha\Delta\phi - \beta|\nabla\phi|^2 + \omega_*, \quad \phi \in \mathbb{R}/(2\pi\mathbb{Z}), \quad \omega_* = 0$$

Has solutions of the form

$$\phi(t, x) = kx - \omega t, \quad \omega = \beta|k|^2.$$

Similar to

- Periodic wave trains: $u(k \cdot x - \omega t)$ in R-D eq.
- Plane waves :$\sqrt{1 - |k|^2} e^{i(k \cdot x - \omega t)}$ in CGL eq.

Chemical Oscillations

Adding a localized inhomogeneity

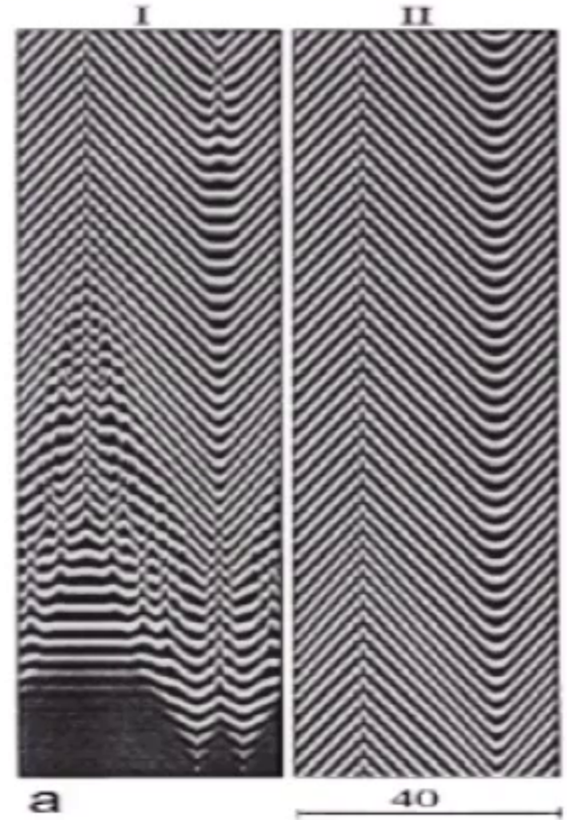
$$\phi_t = \alpha\Delta\phi - \beta|\nabla\phi|^2 + \varepsilon g(x),$$

we expect to obtain target patterns.
For $|x| \to \infty$

$$\phi(t,x) = kx - \omega t, \quad \omega = \beta|k|^2.$$

with group velocity

$$c_g = \frac{\partial\omega}{\partial k}$$
$$c_g = 2\beta k$$



Space time plot for target patterns in chemical reaction. From Zhabotinsky's et. al. in The J. Chem. Phys. 103.23 (1995)

Our Model

Our Model

$$\boxed{\phi_t = -\phi + G * \phi - (J * \phi_x)^2, \quad x \in \mathbb{R}}$$

- Biological oscillator
- Phase dynamics
- An infinite number of oscillators
- Array topology

Our Model

$$\boxed{\phi_t = -\phi + G * \phi - (J * \phi_x)^2, \quad x \in \mathbb{R}}$$

Assumptions on $L\phi = (G(x) - \delta) * \phi$:

- G is continuous, even, and exponentially localized
- $\hat{L}(0) = \hat{L}'(0) = 0, \qquad \hat{L}''(0) \neq 0$ because

$$\int G(x)\, dx = 1, \quad \int x^2 G(x)\, dx > 0,$$

Therefore, $\implies \phi_t = L\phi$ is a diffusive operator for $t \to \infty$.

- J & J_x are continuous and $\int J\, dx = J_0 \neq 0$.

Our Model

Q: What happens when we add a localized inhomogeneity?

$$\boxed{\phi_t = -\phi + G * \phi - (J * \phi_x)^2 + \varepsilon g(x), \quad x \in \mathbb{R}}$$

Results

Results

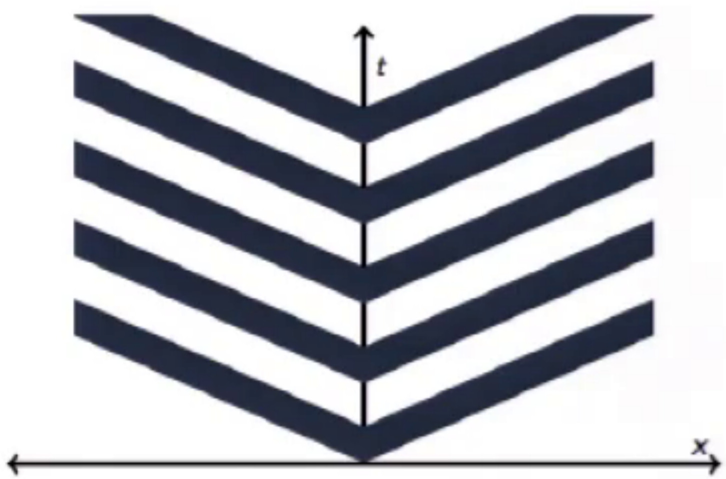
We find solutions of the form

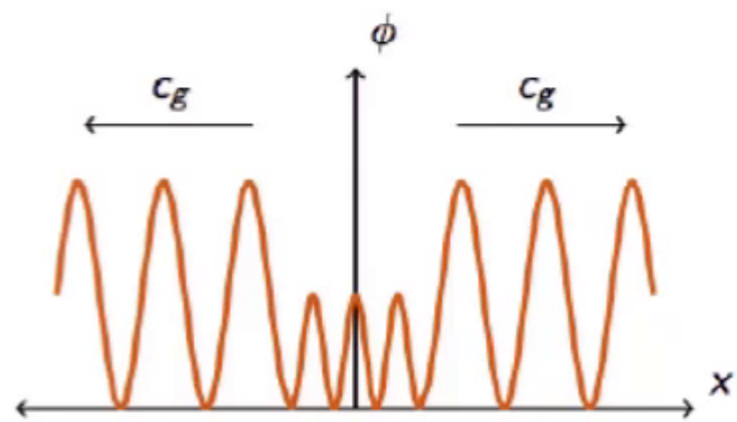
$$\Phi(x, t; \varepsilon) = \phi(x, \varepsilon) + (\phi_0(\varepsilon) + k(\varepsilon)x) \tanh(x) - \omega_{nl}(k(\varepsilon))t.$$

For $x \to \pm\infty$

$$\boxed{\Phi(x, t; \varepsilon) = \phi_0(\varepsilon) \pm k(\varepsilon)x - \omega_{nl}(k(\varepsilon))t}$$

with $c_g = 2J_0^2 \nabla \phi \cdot x > 0$.





Methods

Methods

We would like to use the Implicit Function Theorem.

$$F(u;\varepsilon) = Lu + N(u) + \varepsilon g(x) = 0$$

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots$$

At $O(\varepsilon)$:

$$Lu_1 + g(x) = 0 \quad \Longrightarrow u_1 = L^{-1} g$$

The linear operator L is **not invertible** in regular Sobolev spaces.

But L is a **Fredholm** operator in Kondratiev spaces.

Kondratiev Spaces

Definition: We denote **Kondratiev Spaces** by $M_\gamma^{s,p}(\mathbb{R})$, and define them as the completion of $C_0^\infty(\mathbb{R})$ under the norm,

$$\|u\|^p_{M_\gamma^{s,p}} = \sum_k^s \|(1+|x|^2)^{(\gamma+k)/2}\partial_x^k u\|_{L^p},$$

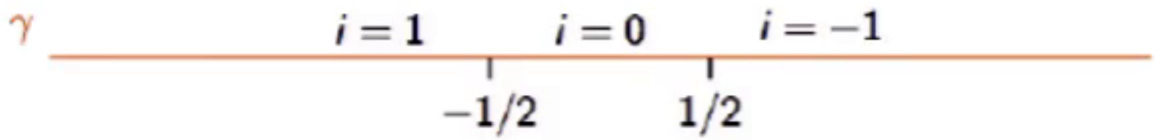
where s is a nonnegative integer, $p \in (1,\infty)$ and $\gamma \in \mathbb{R}$.

- Example: $u(x) = x \in M_\gamma^{2,2}$ if $\gamma < -3/2$
- $M_\gamma^{s,p} \not\subset L^p_{\gamma+s}$
- For $p^{-1} + q^{-1} = 1$, we have $(M_\gamma^{s,p})^* = M_{-\gamma}^{-s,q}$

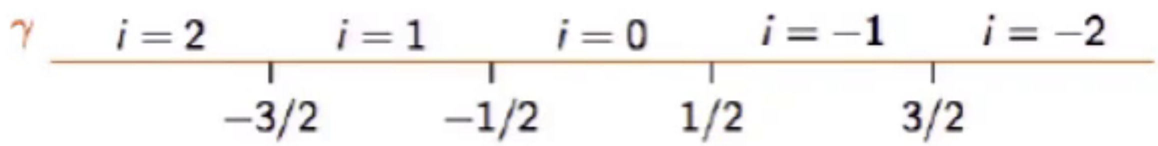
Kondratiev Spaces

Proposition: For the corresponding γ, the following operators are Fredholm with Fredholm index i.

$$\partial_x : M^{1,2}_{\gamma} \longrightarrow L^2_{\gamma+1}$$



$$\partial_{xx} : M^{2,2}_{\gamma} \longrightarrow L^2_{\gamma+2}$$



If ∂_{xx} is Fredholm $\implies L$ is Fredholm

Assume

- Domain $M^{2,2}_\gamma(\mathbb{R})$.
- The F.T. $\hat{L}(k)$ is uniformly bounded in k and satisfies

$$\hat{L}(0) = \hat{L}'(0) = 0, \qquad \hat{L}''(0) \neq 0,$$

Then we can define L as a composition of ∂_{xx} with invertible operators.

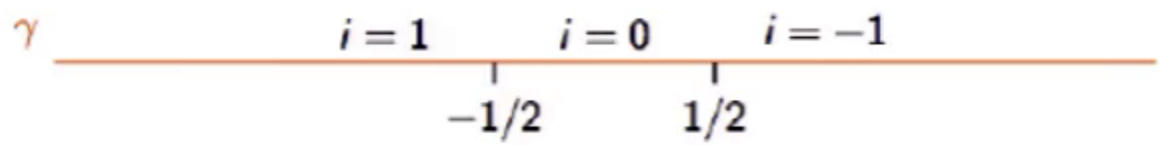
$$M^{2,2}_\gamma \xrightarrow{\partial_{xx}} L^2_{\gamma+2} \xrightarrow{(1-\partial_{xx})^{-1}} H^2_{\gamma+2} \xrightarrow{T} H^2_{\gamma+2},$$

Therefore, If ∂_{xx} is Fredholm $\implies L$ is Fredholm.

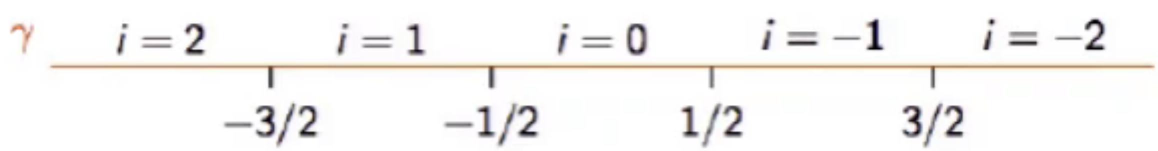
Kondratiev Spaces

Proposition: For the corresponding γ, the following operators are Fredholm with Fredholm index i.

$$\partial_x : M^{1,2}_{\gamma} \longrightarrow L^2_{\gamma+1}$$



$$\partial_{xx} : M^{2,2}_{\gamma} \longrightarrow L^2_{\gamma+2}$$



If ∂_{xx} is Fredholm $\implies L$ is Fredholm

Assume

- Domain $M^{2,2}_{\gamma}(\mathbb{R})$.
- The F.T. $\hat{L}(k)$ is uniformly bounded in k and satisfies

$$\hat{L}(0) = \hat{L}'(0) = 0, \qquad \hat{L}''(0) \neq 0,$$

Then we can define L as a composition of ∂_{xx} with invertible operators.

$$M^{2,2}_{\gamma} \xrightarrow{\partial_{xx}} L^2_{\gamma+2} \xrightarrow{(1-\partial_{xx})^{-1}} H^2_{\gamma+2} \xrightarrow{T} H^2_{\gamma+2},$$

Therefore, If ∂_{xx} is Fredholm $\implies L$ is Fredholm.

Proof

$$L\phi + N(\phi) + \varepsilon g = 0$$

- $L : M^{2,2}_{\gamma} \to H^2_{\gamma+2}$, $\gamma > 3/2$, is Fredholm index $i = -2$.

$$\Longrightarrow \phi = \underbrace{\bar{\phi}}_{M^{2,2}_{\gamma}} + \underbrace{(a + bx)\tanh(x)}_{\text{far field corrections}}$$

$$\Longrightarrow \bar{L}(\bar{\phi}, a, b) + \bar{N}(\bar{\phi}, a, b; \varepsilon) = 0, \quad \bar{N} \notin L^2_{\gamma+2}$$

- If $\bar{L}(\phi_1, a_1, b_1) = g$, then

$$\phi = \varepsilon(\phi_1 + \rho), \quad a = \varepsilon(a_1 + \alpha), \quad b = \varepsilon(b_1 + \beta)$$

gives

$$(\rho, \alpha, \beta) + \underbrace{\tilde{L}^{-1}}_{\text{looses loc.}} \underbrace{\tilde{N}(\rho, \alpha, \beta; \varepsilon)}_{\text{gains loc.}} = 0$$

Theorem

Model $\phi_t = -\phi + G * \phi - (J' * \phi)^2 + \varepsilon g(x), \quad x \in \mathbb{R} \quad (\star)$

Define $g_0 := \int g \, dx, \quad g_b = \int xg \, dx, \quad G_2 = \int x^2 G(x) \, dx.$

Theorem: We can find $\varepsilon_0 > 0$ s.t. for all $0 < |\varepsilon| < \varepsilon_0$ and $\text{sign}(\varepsilon) = -\text{sign}(g_0)$, there exists a solution to $(\star)$ of the form

$$\Phi(x, t; \varepsilon) = \phi(x, \varepsilon) + (\phi_0(\varepsilon) + k(\varepsilon)x) \tanh(x) - \omega_{nl}(k(\varepsilon))t$$

where

1. $\phi_0(\varepsilon), k(\varepsilon)$ are C^1
2. $\phi_0'(0) = -\frac{g_b}{G_2}, \quad k'(0) = \frac{g_0}{G_2},$
3. $|\phi(x, \varepsilon)| \to 0$, for $|x| \to \infty$, uniformly in ε.